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In Search of a Global Belief Model for Discrete-Time Uncertain Processes

Natan T'Joens NATAN.TJOENS@UGENT.BE
Jasper De Bock JASPER.DEBOCK@UGENT.BE
Gert de Cooman GERT.DECOOMAN@UGENT.BE
ELIS – FLip, Ghent University, Belgium

Abstract

To model discrete-time uncertain processes, we argue for the use of a global belief model in the form of an upper expectation that satisfies a number of simple and intuitive axioms. We motivate these axioms on the basis of two possible interpretations for this upper expectation: a behavioural interpretation similar to that of Walley's, and an interpretation in terms of upper envelopes of linear expectations. Subsequently, we show that the most conservative upper expectation satisfying our axioms coincides with a particular version of the game-theoretic upper expectation introduced by Shafer and Vovk. This has two important implications. On the one hand, it guarantees that there is a unique most conservative global belief model satisfying our axioms. On the other hand, it shows that Shafer and Vovk's model can be given an axiomatic characterisation, thereby providing an alternative motivation for adopting this model, even outside their framework.

Keywords: Game-theoretic probability, Upper expectations, Uncertain processes, Coherence

1. Introduction

There are various ways in which discrete-time uncertain processes, such as Markov processes, can be described mathematically. For many, measure theory has been the preferred framework to describe the uncertain dynamics of these processes. Others may use martingales or a game-theoretic approach to do so. The common starting point for all these approaches are the local belief models. They describe the dynamics of the process from one time instant to the next. In a measure-theoretic context, they are given in the form of (sets of) probability charges or (sets of) measures on the local state space; in a game-theoretic context, sets of allowable bets are used. When local state-spaces are assumed finite, as in our case, these descriptions are all mathematically equivalent. However, it is how these local models are extended to a global level that differs greatly from one theory to another. Measure theory uses the concept of sigma-additivity to do this, leading to a mathematically elegant, but rather abstract framework. Apart from being bounded to the constraint of measurability, it moreover relies on the questionable assumption of 'precision', in the sense that imprecision is always regarded as partial information about a 'precise' probability measure or charge. The game-theoretic framework by Shafer and Vovk has no need for such assumptions. However, since it defines global upper expectations in a constructive way using the concept of a 'supermartingale', it lacks a concrete identification in terms of mathematical properties.

Our aim here is to establish a global belief model, in the form of an upper expectation, that extends the information gathered in the local models by using a number of mathematical properties. Notably, this model will not be bound to a single interpretation. Instead, its characterising properties can be justified starting from a number of different interpretations. We here consider and discuss two of the most significant; see Section 4. We will then define the desired global model as the most conservative—the unique model that does not include any additional information apart from what is given—under this particular set of properties. In Section 5, we give a complete axiomatisation for this most conservative model, serving as an alternative definition. Finally and most importantly, we show that the obtained model is equal to a version of the global upper expectation defined by Shafer and Vovk [3, 4]. On the one hand, this serves as an additional motivation for the use of our model. On the other hand, it gives a concrete axiomatisation for this game-theoretic upper expectation.

2. Upper Expectations

We denote the set of all natural numbers, without 0, by $\mathbb{N}$, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The set of extended real numbers is denoted by $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ and is endowed with the usual order topology (corresponding to the two-point compactification of $\mathbb{R}$). The set of positive real numbers is denoted by $\mathbb{R}_{>}$ and the set of non-negative real numbers by $\mathbb{R}_{\geq}$. We also adopt the conventions that $+\infty - \infty = -\infty + \infty = +\infty$ and $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$.

Informally, we will consider a subject that is uncertain about the value that some variable Y assumes in a non-empty set $\mathscr{Y}$. More formally, we call any map on $\mathscr{Y}$ a *variable*; our informal Y is a special case: it corresponds to the identity map on $\mathscr{Y}$. A subject's uncertainty about the unknown value of Y can then be represented by an *upper expectation* $\overline{\mathrm{E}}$: an extended real-valued map on some subset

$\mathscr{D}$ of the set $\overline{\mathscr{L}}(\mathscr{Y})$ of all extended real-valued variables on $\mathscr{Y}$. An element f of $\overline{\mathscr{L}}(\mathscr{Y})$ is simply called an *extended real variable*. We say that f is bounded below if there is a real c such that $f(y) \geq c$ for all $y \in \mathscr{Y}$, and we say that f is bounded above if $-f$ is bounded below. An important role will be reserved for elements f of $\overline{\mathscr{L}}(\mathscr{Y})$ that are bounded, meaning that they are bounded above *and* below. These bounded real-valued variables on $\mathscr{Y}$ are called *gambles*, and we use $\mathscr{L}(\mathscr{Y})$ to denote the set of all of them. The set of all bounded below elements of $\overline{\mathscr{L}}(\mathscr{Y})$ is denoted by $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$.

Consider now the special case that $\overline{\mathrm{E}}$ is at least defined on the set of all bounded real-valued variables; so $\mathscr{L}(\mathscr{Y}) \subseteq \mathscr{D}$. Then we call $\overline{\mathrm{E}}$ *coherent* [9] if it satisfies the following three coherence axioms:

C1. $\overline{\mathrm{E}}(f) \leq \sup f$ for all $f \in \mathscr{L}(\mathscr{Y})$;

C2. $\overline{\mathrm{E}}(f+g) \leq \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(g)$ for all $f, g \in \mathscr{L}(\mathscr{Y})$;

C3. $\overline{\mathrm{E}}(\lambda f) = \lambda \overline{\mathrm{E}}(f)$ for all $\lambda \in \mathbb{R}_{\geq}$ and $f \in \mathscr{L}(\mathscr{Y})$.

If we let $\underline{\mathrm{E}}$ be the conjugate lower expectation associated with $\overline{\mathrm{E}}$, meaning that $\underline{\mathrm{E}}(-f) := -\overline{\mathrm{E}}(f)$ for all $f \in \mathscr{D}$, the following additional properties follow from C1–C3:

C4. $f \leq g \Rightarrow \overline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(g)$ for all $f, g \in \mathscr{L}(\mathscr{Y})$;

C5. $\inf f \leq \underline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(f) \leq \sup f$ for all $f \in \mathscr{L}(\mathscr{Y})$;

C6. $\overline{\mathrm{E}}(f+\mu) = \overline{\mathrm{E}}(f) + \mu$ for all $\mu \in \mathbb{R}$ and all $f \in \mathscr{L}(\mathscr{Y})$;

C7. for any sequence $\{f_n\}_{n \in \mathbb{N}_0}$ in $\mathscr{L}(\mathscr{Y})$:

$$\lim_{n \to +\infty} \sup |f - f_n| = 0 \Rightarrow \lim_{n \to +\infty} \overline{\mathrm{E}}(f_n) = \overline{\mathrm{E}}(f).$$

Proof of C4–C7 We only prove C5. This clearly implies that $\overline{\mathrm{E}}$ is real-valued on $\mathscr{L}(\mathscr{Y})$, and the remaining properties then follow from [9, Section 2.6.1.].

First, note that $\overline{\mathrm{E}}(0) = 0$ because of C3 and our convention that $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$. Therefore, for all $f \in \mathscr{L}(\mathscr{Y})$, it follows from C2 that $0 \leq \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(-f)$, or equivalently, due to our convention that $+\infty - \infty = -\infty + \infty = +\infty$, that $-\overline{\mathrm{E}}(-f) \leq \overline{\mathrm{E}}(f)$. Applying C1 to both sides, we find that $\inf f = -\sup(-f) \leq -\overline{\mathrm{E}}(-f) \leq \overline{\mathrm{E}}(f) \leq \sup f$. The result now follows readily from the definition of $\underline{\mathrm{E}}$. ■

A gamble f is typically interpreted as an uncertain reward or gain that depends on the value that Y takes in $\mathscr{Y}$; if Y takes the value y, the (possibly negative) gain is $f(y)$. Then, according to Walley's behavioural interpretation [9], the upper expectation $\overline{\mathrm{E}}(f)$ of a gamble f is a subject's infimum selling price for the gamble f. Axioms C1–C3 are then called rationality axioms, since they ensure that these selling prices are chosen in a rational way. However, any coherent upper expectation on $\mathscr{L}(\mathscr{Y})$ can equivalently be represented as an upper envelope of a set of *linear expectations* [9, Section 3.3.3]: coherent upper expectations on $\mathscr{L}(\mathscr{Y})$ that are self-conjugate, meaning that $\overline{\mathrm{E}}(f) = -\overline{\mathrm{E}}(-f)$ for all $f \in \mathscr{L}(\mathscr{Y})$. According to [1, Theorem 8.15], linear expectations on $\mathscr{L}(\mathscr{Y})$ are on their turn in a one-to-one relation with *probability charges* on the powerset $\mathbb{P}(\mathscr{Y})$ of $\mathscr{Y}$, being maps $\mu : \mathbb{P}(\mathscr{Y}) \to \mathbb{R}_{\geq}$ that are finitely additive and where $\mu(\mathscr{Y}) = 1$ and $\mu(\emptyset) = 0$. These probability charges are more general than the conventional notion of a *probability measure*, which additionally requires σ-additivity. If $\mathscr{Y}$ is finite, however, this distinction disappears.

It follows from the discussion above, that coherent upper expectations can be interpreted in two possible ways: in a direct behavioural way in terms of selling prices for gambles, or as a supremum over—an upper envelope of—a set of linear expectations. In this paper, we will not bound ourselves to any of these two interpretations. Instead, we will motivate the defining properties of our proposed global belief model in terms of both of these interpretations.

We conclude this section by introducing a method for extending the domain of a coherent upper expectation on $\mathscr{L}(\mathscr{Y})$ to the set $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ of all extended real variables that are bounded below. This will prove to be particularly useful when we introduce game-theoretic upper expectations in Section 6. We limit ourselves to the special case where $\mathscr{Y}$ is finite. To obtain the desired upper expectation $\overline{\mathrm{E}}'$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$, we will impose the following continuity property:

C8. For any non-decreasing sequence $\{f_n\}_{n \in \mathbb{N}_0}$ in $\mathscr{L}(\mathscr{Y})$:

$$\lim_{n \to +\infty} \overline{\mathrm{E}}'(f_n) = \overline{\mathrm{E}}'(\lim_{n \to +\infty} f_n).$$[1]

If $\lim_{n \to +\infty} f_n$ is a gamble, then if $\mathscr{Y}$ is finite, C8 is implied by C7, and therefore a consequence of coherence (C1–C3). Property C8 can therefore be regarded as a generalisation of C7 to extended real variables that are bounded below. Moreover, any coherent upper expectation $\overline{\mathrm{E}}$ on $\mathscr{L}(\mathscr{Y})$ can be uniquely extended to $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ if we impose C8.

Proposition 1 *Consider any* finite *set* $\mathscr{Y}$ *and a coherent upper expectation* $\overline{\mathrm{E}}$ *on* $\mathscr{L}(\mathscr{Y})$*. Then there exists a unique coherent upper expectation* $\overline{\mathrm{E}}'$ *on* $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ *that satisfies* C8 *and coincides with* $\overline{\mathrm{E}}$ *on* $\mathscr{L}(\mathscr{Y})$.

Proof Throughout this proof, for any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$, we let $\{f^{\wedge n}\}_{n \in \mathbb{N}_0}$ be the sequence defined by $f^{\wedge n}(y) := \min\{f(y), n\}$ for all $n \in \mathbb{N}_0$ and all $y \in \mathscr{Y}$. We first prove the *existence* of $\overline{\mathrm{E}}'$. Let $\overline{\mathrm{E}}'$ be the extended real-valued map defined by $\overline{\mathrm{E}}'(f) = \lim_{n \to +\infty} \overline{\mathrm{E}}(f^{\wedge n})$ for all $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$. Since $\overline{\mathrm{E}}$ satisfies C4 [because it is coherent] and $\{f^{\wedge n}\}_{n \in \mathbb{N}_0}$ is non-decreasing, the limit $\lim_{n \to +\infty} \overline{\mathrm{E}}(f^{\wedge n})$ exists, implying that $\overline{\mathrm{E}}'$ is well defined. That it coincides with $\overline{\mathrm{E}}$ on $\mathscr{L}(\mathscr{Y})$ follows from the fact that for all $f \in \mathscr{L}(\mathscr{Y})$ and

1. Here, as well as in what follows, limits of variables are always intended to be taken *pointwise*.

$n \geq \max f$, $f^{\wedge n} = f$. Since $\overline{\mathrm{E}}$ is a coherent upper expectation, this immediately implies that $\overline{\mathrm{E}}'$ is a coherent upper expectation as well. We now show that it satisfies C8.

Consider any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ and a non-decreasing sequence $\{g_n\}_{n\in\mathbb{N}_0}$ in $\mathscr{L}(\mathscr{Y})$ that converges pointwise to f. Since $\overline{\mathrm{E}}'$ satisfies C4 and $\{g_n\}_{n\in\mathbb{N}_0}$ is non-decreasing, the limit $\lim_{n\to+\infty}\overline{\mathrm{E}}'(g_n)$ exists. Fix any $n \in \mathbb{N}_0$. Because $g_n \leq f$, we have that $g_n \leq f^{\wedge n'}$ for all $n' \in \mathbb{N}_0$ such that $n' \geq \max g_n$. This implies by C4 that $\overline{\mathrm{E}}'(g_n) \leq \overline{\mathrm{E}}'(f^{\wedge n'})$ for all $n' \in \mathbb{N}_0$ such that $n' \geq \max g_n$, so $\overline{\mathrm{E}}'(g_n) \leq \lim_{n'\to+\infty}\overline{\mathrm{E}}'(f^{\wedge n'}) = \lim_{n'\to+\infty}\overline{\mathrm{E}}(f^{\wedge n'}) = \overline{\mathrm{E}}'(f)$. Since this holds for any $n \in \mathbb{N}_0$, we have that $\lim_{n\to+\infty}\overline{\mathrm{E}}'(g_n) \leq \overline{\mathrm{E}}'(f)$. To prove the other inequality, fix any $\varepsilon > 0$ and any $n' \in \mathbb{N}_0$. Since $\{g_n\}_{n\in\mathbb{N}_0}$ increases pointwise to f, and $f^{\wedge n'}$ is a gamble such that $f^{\wedge n'} \leq f$, there is an index $n^*(y)$ for each $y \in \mathscr{Y}$ such that $f^{\wedge n'}(y) - \varepsilon \leq g_n(y)$ for all $n \geq n^*(y)$. Let $n_{\max} := \max_{y\in\mathscr{Y}} n^*(y)$. Since $\mathscr{Y}$ is finite, this maximum is attained. So we have that $f^{\wedge n'} - \varepsilon \leq g_n$ for all $n \geq n_{\max}$. By C4 and C6, this in turn implies that $\overline{\mathrm{E}}'(f^{\wedge n'}) - \varepsilon \leq \overline{\mathrm{E}}'(g_n)$ for all $n \geq n_{\max}$. Hence, $\overline{\mathrm{E}}'(f^{\wedge n'}) - \varepsilon \leq \lim_{n\to+\infty}\overline{\mathrm{E}}'(g_n)$. This holds for any $n' \in \mathbb{N}_0$, implying that

$$\overline{\mathrm{E}}'(f) - \varepsilon = \lim_{n'\to+\infty}\overline{\mathrm{E}}'(f^{\wedge n'}) - \varepsilon \leq \lim_{n\to+\infty}\overline{\mathrm{E}}'(g_n),$$

and since this holds for any $\varepsilon > 0$, we indeed find that $\overline{\mathrm{E}}'(f) \leq \lim_{n\to+\infty}\overline{\mathrm{E}}'(g_n)$. Hence, we conclude that $\overline{\mathrm{E}}'$ satisfies C8, establishing the existence of $\overline{\mathrm{E}}'$.

To prove the uniqueness of $\overline{\mathrm{E}}'$, consider two coherent upper expectations $\overline{\mathrm{E}}'_1$ and $\overline{\mathrm{E}}'_2$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ that both satisfy C8 and coincide with $\overline{\mathrm{E}}$ on $\mathscr{L}(\mathscr{Y})$. Consider now any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ and a non-decreasing sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\mathscr{L}(\mathscr{Y})$ that converges pointwise to f. Since $\overline{\mathrm{E}}'_1$ and $\overline{\mathrm{E}}'_2$ coincide with $\overline{\mathrm{E}}$ on all gambles, we directly have that $\lim_{n\to+\infty}\overline{\mathrm{E}}'_1(f_n) = \lim_{n\to+\infty}\overline{\mathrm{E}}'_2(f_n)$. Applying C8 to $\overline{\mathrm{E}}'_1$ and $\overline{\mathrm{E}}'_2$ therefore implies that $\overline{\mathrm{E}}'_1(f) = \overline{\mathrm{E}}'_2(f)$, proving the uniqueness of $\overline{\mathrm{E}}'$. ■

3. Upper Expectations in Discrete-Time Uncertain Processes

We consider a *discrete-time uncertain process*, being a sequence $X_1, X_2, \ldots, X_n, \ldots$ of uncertain states, where the state X_k at each discrete time $k \in \mathbb{N}$ takes values in a fixed non-empty finite set $\mathscr{X}$, called the *state space*.

Let a *situation* $x_{1:n}$ be any finite string $(x_1, \ldots, x_n) \in \mathscr{X}_{1:n} := \mathscr{X}^n$ of possible state values. In particular, the unique empty string $x_{1:0}$, denoted by $\Box$, is called the *initial situation*: $\mathscr{X}_{1:0} := \{\Box\}$. We denote the set of all situations by $\mathscr{X}^* := \cup_{n\in\mathbb{N}_0}\mathscr{X}_{1:n}$. We also use the generic notations s and t to denote any situation.

To model our uncertainty about the dynamics of an uncertain process, we associate, with every situation $x_{1:n} \in \mathscr{X}^*$, a coherent upper expectation $\overline{\mathrm{Q}}_{x_{1:n}}$ on $\mathscr{L}(\mathscr{X})$. This upper expectation expresses a subject's beliefs about the uncertain value of the next state X_{n+1} when she has observed that $X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}$ and $X_n = x_n$. As discussed in the previous section, these upper expectations can be given both a behavioural interpretation or an interpretation in terms of upper envelopes of expectations. We will also refer to $\overline{\mathrm{Q}}_{x_{1:n}}$ as the *local* model or upper expectation associated with $x_{1:n}$, because it gives information about how the process changes from one time instant to the next. An *imprecise probability tree* $\mathscr{T}$ is a function that maps any situation s to its corresponding coherent upper expectation $\overline{\mathrm{Q}}_s$. Hence, an imprecise probability tree $\mathscr{T}$ models the dynamics of the uncertain process as a whole.

In practice, the local behaviour of an uncertain process is typically learned from physical measurements or elicited from experts. For instance, we usually have some information about *'The probability of throwing heads on the next coin toss'*, *'The expected amount of goods that are sold by a certain shop on a single day'*, *'The probability of rain tomorrow'*, ... However, it is less straightforward how we should gather information about other, more general inferences such as *'The expected number of tosses until the first tails is thrown'*, *'The probability of being out of stock on a given day'*, ... Moreover, even if we could in principle do so, it is often not possible or feasible to gather all the necessary information because of time and budget limitations. Hence, the question arises: 'How and to which extent, can we extend the information captured in the local models towards global information about the entire process?'. To answer this question, we will represent this global information with a so-called *global belief model*, being a particular kind of upper expectation. Before we proceed to do so, we finish this section with some further notation about uncertain processes.

An infinite sequence $x_1x_2x_3\cdots$ of state values is called a *path*, which we denote by $\omega = x_1x_2x_3\cdots$. We gather all paths in the *sample space* $\Omega := \mathscr{X}^{\mathbb{N}}$. For any path $\omega \in \Omega$, the situation $x_{1:n} = x_1x_2\cdots x_n$ that consists of its first n state values is denoted by $\omega^n \in \mathscr{X}_{1:n}$. The state value x_n at time n is denoted by $\omega_n \in \mathscr{X}$. An *event* $A \subseteq \Omega$ is a collection of paths, and in particular, the *cylinder event* $\Gamma(x_{1:n}) := \{\omega \in \Omega \colon \omega^n = x_{1:n}\}$ of some situation $x_{1:n} \in \mathscr{X}^*$, is the set of all paths $\omega \in \Omega$ that go through the situation $x_{1:n}$.

A variable on Ω is called a *global* variable and we gather all extended real ones in $\overline{\mathbb{V}} := \overline{\mathscr{L}}(\Omega)$. Similarly, we let $\overline{\mathbb{V}}_{\mathrm{b}} := \overline{\mathscr{L}}_{\mathrm{b}}(\Omega)$ and $\mathbb{V} := \mathscr{L}(\Omega)$. For any natural $k \leq \ell$, we let $X_{k:\ell}$ be the global variable defined by $X_{k:\ell}(\omega) := (\omega_k, \ldots, \omega_\ell)$ for all $\omega \in \Omega$. As such, the state $X_k = X_{k:k}$ at time k can also be regarded as a global variable. Moreover, for any natural $k \leq \ell$ and any map $f \colon \mathscr{X}^{\ell-k+1} \to \overline{\mathbb{R}}$, we will write $f(X_{k:\ell})$ to denote the global extended real variable defined by $f(X_{k:\ell}) := f \circ X_{k:\ell}$. We call a global extended real variable f *n-measurable* for some $n \in \mathbb{N}_0$, if it only depends on the initial n state values; so $f(\omega_1) = f(\omega_2)$ for any two paths ω_1 and ω_2 such that $\omega_1^n = \omega_2^n$. We will then use the notation $f(x_{1:n})$ for its constant value $f(\omega)$ on all paths $\omega \in \Gamma(x_{1:n})$. The *indicator* $\mathbb{I}_A$ of an event A is defined as the global variable that assumes the value 1 on A and 0 elsewhere. Hence, the *indicator* $\mathbb{I}_{x_{1:n}} := \mathbb{I}_{\Gamma(x_{1:n})}$ of the cylinder event $\Gamma(x_{1:n})$ for some $x_{1:n} \in \mathscr{X}^*$ is clearly an n-measurable variable. Finally, we call any $f \in \overline{\mathbb{V}}$ *finitary* if it is n-measurable for some $n \in \mathbb{N}_0$. We gather all finitary gambles in $\mathbb{V}_{\text{fin}}$.

4. In Search of a Global Belief Model

Any extended real-valued map $\overline{\mathrm{E}}\colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ will be called a *global upper expectation*. Given an imprecise probability tree $\mathscr{T}$ that associates a local upper expectation $\overline{\mathrm{Q}}_s$ with every situation $s \in \mathscr{X}^*$, we aim to define a global upper expectation $\overline{\mathrm{E}}$ that extends the information included in these local models in a rational way. To do so, we will impose the following properties.

P1. For any $f \in \mathscr{L}(\mathscr{X})$ and any $x_{1:n} \in \mathscr{X}^*$,

$$\overline{\mathrm{E}}(f(X_{n+1})|x_{1:n}) = \overline{\mathrm{Q}}_{x_{1:n}}(f).$$

P2. For any $f \in \mathbb{V}_{\text{fin}}$ and any $s \in \mathscr{X}^*$,

$$\overline{\mathrm{E}}(f|s) = \overline{\mathrm{E}}(f\mathbb{I}_s|s).$$

P3. For any $f \in \mathbb{V}_{\text{fin}}$ and any $k \in \mathbb{N}_0$,

$$\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k}).$$

P4. For any $f, g \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$,

$$f \leq g \Rightarrow \overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(g|s).$$

P5. For any sequence $\{f_n\}_{n\in\mathbb{N}_0}$ of finitary gambles that is uniformly bounded below and any $s \in \mathscr{X}^*$:

$$\lim_{n\to+\infty} f_n = f \Rightarrow \limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) \geq \overline{\mathrm{E}}(f|s).$$

Here, as well as further on, we call a sequence $\{f_n\}_{n\in\mathbb{N}_0}$ of extended real variables uniformly bounded below if there is a real c such that $f_n(\omega) \geq c$ for all $n \in \mathbb{N}_0$ and $\omega \in \Omega$.

To motivate the use of P1–P5, we need to link some interpretational meaning to $\overline{\mathrm{E}}$. We here consider two particular ones, similar to what we have done for the coherent upper expectations in Section 2.

We start from the interpretation of a global gamble $f \in \mathbb{V}$ as an uncertain reward depending on the path ω that the uncertain process takes in Ω. However, it is not clear what this exactly means if the gamble f depends on the entire length of the path. Indeed, the gamble f depends on an infinite number of subsequent state values, so there is no point in time when we can actually exchange the reward linked to the gamble f. The same interpretational problem arises when considering unbounded or extended real variables (on a general set $\mathscr{Y}$). The simple interpretation of an uncertain reward does not suffice here because the reward itself can be unbounded or infinite, which is unrealistic—or even meaningless—in practice. For that reason, we prefer to only attach a direct practical meaning to the value $\overline{\mathrm{E}}(f|s)$ of a global upper expectation $\overline{\mathrm{E}}$ for a finitary gamble $f \in \mathbb{V}_{\text{fin}}$ conditional on a situation $s \in \mathscr{X}^*$. Indeed, these finitary gambles themselves can be given a meaningful interpretation because they take real values and only depend on the state at a finite number of time instances.

We distinguish the following two ways for interpreting the global upper expectation $\overline{\mathrm{E}}(f|s)$ of a finitary gamble $f \in \mathbb{V}_{\text{fin}}$ conditional on a situation $s \in \mathscr{X}^*$:

- **Behavioural interpretation.** It is a subject's infimum selling price for f *contingent* on the event $\Gamma(s)$, meaning that, for any $\alpha > \overline{\mathrm{E}}(f|s)$, she is willing to accept the uncertain reward associated with the gamble $\mathbb{I}_s(\alpha - f)$.
- **An upper envelope.** It is the supremum value of $\mathrm{E}(f|s)$, where E belongs to some given set $\mathbb{E}$ of conditional linear expectation operators: $\overline{\mathrm{E}}(f|s) = \sup\{\mathrm{E}(f|s) \colon \mathrm{E} \in \mathbb{E}\}$.

Since P1–P3 only apply to finitary gambles, a direct justification for these axioms can easily be given for each of the above interpretations. Property P1 imposes that the global model $\overline{\mathrm{E}}$ should be compatible with the local models $\overline{\mathrm{Q}}_s$. The desirability of this property is self-evident, no matter which interpretation is used. Property P2 says that the upper expectation of a finitary gamble f conditional on s should only depend on the value of f on the paths $\omega \in \Gamma(s)$. This property is clearly desirable when using the behavioural interpretation, because $\mathbb{I}_s(\alpha - f)$ only depends on the restriction of f to $\Gamma(s)$. It is also quite evident that this property is desirable for an upper envelope of conditional linear expectations, because P2 is true for these conditional linear expectations themselves. Similarly, that P3 should hold under the upper envelope interpretation, is motivated by the fact that conditional linear expectations satisfy P3 with equality—then better known as the law of iterated expectation; an upper envelope of conditional expectations is therefore guaranteed to satisfy P3. In order to see that property P3 is also desirable according to the behavioural interpretation, one requires a conditional version of the notion of coherence that we discussed in Section 2 [9, 10, 1].[2] Explaining why this is the case, and what this conditional notion of coherence exactly entails, would however lead us too far. Basically, if P3 would not hold, the selling prices that are implied by $\overline{\mathrm{E}}$ would allow for a Dutch book—a loss no matter the outcome—which we consider irrational.

2. There are several versions of conditional coherence [9, 10, 1], however, in the case where variables take values in a finite set, all these different versions are mathematically equivalent.

Having attached an interpretation to finitary gambles and their conditional upper expectations, we now proceed to do the same for more general variables. We have already argued that no direct practical meaning can be given to such variables. However, this should not be taken to imply that an uncertainty model should not be able to deal with them. In fact, they can serve as useful abstract idealisations of (sequences of) variables that can be given a direct practical meaning. In particular, any extended real variable f that is bounded below and that can be written as the pointwise limit $\lim_{n\to+\infty} f_n$ of some sequence of finitary gambles $\{f_n\}_{n\in\mathbb{N}_0}$, we will regard as an abstract idealisation of f_n for *large* n. We gather these limits in the set

$$\overline{\mathbb{V}}_{\mathrm{b,lim}} := \{f \in \overline{\mathbb{V}}_{\mathrm{b}} \colon f = \lim_{n\to+\infty} f_n \text{ for some sequence } \{f_n\}_{n\in\mathbb{N}_0} \text{ in } \mathbb{V}_{\mathrm{fin}}\}.$$

Since P5 applies to precisely these kinds of variables, this axiom can be justified by extending the above idealisation from the variables f to their upper expectations $\overline{\mathrm{E}}(f|s)$. Basically, since f is an abstract idealisation of f_n for large n, $\overline{\mathrm{E}}(f|s)$ should be an abstract idealisation of $\overline{\mathrm{E}}(f_n|s)$ for large n. The practical benefit of this is that we can then use $\overline{\mathrm{E}}(f|s)$ to reason about $\overline{\mathrm{E}}(f_n|s)$ for a generic large value of n, without having to specify the specific value of n. The problem, however, is that the sequence $\{\overline{\mathrm{E}}(f_n|s)\}_{n\in\mathbb{N}_0}$ may not converge. What we then do know for sure, however, is that as n approaches infinity, $\overline{\mathrm{E}}(f_n|s)$ will oscillate between the limit superior and inferior of $\{\overline{\mathrm{E}}(f_n|s)\}_{n\in\mathbb{N}_0}$. Since we want $\overline{\mathrm{E}}(f|s)$ to serve as an idealisation of $\overline{\mathrm{E}}(f_n|s)$ for generic large values of n, $\overline{\mathrm{E}}(f|s)$ should therefore definitely not exceed the limit superior, as this would result in an unwarranted loss of information. We therefore impose P5.

The final property that we impose is P4, which states that $\overline{\mathrm{E}}$ should be monotone. For finitary gambles f and g, this follows easily from either of our two different interpretations for $\overline{\mathrm{E}}(f|s)$ and $\overline{\mathrm{E}}(g|s)$. Under a behavioural interpretation, since the reward associated with g is at least as high as that of f, the same should be true for a subject's selling prices for these two gambles. Under an interpretation in terms of upper envelopes of expectations, monotonicity of the envelope is implied by the monotonicity of each of the individual expectations. If f and g are more general variables—so not necessarily finitary gambles—the motivation for P4 is that still, higher rewards—even if abstract and idealized—should correspond to higher upper expectations. It is also worth noting that the combination of P4 and P5 implies that $\overline{\mathrm{E}}$ is continuous with respect to non-decreasing sequences of finitary gambles. In a measure-theoretic context, this kind of continuity is usually obtained as a consequence of the assumption of σ-additivity [6, 2].

We will show in Section 6 that P1–P5 are compatible, in the sense that if the local models $\overline{\mathrm{Q}}_s$ are coherent, there always is at least one global upper expectation $\overline{\mathrm{E}}$ satisfying P1–P5. This would, for example, not be the case if we replace P5 by the stronger property of continuity with respect to pointwise convergence.

However, there may be more than one global upper expectation $\overline{\mathrm{E}}$ satisfying P1–P5. In that case, the best thing to do, we think, is to choose the most conservative model among those that satisfy P1–P5, as choosing any other would mean adding information that is not implied by our axioms. We will denote this most conservative global upper expectation by $\overline{\mathrm{E}}^*$. As we will see in Section 6, $\overline{\mathrm{E}}^*$ is guaranteed to exist, and furthermore coincides with a particular version of the game-theoretic upper expectation defined by Shafer and Vovk [3, 5].

Of course, in order for our definition of $\overline{\mathrm{E}}^*$ to make sense, we need to know what it means for an upper expectation $\overline{\mathrm{E}}^*$ to be more conservative than some other upper expectation $\overline{\mathrm{E}}$. We here take this to mean that $\overline{\mathrm{E}}^*$ is higher than $\overline{\mathrm{E}}$. So higher upper expectations are more conservative, or less informative. In fact, we already used this implicitly in our motivation for P5, when we talked about an unwarranted 'loss of information'. That it is indeed reasonable to regard higher expectations as more conservative, can again be motivated using either of the two interpretations that we considered before. Under the behavioural interpretation, higher upper expectations means higher selling prices, which is clearly more conservative. Using an interpretation in terms of upper envelopes of expectations, higher upper expectations correspond to larger sets of expectations, which is again less informative and hence more conservative.

All of that said, we would like to stress that—despite our extensive use of them to motivate our axioms—none of the results that we are about to develop hinge on a particular interpretation for upper expectations. Other interpretations could also be adopted, or perhaps even no interpretation at all. All that is needed is to agree on P1–P5 and on the fact that higher upper expectations are more conservative.

5. An Axiomatisation of $\overline{\mathrm{E}}^*$

For a given imprecise probability tree $\mathscr{T}$, let $\overline{\mathbb{E}}_{1-2}(\mathscr{T})$ be the set of all global upper expectations satisfying P1–P2, and similarly for $\overline{\mathbb{E}}_{1-4}(\mathscr{T})$ and $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$. In this section, we introduce sufficient conditions for a global upper expectation $\overline{\mathrm{E}}$ to be the most conservative among all upper expectations in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$. We will start by considering the domain of finitary gambles and then, step by step, extend the domain and introduce additional conditions on $\overline{\mathrm{E}}$ such that it is the most conservative on this extended domain.

For any situation $x_{1:n} \in \mathscr{X}^*$ and any $(n+1)$-measurable gamble f, we use $f(x_{1:n}\cdot)$ to denote the gamble on $\mathscr{X}$ that assumes the value $f(x_{1:n+1})$ on $x_{n+1} \in \mathscr{X}$, and then use $f(x_{1:n}X_{n+1})$ as a shorthand for $f(x_{1:n}\cdot)(X_{n+1})$. The following lemma establishes compatibility with the local models in a stronger way than P1 does.

Lemma 2 *Consider any $\overline{\mathrm{E}} \in \overline{\mathbb{E}}_{1-2}(\mathscr{T})$. Then, for any situation $x_{1:n} \in \mathscr{X}^*$ and any $(n+1)$-measurable gamble f,*

$$\overline{\mathrm{E}}(f|x_{1:n}) = \overline{\mathrm{Q}}_{x_{1:n}}(f(x_{1:n}\cdot)).$$

Proof Fix any $x_{1:n} \in \mathscr{X}^*$ and any $(n+1)$-measurable gamble f. Note that $f(x_{1:n}X_{n+1})\mathbb{I}_{x_{1:n}} = f\mathbb{I}_{x_{1:n}}$ and hence, because of P2, $\overline{\mathrm{E}}(f|x_{1:n}) = \overline{\mathrm{E}}(f(x_{1:n}X_{n+1})|x_{1:n})$. P1 therefore implies that $\overline{\mathrm{E}}(f|x_{1:n}) = \overline{\mathrm{Q}}_{x_{1:n}}(f(x_{1:n}\cdot))$. ∎

To ensure that $\overline{\mathrm{E}}$ is the most conservative upper expectation on the domain of all finitary gambles, we impose the following property, known as *the law of iterated upper expectations*:

P3'. For any $f \in \mathbb{V}_{\mathrm{fin}}$ and any $k \in \mathbb{N}_0$,

$$\overline{\mathrm{E}}(f|X_{1:k}) = \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k}).$$

Proposition 3 *Consider any $\overline{\mathrm{E}} \in \overline{\mathbb{E}}_{1-4}(\mathscr{T})$ that satisfies P3'. Then, for any $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-4}(\mathscr{T})$, we have that*

$$\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}'(f|s) \text{ for all } f \in \mathbb{V}_{\mathrm{fin}} \text{ and all } s \in \mathscr{X}^*.$$

Proof Fix any $f \in \mathbb{V}_{\mathrm{fin}}$ and any $x_{1:m} \in \mathscr{X}^*$. We show that $\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}(f|x_{1:m})$ for all $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-4}(\mathscr{T})$. Since f is finitary, it is n-measurable for some $n \in \mathbb{N}_0$. We can assume that $m+2 < n$ without loss of generality, because f is obviously also p-measurable for every $p \geq n$. Now, it follows from P3 that

$$\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}'(\overline{\mathrm{E}}'(f|X_{1:m+1})|x_{1:m}).$$

Repeating this argument and using P4, gives us

$$\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}'(\overline{\mathrm{E}}'(\cdots\overline{\mathrm{E}}'(f|X_{1:n-1})\cdots|X_{1:m+1})|x_{1:m}). \tag{1}$$

For any $p \in \mathbb{N}_0$ and any $(p+1)$-measurable gamble g, we now let $\overline{\mathrm{Q}}_{X_{1:p}}(g)$ be the p-measurable gamble defined by

$$\overline{\mathrm{Q}}_{X_{1:p}}(g)(\omega) := \overline{\mathrm{Q}}_{\omega^p}(g(\omega^p\cdot)) \text{ for all } \omega \in \Omega.$$

Note that $\overline{\mathrm{Q}}_{X_{1:p}}(g)$ is indeed a gamble because of coherence [C5] and the fact that g is a gamble. Then, because $\overline{\mathrm{E}}'$ satisfies P1 and P2, it follows from Lemma 2 that

$$\overline{\mathrm{E}}'(g|X_{1:p}) = \overline{\mathrm{Q}}_{X_{1:p}}(g).$$

Applying this to Equation (1) gives us

$$\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}'(\overline{\mathrm{E}}'(\cdots\overline{\mathrm{Q}}_{X_{1:n-1}}(f)\cdots|X_{1:m+1})|x_{1:m}),$$

and repeating the argument results in

$$\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{Q}}_{x_{1:m}}\left(\overline{\mathrm{Q}}_{X_{1:m+1}}\left(\cdots\overline{\mathrm{Q}}_{X_{1:n-1}}(f)\cdots\right)\right).$$

Since $\overline{\mathrm{E}}$ also satisfies P1 and P2, we can reverse these steps [again using Lemma 2], to find that

$$\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\cdots\overline{\mathrm{E}}(f|X_{1:n-1})\cdots|X_{1:m+1})|x_{1:m}),$$

and since $\overline{\mathrm{E}}$ satisfies P3', this results in $\overline{\mathrm{E}}'(f|x_{1:m}) \leq \overline{\mathrm{E}}(f|x_{1:m})$. ∎

Next, we consider the domain $\overline{\mathbb{V}}_{\mathrm{b,lim}} \subset \overline{\mathbb{V}}$ of all extended real variables that are bounded below and that can be written as the pointwise limit of a sequence of finitary gambles. The following condition, together with P3', is sufficient for an upper expectation $\overline{\mathrm{E}}$ to be the most conservative on $\overline{\mathbb{V}}_{\mathrm{b,lim}}$ among all upper expectations in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$.

P6. For any $f \in \overline{\mathbb{V}}_{\mathrm{b,lim}}$ and any $s \in \mathscr{X}^*$, there is a sequence $\{f_n\}_{n\in\mathbb{N}_0}$ of n-measurable gambles that is uniformly bounded below and that converges pointwise to f such that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s).$$

Proposition 4 *Consider any $\overline{\mathrm{E}} \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$ that satisfies P3' and P6. Then, for any $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$, we have that*

$$\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}'(f|s) \text{ for all } f \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and all } s \in \mathscr{X}^*.$$

Proof Fix any $f \in \overline{\mathbb{V}}_{\mathrm{b,lim}}$, any $s \in \mathscr{X}^*$ and any $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$. According to P6, there is a sequence $\{f_n\}_{n\in\mathbb{N}_0}$ of n-measurable gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{n\to+\infty}\overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s)$, and therefore also

$$\limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s). \tag{2}$$

Because all f_n are finitary gambles and both $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}'$ are upper expectations in $\overline{\mathbb{E}}_{1-4}(\mathscr{T})$, with $\overline{\mathrm{E}}$ additionally satisfying P3', we can apply Proposition 3 to find that

$$\limsup_{n\to+\infty} \overline{\mathrm{E}}'(f_n|s) \leq \limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s). \tag{3}$$

Furthermore, since $\{f_n\}_{n\in\mathbb{N}_0}$ is a sequence of finitary gambles that is uniformly bounded below and that converges pointwise to f, P5 implies that

$$\overline{\mathrm{E}}'(f|s) \leq \limsup_{n\to+\infty} \overline{\mathrm{E}}'(f_n|s).$$

Combining this with Equations (2) and (3), we find that $\overline{\mathrm{E}}'(f|s) \leq \overline{\mathrm{E}}(f|s)$. ∎

Finally, we consider the entire domain $\overline{\mathbb{V}}$. Then, in order for an upper expectation to be the most conservative, it suffices to additionally impose the following property.

P7. For any $f \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$,

$$\overline{\mathrm{E}}(f|s) = \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and } g \geq f\right\}.$$

Theorem 5 *Consider any* $\overline{\mathrm{E}} \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$ *that satisfies* P3', P6 *and* P7. *Then, for any* $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$*, we have that*

$$\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}'(f|s) \text{ for all } f \in \overline{\mathbb{V}} \text{ and all } s \in \mathscr{X}^*.$$

Proof Fix any $f \in \overline{\mathbb{V}}$, any $s \in \mathscr{X}^*$ and any $\overline{\mathrm{E}}' \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$. According to P7, we have that

$$\overline{\mathrm{E}}(f|s) = \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and } g \geq f\right\}.$$

Then, using Proposition 4, we get

$$\inf\left\{\overline{\mathrm{E}}'(g|s) \colon g \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and } g \geq f\right\} \leq \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and } g \geq f\right\} = \overline{\mathrm{E}}(f|s).$$

Since, for any $g \in \overline{\mathbb{V}}_{\mathrm{b,lim}}$ such that $f \leq g$, we have that $\overline{\mathrm{E}}'(f|s) \leq \overline{\mathrm{E}}'(g|s)$ because $\overline{\mathrm{E}}'$ satisfies P4, we find that

$$\overline{\mathrm{E}}'(f|s) \leq \inf\left\{\overline{\mathrm{E}}'(g|s) \colon g \in \overline{\mathbb{V}}_{\mathrm{b,lim}} \text{ and } g \geq f\right\} \leq \overline{\mathrm{E}}(f|s),$$

proving the stated. ∎

Hence, according to Theorem 5, if there is an upper expectation $\overline{\mathrm{E}} \in \overline{\mathbb{E}}_{1-5}(\mathscr{T})$ satisfying P3', P6 and P7, it is the unique most conservative in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ and therefore equal to the global belief model $\overline{\mathrm{E}}^*$ that we are after. So it remains to show that there is at least one (and then necessarily unique) global upper expectation in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ that additionally satisfies P3', P6 and P7.

6. Game-Theoretic Upper Expectations

In this section, we show that a particular version of the game-theoretic upper expectation defined by Shafer and Vovk [3, 5] belongs to $\overline{\mathbb{E}}_{1-5}$ and furthermore satisfies P3', P6 and P7, thereby implying the existence of $\overline{\mathrm{E}}^*$. This game-theoretic upper expectation relies on the concept of a *supermartingale* (first introduced by Ville [8]), which is a capital process—the evolution of a subject's capital—that is obtained by betting against a system. The system—called forecaster in Shafer and Vovk's framework—determines for each situation a number of allowable bets which a subject—called skeptic in Shafer and Vovk's framework—can choose from. These allowable bets define the model, similar to what we here do by defining an imprecise probability tree. In general however, they allow for more general settings where local models need not be coherent and state spaces can be infinite. We here consider one particular version.

Once more, we assume that a local coherent upper expectation $\overline{\mathrm{Q}}_s$ on $\mathscr{L}(\mathscr{X})$ is given for every situation $s \in \mathscr{X}^*$. However, to introduce the game-theoretic upper expectation in its desired form, the local models should be defined on the extended domain $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ of all extended real variables that are bounded below. In particular, Shafer and Vovk [4, 5] start from local models that satisfy a modified version of the coherence axioms C1–C3, generalised to extended real variables. We show in [7] that, for finite state spaces, these modified axioms are equivalent to a combination of coherence with C8. In order to adhere to their framework, we therefore need to extend the domain of our local models from $\mathscr{L}(\mathscr{X})$ to $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ in such a way that this extension satisfies C8. According to Proposition 1, such an extension always exists and is furthermore unique. Without loss of generality, we can therefore henceforth assume that our local models $\overline{\mathrm{Q}}_s$ are defined on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, are coherent and satisfy C8.[3]

For a given imprecise probability tree $\mathscr{T}$, let $\overline{\mathbb{M}}_{\mathrm{b}}$ be the set of all *supermartingales*, being maps $\mathscr{M} \colon \mathscr{X}^* \to \mathbb{R} \cup \{+\infty\}$ that are uniformly bounded below, i.e. there is a real c such that $\mathscr{M}(s) \geq c$ for all $s \in \mathscr{X}^*$, and that satisfy $\overline{\mathrm{Q}}_s(\mathscr{M}(s\cdot)) \leq \mathscr{M}(s)$ for all $s \in \mathscr{X}^*$. In the last expression we used the notation $\mathscr{M}(s\cdot)$ to denote the variable in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ that takes the value $\mathscr{M}(sx)$ for each $x \in \mathscr{X}$. Note that we here indeed use the fact that the local models are defined on the domain $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. For any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{b}}$, we also let $\liminf \mathscr{M}$ be the extended real variable that takes the value $\liminf_{n\to+\infty} \mathscr{M}(\omega^n)$ for each $\omega \in \Omega$. Furthermore, for any two f and g in $\overline{\mathbb{V}}$ we write $f \geq_s g$ if $f(\omega) \geq g(\omega)$ for all paths $\omega \in \Gamma(s)$. The global game-theoretic upper expectation $\overline{\mathrm{E}}_{\mathrm{V}} \colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ is now defined by

$$\overline{\mathrm{E}}_{\mathrm{V}}(f|s) := \inf\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{b}} \text{ and } \liminf \mathscr{M} \geq_s f\},$$

for all $f \in \overline{\mathbb{V}}$ and all $s \in \mathscr{X}^*$. Crucially, this global game-theoretic upper expectation $\overline{\mathrm{E}}_{\mathrm{V}}$ is an element of $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ satisying P3', P6 and P7.

Proposition 6 $\overline{\mathrm{E}}_{\mathrm{V}}$ *is an element of* $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ *and furthermore satisfies* P3', P6 *and* P7.

Proof We prove that $\overline{\mathrm{E}}_{\mathrm{V}}$ satisfies P1–P5, P3', P6 and P7. Axiom P1 follows immediately from [7, Proposition 15]. Indeed, consider any $h \in \mathscr{L}(\mathscr{X})$. Since $h(X_{n+1})$ is an $(n+1)$-measurable gamble, [7, Proposition 15] tells us that

$$\overline{\mathrm{E}}(h(X_{n+1})|x_{1:n}) = \overline{\mathrm{Q}}_{x_{1:n}}(h(X_{n+1})(x_{1:n}\cdot)) = \overline{\mathrm{Q}}_{x_{1:n}}(h).$$

To prove P2, observe that $\liminf \mathscr{M} \geq_s f$ if and only if $\liminf \mathscr{M} \geq_s f\mathbb{I}_s$ for all $f \in \overline{\mathbb{V}}$ and all $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{b}}$. Then the desired equality follows directly from the definition of $\overline{\mathrm{E}}_{\mathrm{V}}$. P3 and P3' follow immediately from [7, Theorem 16]. Properties P4, P6 and P7 immediately follow from respectively

3. Apart from the fact that imposing C8 allows us to apply the framework of Shafer and Vovk, it is also consistent with our requirement that a global upper expectation should satisfy P4 and P5 because—as mentioned in Section 4—these imply continuity with respect to non-decreasing sequences of finitary gambles. Hence, if we let the extended local models be defined as restrictions of the global model to local variables, they clearly coincide with the original local models and moreover satisfy C8.

Proposition 13 (V4), Proposition 33 and Proposition 34 in [7]. Note that in these results $\overline{\mathbb{V}}_{\mathrm{b,lim}}$ has a slightly different definition, as it there also includes pointwise limits of possibly extended real finitary variables. However, according to [7, Lemma 8], this alternative definition is equivalent with the definition of $\overline{\mathbb{V}}_{\mathrm{b,lim}}$ that we have adopted here, allowing us to directly apply the results in [7]. Finally, property P5 follows from [7, Lemma 27], which says that, for any sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_{\mathrm{b}}$ that is uniformly bounded below,

$$\overline{\mathrm{E}}_{\mathrm{V}}(\liminf_{n\to+\infty} f_n|s) \leq \liminf_{n\to+\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_n|s).$$

Indeed, in the special case that $\{f_n\}_{n\in\mathbb{N}_0}$ is a sequence of finitary gambles that is uniformly bounded below and converges pointwise to some variable $f \in \overline{\mathbb{V}}_{\mathrm{b}}$, this implies

$$\overline{\mathrm{E}}_{\mathrm{V}}(f|s) \leq \liminf_{n\to+\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_n|s) \leq \limsup_{n\to+\infty} \overline{\mathrm{E}}_{\mathrm{V}}(f_n|s).$$

■

Theorem 7 *The set $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ is non-empty. Moreover, there is a unique most conservative upper expectation $\overline{\mathrm{E}}^*$ in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$ and it is equal to the global game-theoretic upper expectation $\overline{\mathrm{E}}_{\mathrm{V}}$.*

Proof Immediate from Theorem 5 and Proposition 6. ■

7. Discussion

Shafer and Vovk define their game-theoretic upper expectations using supermartingales. Their definition is therefore constructive and can be given a clear interpretation in terms of capital processes and betting behaviour. However, it requires that one allows unbounded and even infinite-valued bets, which we find questionable from an interpretational point of view. Furthermore, a complete axiomatisation of $\overline{\mathrm{E}}_{\mathrm{V}}$ in terms of its mathematical properties has, according to the best of our knowledge, been absent until this point. Theorem 7 addresses both of these issues. First, it provides an abstract axiomatisation for $\overline{\mathrm{E}}_{\mathrm{V}}$ using P3', P6 and P7 in addition to P1–P5. Most importantly, however, Theorem 7 provides an alternative definition— and interpretation—for $\overline{\mathrm{E}}_{\mathrm{V}}$ as the most conservative upper expectation under a limited set of intuitive properties: P1–P5. This strengthens the choice of using $\overline{\mathrm{E}}_{\mathrm{V}}$ as a global upper expectation, because it can now be motivated from both a game-theoretic point of view and from a purely axiomatic point of view. A reader should not even be familiar with the concepts of game-theoretic probability in order to use $\overline{\mathrm{E}}_{\mathrm{V}}$ as global upper expectation. Put simply, he only has to agree on the axioms P1–P5. And even if he preferred to impose additional axioms, then still, $\overline{\mathrm{E}}_{\mathrm{V}}$ would serve as a conservative upper bound for his desired upper expectation.

8. Additional Properties

Of course, there is more to an uncertainty model than only a compelling axiomatisation or interpretation. Te be practically useful, its mathematical properties should also be sufficiently powerful. For instance, the popularity of the Lebesgue integral as a tool for defining expected values, is in part due to its strong continuity properties (e.g. the Dominated Convergence Theorem [6, 2]). What may perhaps be somewhat surprising, is that despite the simplicity of our axioms P1–P5, our most conservative model $\overline{\mathrm{E}}_{\mathrm{V}} = \overline{\mathrm{E}}^*$ scores well on this account as well. For example, it satisfies the following generalisation of coherence.

Proposition 8 [7, Proposition 13] *For all $f,g \in \overline{\mathbb{V}}$, all $\lambda \in \mathbb{R}_{\geq}$, all $\mu \in \mathbb{R}$ and all $s \in \mathscr{X}^*$, $\overline{\mathrm{E}}^*$ satisfies*

V1. $\inf_{\omega\in\Gamma(s)} f(\omega) \leq \overline{\mathrm{E}}^*(f|s) \leq \sup_{\omega\in\Gamma(s)} f(\omega)$;

V2. $\overline{\mathrm{E}}^*(f+g|s) \leq \overline{\mathrm{E}}^*(f|s) + \overline{\mathrm{E}}^*(g|s)$;

V3. $\overline{\mathrm{E}}^*(\lambda f|s) = \lambda \overline{\mathrm{E}}^*(f|s)$;

V4. $\overline{\mathrm{E}}^*(f+\mu|s) = \overline{\mathrm{E}}^*(f|s) + \mu$.

Another important result, better known as Fatou's Lemma, shows that the upper bound imposed by property P5 for a global upper expectation $\overline{\mathrm{E}}(f|s)$ of a variable $f \in \overline{\mathbb{V}}_{\mathrm{b,lim}}$, is not attained by the most conservative upper expectation $\overline{\mathrm{E}}^*$ in $\overline{\mathbb{E}}_{1-5}(\mathscr{T})$. Interestingly enough, it can be replaced by a tighter one.

Lemma 9 [7, Lemma 27] *For any $s \in \mathscr{X}^*$ and any sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_{\mathrm{b}}$ that is uniformly bounded below:*

$$\overline{\mathrm{E}}^*(\liminf_{n\to+\infty} f_n|s) \leq \liminf_{n\to+\infty} \overline{\mathrm{E}}^*(f_n|s).$$

The following two theorems show that $\overline{\mathrm{E}}^*$ satisfies continuity with respect to both non-decreasing and non-increasing sequences of variables. However, continuity with respect to non-increasing sequences is only guaranteed if we consider sequences of finitary gambles.

Theorem 10 [7, Theorem 24] *For any $s \in \mathscr{X}^*$ and any non-decreasing sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_{\mathrm{b}}$ that converges point-wise to $f \in \overline{\mathbb{V}}_{\mathrm{b}}$, we have that*

$$\overline{\mathrm{E}}^*(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^*(f_n|s).$$

Theorem 11 [7, Corollary 31] *For any $s \in \mathscr{X}^*$ and any non-increasing sequence $\{f_n\}_{n\in\mathbb{N}_0}$ of finitary gambles that converges point-wise to a variable $f \in \overline{\mathbb{V}}$, we have that*

$$\overline{\mathrm{E}}^*(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^*(f_n|s).$$

Apart from the brief overview above, $\overline{\mathrm{E}}_{\mathrm{V}} = \overline{\mathrm{E}}^*$ also satisfies a weak and a strong law of large numbers, a law of the iterated logarithm, Lévy's zero-one law, and many more surprisingly strong properties; we refer the interested reader to the work of Shafer and Vovk [3, 5, 4].

9. Conclusion

We have put forward a small set of simple axioms P1–P5 for a global upper expectation that models a subject's uncertainty about a discrete-time process. We have established the existence of a unique most conservative model under these axioms, and additionally gave sufficient conditions to uniquely characterise this most conservative model. Moreover, this most conservative upper expectation was shown to coincide with a version of the game-theoretic upper expectation used by Shafer and Vovk, and therefore has particularly powerful mathematical properties, despite the simplicity of our defining axioms.

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