Supplementary Materials for Hyperbolic Ordinal Embedding

Appendix A. Proof of Theorem 3

We first introduce a Sarker’s \((1 + \epsilon)\) distortion embedding given by Algorithm 5 in (Sarkar, 2011), before we prove that the embedding is a concrete instance of Theorem 3. In the following, the geodesic path in \(H^2\) from \(x \in H^2\) to \(y \in H^2\) is denoted by \(c(x, y)\). Let \(G = ([N], E)\) be a tree. Let \(\deg(v)\) denotes the degree of \(v \in [N]\), defined by

\[
\deg(v) := |\{u \in [N] \mid (u, v) \in E\}|. \tag{27}
\]

Let the maximum degree of any vertex in \(G\) denoted by \(\deg(G)\), which is defined by

\[
\deg(G) := \max \{\deg(v) \mid v \in [N]\}. \tag{28}
\]

We denote the graph distance of graph \(G = ([N], E)\) by \(d_G : [N] \times [N] \rightarrow \mathbb{Z}_{\geq 0}\). In the following, we introduce Sarker’s \((1 + \epsilon)\) distortion embedding for tree \(G\), with distortion parameter \(\epsilon \in \mathbb{R}_{\geq 0}\). By regarding object \(N\) as the root, we can regard \(G\) as a rooted tree. For \(v \in [N - 1]\), let the parent of \(v\) be denoted by \(\text{ch}(1; v)\) and let the \(k - 1\)-th child of \(v\) be denoted by \(\text{ch}(k; v)\). For the root, let the \(k\)-th child of \(v\) be denoted by \(\text{ch}(k; v)\). Here \(k \in [\deg(v)]\) if \(v = N\), and \(k \in [\deg(v) - 1]\) otherwise. In particular, \(k \in [\deg(G)]\). Fix \(\beta \in \left(0, \frac{\pi}{\deg(G)}\right)\). Let \(\alpha = \frac{2\pi}{\deg(G)} - \beta\), \(\nu = -2\ln\left(\tan\frac{\beta}{2}\right)\), and \(\tau = \nu \frac{1 + \epsilon}{\epsilon}\). For the root \(v = N\), first, arbitrarily place \(x_N\) in \(H^2\), then \(x_{\text{ch}(1; v)}\) so that \(d_{H^2}(x_N, x_{\text{ch}(1; v)}) = \tau\). Then, recursively, for all objects \(v \in [N]\) whose embedding has been already placed, we place the embeddings \(x_{\text{ch}(k; v)}\) \((k = 2, 3, \ldots, \deg(N))\) of the children of \(v\) so that the following conditions are satisfied.

- \(d_{H^2}(x_v, x_{\text{ch}(k; v)}) = \tau\).
- The angles \(\{\angle x_{\text{ch}(k; v)} x_v x_{\text{ch}(1; v)} \mid k = 2, 3, \ldots, [\deg(v)]\}\) are mutually exclusive and located in open intervals \(\left\{(\frac{2\pi}{\deg(G)} - \alpha, \frac{2\pi}{\deg(G)} + \alpha) \mid \tau \in [\deg(G) - 1]\right\}\), where \(\angle x_{\text{ch}(k; v)} x_v x_{\text{ch}(1; v)}\) is the angle that \(c(x_v, x_{\text{ch}(k; v)})\) makes with \(c(x_{\text{ch}(1; v)})\).

In the following, the embedding given by the above algorithm is called Sarker’s \((1 + \epsilon)\) distortion embedding. For any Sarker’s \((1 + \epsilon)\) distortion embedding, the following holds.

**Theorem 12 (Theorem 6 in (Sarkar, 2011))** Let \(G = ([N], E)\) be a tree. For all \(\epsilon \in \mathbb{R}_{\geq 0}\) and all embeddings \((x_n)_{n \in [N]}\) given by Sarker’s \((1 + \epsilon)\) distortion embedding, the following holds: for any object pair \((u, v) \in [N] \times [N]\),

\[
\frac{1}{1 + \epsilon} \tau d_G(u, v) < d_{H^2}(x_u, x_v) < \tau d_G(u, v),
\]

where \(\tau = \nu \frac{1 + \epsilon}{\epsilon}\).

The previous theorem directly proves Theorem 3.

**Proof** [Proof of Theorem 3] Let \(\text{diam}(G)\) denote the diameter of \(G\), defined by

\[
\text{diam}(G) := \max \{d_G(u, v) \mid u, v \in [N]\}. \tag{29}
\]
According to Theorem 12 in (Sarkar, 2011), for any \( \epsilon > 0 \), there exists embedding \( (x_n)_{n \in [N]} \) and factor \( \tau > 0 \) such that for any object pair \( (u, v) \in [N] \times [N], \frac{1}{1+\epsilon} \tau d_G(u, v) < d_{SS}(x_u, x_v) < \tau d_G(u, v) \). By setting \( \epsilon = \frac{1}{\text{diam}(G)} \), we have embedding \( (x_n)_{n \in [N]} \) such that for any object pair \( (u, v) \in [N] \times [N], \tau [d_G(u, v) - 1] < d_{SS}(x_u, x_v) < \tau d_G(u, v) \), which completes the proof.

\[ \blacksquare \]

Appendix B. Proof of Theorem 8

Definition 13 We say that \( G = ([N], E) \) includes a \( m \)-star if there exists a set of vertices \( v_0, v_1, \ldots, v_m \in [N] \) such that for all \( i = 1, 2, \ldots, m, (v_0, v_i) \in E \) and for all \( i, j = 1, 2, \ldots, m \) such that \( i \neq j \), \( (v_i, v_j) \notin E \).

The following trivial proposition states the relation between Definition 13 and tree.

Proposition 14 If a graph \( G \) is a tree and \( \text{deg}(G) = m \), then \( G \) includes a \( m \)-star.

Proof [Proof of Theorem 8] Assume that the embedding \( (x_n)_{n \in [N]} \) in \( \mathbb{R}^D \) that is non-contradictory to the complete ordinal triplet data of \( G \). According to Proposition 14, \( G \) includes a \( m \)-star. In this proof, the center of the sub \( m \)-star is relabeled \( m + 1 \) and the vertices that has an edge to \( m + 1 \) are relabeled \( 1, 2, \ldots, m \). In the following, \( \| \cdot \|_2 \) denotes the 2-norm defined by \( \| x \|_2 : = \sqrt{x^\top x} \), and the closed ball with center \( x \in \mathbb{R}^D \) and radius \( R \in \mathbb{R}_{>0} \) is denoted by \( B_R(x) \), defined by \( B_R(x) := \{ x' \in \mathbb{R}^D \mid \| x' - x \|_2 \leq R \} \). Without loss of generality, we can set \( x_{m+1} = 0 \). Let \( R := \min \{ \| x_n \|_2 \mid n \in [m] \} \). By the assumption of non-contradiction of embedding, for all \( n, n' \in [m] \) such that \( n \neq n' \), it holds that \( x_n' \notin B_{\| x_n \|_2} x_n \). Define \( \bar{x}_n : = \frac{1}{\| x_n \|_2} x_n \). For fixed \( n, n' \in [m] \) such that they satisfy \( n \neq n' \) and \( \| x_n \|_2 \geq \| x_n' \|_2 \), define \( x'_n : = \frac{\| x_n' \|_2}{\| x_n \|_2} x_n \). As \( x_n' \notin B_{\| x_n \|_2} x_n \) and \( B_{\| x_n \|_2} x_n \subseteq B_{\| x_n \|_2} \), it follows that \( x_n' \notin B_{\| x_n \|_2} x_n \). By multiplying factor \( \frac{1}{\| x_n \|_2} \), we have \( \bar{x}_n' \notin B_1 \bar{x}_n \). Hence, it holds that \( d_{S^D}(\bar{x}_n, \bar{x}_n') > 1 \). If we regard \( \bar{x}_1, \bar{x}_1, \ldots, \bar{x}_m \) as points in the \( D - 1 \) dimensional unit sphere, for all \( n, n' \in [m] \) such that \( n \neq n' \), it holds that \( d_{S^{D-1}}(\bar{x}_n, \bar{x}_n') > \frac{\pi}{3} \). Therefore, \( m \) cannot be larger than the \( \frac{\pi}{3} \) packing number of \( S^{D-1} \). \( \blacksquare \)