Supplementary Materials for Hyperbolic Ordinal Embedding

Appendix A. Proof of Theorem 3

We first introduce a Sarker's $(1+\epsilon)$ distortion embedding given by Algorithm 5 in (Sarkar, 2011), before we prove that the embedding is a concrete instance of Theorem 3. In the following, the geodesic path in $\mathcal{H}^2$ from $\boldsymbol{x} \in \mathbb{H}^2$ to $\boldsymbol{y} \in \mathbb{H}^2$ is denoted by $c(\boldsymbol{x}, \boldsymbol{y})$. Let $\mathcal{G} = ([N], \mathcal{E})$ be a tree. Let $\deg(v)$ denotes the degree of $v \in [N]$, defined by

$$\deg(v) \coloneqq |\{u \in [N] \mid (u, v) \in \mathcal{E}\}|. \tag{27}$$

Let the maximum degree of any vertex in $\mathcal{G}$ denoted by $\deg(\mathcal{G})$, which is defined by

$$\deg(\mathcal{G}) \coloneqq \max\{\deg(v) \mid v \in [N]\}. \tag{28}$$

We denote the graph distance of graph $\mathcal{G} = ([N], \mathcal{E})$ by $d_{\mathcal{G}} : [N] \times [N] \to \mathbb{Z}_{\geq 0}$. In the following, we introduce Sarker's $(1+\epsilon)$ distortion embedding for tree $\mathcal{G}$, with distortion parameter $\epsilon \in \mathbb{R}_{>0}$. By regarding object N as the root, we can regard $\mathcal{G}$ as a rooted tree. For $v \in [N-1]$, let the parent of v be denoted by $\mathrm{ch}(1; v)$ and let the $k-1$-th child of v be denoted by $\mathrm{ch}(k; v)$. For the root, let the k-th child of v be denoted by $\mathrm{ch}(k; v)$. Here $k \in [\deg(v)]$ if $v = N$, and $k \in [\deg(v) - 1]$ otherwise. In particular, $k \in [\deg(\mathcal{G})]$. Fix $\beta \in \left(0, \frac{\pi}{\deg(\mathcal{G})}\right)$. Let $\alpha = \frac{2\pi}{\deg(\mathcal{G})} - \beta$, $\nu = -2\ln\left(\tan\frac{\beta}{2}\right)$, and $\tau = \nu\frac{1+\epsilon}{\epsilon}$. For the root $v = N$, first, arbitrarily place $\boldsymbol{x}_N$ in $\mathbb{H}^2$, then $\boldsymbol{x}_{\mathrm{ch}(1;v)}$ so that $d_{\mathbb{H}^2}\left(\boldsymbol{x}_N, \boldsymbol{x}_{\mathrm{ch}(1;v)}\right) = \tau$. Then, recursively, for all objects $v \in [N]$ whose embedding has been already placed, we place the embeddings $\boldsymbol{x}_{\mathrm{ch}(k;N)}$ $(k = 2, 3, \dots, \deg(N))$ of the children of v so that the following conditions are satisfied.

- $d_{\mathbb{H}^2}\left(\boldsymbol{x}_v, \boldsymbol{x}_{\mathrm{ch}(k;v)}\right) = \tau$.
- The angles $\left\{\angle \boldsymbol{x}_{\mathrm{ch}(k;v)} \boldsymbol{x}_v \boldsymbol{x}_{\mathrm{ch}(1;v)} \mid k = 2, 3, \dots, [\deg(v)]\right\}$ are mutually exclusively located in open intervals $\left\{\left(\frac{2\ell\pi}{\deg(\mathcal{G})} - \alpha, \frac{2\ell\pi}{\deg(\mathcal{G})} + \alpha\right) \,\middle|\, \ell \in [\deg(\mathcal{G}) - 1]\right\}$, where $\angle \boldsymbol{x}_{\mathrm{ch}(k;v)} \boldsymbol{x}_v \boldsymbol{x}_{\mathrm{ch}(1;v)}$ is the angle that $c\left(\boldsymbol{x}_v, \boldsymbol{x}_{\mathrm{ch}(k;v)}\right)$ makes with $c\left(\boldsymbol{x}_{\mathrm{ch}(1;v)}\right)$.

In the following, the embedding given by the above algorithm is called Sarker's $(1+\epsilon)$ distortion embedding. For any Sarker's $(1+\epsilon)$ distortion embedding, the following holds.

Theorem 12 (Theorem 6 in (Sarkar, 2011)) *Let $\mathcal{G} = ([N], \mathcal{E})$ be a tree. For all $\epsilon \in \mathbb{R}_{>0}$ and all embeddings $(\boldsymbol{x}_n)_{n \in [N]}$ given by Sarker's $(1+\epsilon)$ distortion embedding, the following holds: for any object pair $(u, v) \in [N] \times [N]$, $\frac{1}{1+\epsilon}\tau d_{\mathcal{G}}(u, v) < d_{\mathbb{H}^2}(x_u, x_v) < \tau d_{\mathcal{G}}(u, v)$, where $\tau = \nu\frac{1+\epsilon}{\epsilon}$.*

The previous theorem directly proves Theorem 3.
Proof [Proof of Theorem 3] Let $\mathrm{diam}(\mathcal{G})$ denote the diameter of $\mathcal{G}$, defined by

$$\mathrm{diam}(\mathcal{G}) \coloneqq \max\{d_{\mathcal{G}}(u, v) \mid u, v \in [N]\}. \tag{29}$$

According to Theorem 12 in (Sarkar, 2011), for any $\epsilon > 0$, there exists embedding $(\boldsymbol{x}_n)_{n\in[N]}$ and factor $\tau > 0$ such that for any object pair $(u, v) \in [N]\times[N]$, $\frac{1}{1+\epsilon}\tau d_{\mathcal{G}}(u, v) < d_{\mathbb{H}^2}(x_u, x_v) < \tau d_{\mathcal{G}}(u, v)$. By setting $\epsilon = \frac{1}{\operatorname{diam}(\mathcal{G})}$, we have embedding $(\boldsymbol{x}_n)_{n\in[N]}$ such that for any object pair $(u, v) \in [N]\times[N]$, $\tau[d_{\mathcal{G}}(u, v) - 1] < d_{\mathbb{H}^2}(x_u, x_v) < \tau d_{\mathcal{G}}(u, v)$, which completes the proof. ■

Appendix B. Proof of Theorem 8

Definition 13 *We say that $\mathcal{G} = ([N], \mathcal{E})$ includes a m-star if there exists a set of vertices $v_0, v_1, \ldots, v_m \in [N]$ such that for all $i = 1, 2, \ldots, m$, $(v_0, v_i) \in \mathcal{E}$ and for all $i, j = 1, 2, \ldots, m$ such that $i \neq j$, $(v_i, v_j) \notin \mathcal{E}$.*

The following trivial proposition states the relation between Definition 13 and tree.

Proposition 14 *If a graph $\mathcal{G}$ is a tree and $\deg(\mathcal{G}) = m$, then $\mathcal{G}$ includes a m-star.*

Proof [Proof of Theorem 8] Assume that the embedding $(\boldsymbol{x}_n)_{n\in[N]}$ in $\mathbb{R}^D$ that is non-contradictory to the complete ordinal triplet data of $\mathcal{G}$. According to Proposition 14, $\mathcal{G}$ includes a m-star. In this proof, the center of the sub m-star is relabeled $m + 1$ and the vertices that has an edge to $m + 1$ are relabeled $1, 2, \ldots, m$. In the following, $\|\cdot\|_2$ denotes the 2-norm defined by $\|\boldsymbol{x}\|_2 \coloneqq \sqrt{\boldsymbol{x}^\top \boldsymbol{x}}$, and the closed ball with center $\boldsymbol{x} \in \mathbb{R}^D$ and radius $R \in \mathbb{R}_{\geq 0}$ is denoted by $B_R[\boldsymbol{x}]$, defined by $B_R[\boldsymbol{x}] \coloneqq \left\{\boldsymbol{x}' \in \mathbb{R}^D \;\middle|\; \|\boldsymbol{x}' - \boldsymbol{x}\|_2 \leq R\right\}$. Without loss of generality, we can set $\boldsymbol{x}_{m+1} = \mathbf{0}$. Let $R \coloneqq \min\left\{\|\boldsymbol{x}_n\|_2 \mid n \in [m]\right\}$. By the assumption of non-contradiction of embedding, for all $n, n' \in [m]$ such that $n \neq n'$, it holds that $\boldsymbol{x}_{n'} \notin B_{\|\boldsymbol{x}_n\|_2}[\boldsymbol{x}_n]$. Define $\tilde{\boldsymbol{x}}_n \coloneqq \frac{1}{\|\boldsymbol{x}_n\|_2}\boldsymbol{x}_n$. For fixed $n, n' \in [m]$ such that they satisfy $n \neq n'$ and $\|\boldsymbol{x}_n\|_2 \geq \|\boldsymbol{x}_{n'}\|_2$, define $\boldsymbol{x}'_n \coloneqq \frac{\|\boldsymbol{x}_{n'}\|_2}{\|\boldsymbol{x}_n\|_2}\boldsymbol{x}_n$. As $\boldsymbol{x}_{n'} \notin B_{\|\boldsymbol{x}_n\|_2}[\boldsymbol{x}_n]$ and $B_{\|\boldsymbol{x}'_n\|_2}[\boldsymbol{x}'_n] \subset B_{\|\boldsymbol{x}_n\|_2}[\boldsymbol{x}_n]$, it follows that $\boldsymbol{x}_{n'} \notin B_{\|\boldsymbol{x}'_n\|_2}[\boldsymbol{x}'_n]$. By multiplying factor $\frac{1}{\|\boldsymbol{x}'_n\|_2}$, we have $\tilde{\boldsymbol{x}}_{n'} \notin B_1[\tilde{\boldsymbol{x}}_n]$. Hence, it holds that $d_{\mathbb{R}^D}(\tilde{\boldsymbol{x}}_n, \tilde{\boldsymbol{x}}_{n'}) > 1$. If we regard $\tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_m$ as points in the $D-1$ dimensional unit sphere, for all $n, n' \in [m]$ such that $n \neq n'$, it holds that $d_{\mathbb{S}^{D-1}}(\tilde{\boldsymbol{x}}_n, \tilde{\boldsymbol{x}}_{n'}) > \frac{\pi}{3}$. Therefore, m cannot be larger than the $\frac{\pi}{3}$-packing number of $\mathbb{S}^{D-1}$. ■