Supplementary Materials

A. Proof of Theorem 1

Proof. Let $\lambda_{i,t+1}$ be the ith largest eigenvalue of $(\widetilde{A}_{\mathcal{J}_{t+1}}, \widetilde{B}_{\mathcal{J}_{t+1}})$, $\hat{\rho}^{(t+1)}$ be the same as in Lemma 3. By the definition of $\eta_s^{(2)}$, we know that $\eta_s^{(2)} \geq \lambda_{2,t+1}$. Together with $\rho^{(t)} > \eta_s^{(2)}$, we have $\rho^{(t)} > \lambda_{2,t+1}$. On the other hand, using $|\mathcal{J}_t \cap \operatorname{supp}(v_1)| < k$, we know that $\rho^{(t)} \leq \eta_{s,k-1}^{(1)}$. Then by Lemma 3, we have

$$\lambda_{1,t+1} - \hat{\rho}^{(t+1)} \leq (\lambda_{1,t+1} - \rho^{(t)})\epsilon_m^2 + \mathcal{O}((\lambda_{1,t+1} - \rho^{(t)})^{\frac{3}{2}}),$$

where ϵ_m is the same as in Lemma 2. By the definition of ϵ_*, we know that $\epsilon_* \geq \epsilon_m$, it follows that

$$\lambda_{1,t+1} - \hat{\rho}^{(t+1)} \leq (\lambda_{1,t+1} - \rho^{(t)})\epsilon_*^2 + \mathcal{O}((\lambda_{1,t+1} - \rho^{(t)})^{\frac{3}{2}}),$$

Now using Lemma 4, we get the conclusion. □

B. Proof of Theorem 2

Proof. Noticing that $|\operatorname{supp}(v^{(t)})| \leq s$, using the definition of $\eta_{s,\ell}^{(1)}$, we know that if $\rho^{(t)} > \eta_{s,k-1}^{(1)}$, then

$$|\operatorname{supp}(v^{(t)}) \cap \operatorname{supp}(v_1)| = k = |\operatorname{supp}(v_1)|.$$

The conclusion follows immediately. □

C. Proof of Theorem 3

In order to show Theorem 3, we need the following lemmas.

Lemma 6 *Suppose (A, B) is a symmetric-definite pair. Let E, F be two symmetric matrices with $\epsilon = \sqrt{\|E\|_2^2 + \|F\|_2^2} < c(A, B)$. Let (λ, x) and $(\tilde{\lambda}, \tilde{x})$ be the leading eigenpairs of (A, B) and $(A + E, B + F)$, respectively. Suppose $\tilde{\lambda}$ is simple, and denote the smallest nonzero singular value of $(A + E) - \tilde{\lambda}(B + F)$ by g. If $|\tilde{\lambda}|\epsilon < c(A, B)$, then*

$$\sin\theta(x, \tilde{x}) \leq \frac{\|B\|_2\delta + \sqrt{1 + \tilde{\lambda}^2}\epsilon}{g},$$

where

$$\delta = \frac{(1 + \tilde{\lambda}^2)\epsilon}{c(A, B) - |\tilde{\lambda}|\epsilon}. \tag{1}$$

Proof. First, since $\epsilon < c(A, B)$, by Lemma 1, $(A + E, B + F)$ is a definite pair and

$$\arctan(\tilde{\lambda}) - \arctan(\epsilon/c(A, B)) \leq \arctan(\lambda) \leq \arctan(\tilde{\lambda}) + \arctan(\epsilon/c(A, B)). \tag{2}$$

Using $|\tilde{\lambda}|\epsilon < c(A, B)$, we know that $\arctan(\epsilon/c(A, B)) < \arctan(1/|\tilde{\lambda}|) = \frac{\pi}{2} - \arctan(|\tilde{\lambda}|)$, which implies that the left hand side and righthand side of (2) are larger than $-\frac{\pi}{2}$ and smaller than $\frac{\pi}{2}$, respectively. Then it follows from (2) that

$$\frac{\tilde{\lambda}c(A, B) - \epsilon}{c(A, B) + \tilde{\lambda}\epsilon} \leq \lambda \leq \frac{\tilde{\lambda}c(A, B) + \epsilon}{c(A, B) - \tilde{\lambda}\epsilon}.$$

Therefore,

$$|\tilde{\lambda} - \lambda| \leq \frac{(1 + \tilde{\lambda}^2)\epsilon}{c(A, B) - |\tilde{\lambda}|\epsilon} = \delta. \tag{3}$$

Second, without loss of generosity, we set $\|x\|_2 = \|\tilde{x}\|_2 = 1$, let $r = [(A+E) - \tilde{\lambda}(B+F)]x$. Direct calculations give rise to

$$\begin{aligned}\|r\|_2 &= \|(A - \tilde{\lambda}B)x + (E - \tilde{\lambda}F)x\|_2 \leq \|(A - \lambda B)x\|_2 + |\tilde{\lambda} - \lambda|\|Bx\|_2 + \|(E - \tilde{\lambda}F)x\|_2 \\ &\leq \|B\|_2\delta + \|E\|_2 + |\tilde{\lambda}|\|F\|_2 \leq \|B\|_2\delta + \sqrt{1+\tilde{\lambda}^2}\epsilon. \end{aligned} \tag{4}$$

On the other hand, the spectral decomposition of $(A+E) - \tilde{\lambda}(B+F)$ can be given by $(A+E) - \tilde{\lambda}(B+F) = V \operatorname{diag}(0, \gamma_2, \dots, \gamma_p)V^{\mathrm{T}}$, where $V = [\tilde{x}, V_2]$ is orthogonal, $0 > \gamma_2 \geq \cdots \geq \gamma_p$ are the eigenvalues of $(A+E) - \tilde{\lambda}(B+F)$. Here we used the assumption that $\tilde{\lambda}$ is simple. Then it follows that

$$V_2^{\mathrm{T}} r = V_2^{\mathrm{T}}[(A+E) - \tilde{\lambda}(B+F)]x = \Gamma_2 V_2^{\mathrm{T}} x, \tag{5}$$

where $\Gamma_2 = \operatorname{diag}(\gamma_2, \dots, \gamma_p)$. Using (4) and (5), we get

$$\sin\theta(x, \tilde{x}) = \|V_2^{\mathrm{T}} x\|_2 = \|\Gamma_2^{-1} V_2^{\mathrm{T}} r\|_2 \leq \frac{\|r\|_2}{|\gamma_2|} \leq \frac{\|B\|_2\delta + \sqrt{1+\tilde{\lambda}^2}\epsilon}{g},$$

which completes the proof. □

Proof of Theorem 3. Notice that $(\lambda_1, (v_1)_{\mathcal{J}_t})$ and $(\rho^{(t)}, (v^{(t)})_{\mathcal{J}_t})$ are the leading eigenpairs of $(A_{\mathcal{J}_t}, B_{\mathcal{J}_t})$ and $(\widetilde{A}_{\mathcal{J}_t}, \widetilde{B}_{\mathcal{J}_t})$, respectively. Then (a) and (b) follow from Lemma 1 and Lemma 6, respectively. This completes the proof. □