A PROOFS

Proof for Definition 1 We write a proof in the case where $\mathcal{E} = \{\mathbb{R}^d\}$. If $K > 1$, the proof can be applied independently on each set of the partition.

Let $(f_n)_{n\in\mathbb{N}}$ be such that $f_n(0) = 0$ for all $n \in \mathbb{N}$ and

$$W_2\left[(\nabla f_n)_\sharp\mu, \nu\right] \leq \frac{1}{n+1} + \inf_{f\in\mathcal{F}_{\ell,L}} W_2\left[(\nabla f)_\sharp\mu, \nu\right].$$

Let $x_0 \in \operatorname{supp}(\mu)$. Then there exists $C > 0$ such that for all $n \in \mathbb{N}$, $\|\nabla f_n(x_0)\| \leq C$. Indeed, suppose this is not true. Take $r > 0$ such that $V := \mu[B(x_0, r)] > 0$. By Prokhorov theorem, there exists $R > 0$ such that $\nu\left[B(0, R)\right] \geq 1 - \frac{V}{2}$. Then for $C > 0$ large enough, there exists an $n \in \mathbb{N}$ such that:

$$\begin{aligned} W_2^2[(\nabla f_n)_\sharp\mu, \nu] &= \min_{\pi\in\Pi(\mu,\nu)} \int \|\nabla f_n(x) - y\|^2 \, d\pi(x, y) \\ &\geq \int \|\nabla f_n(x) - \operatorname{proj}_{B(0,R)}\left[\nabla f_n(x)\right]\|^2 \, d\mu(x) \\ &\geq \int_{B(x_0,r)\cap\operatorname{supp}(\mu)} \|\nabla f_n - \operatorname{proj}_{B(0,R)}\left[\nabla f_n\right]\|^2 \, d\mu \\ &\geq \frac{1}{2} V \min_{\substack{x\in B(x_0,r)\\ y\in B(0,R)}} \|\nabla f_n(x) - y\|^2 \\ &\geq \frac{1}{2} V (C - Lr - R) \end{aligned}$$

which contradicts the definition of f_n when C is sufficiently large.
Then for $x \in \mathbb{R}^d$,

$$\|\nabla f_n(x)\| \leq L\|x - x_0\| + \|\nabla f_n(x_0)\| \leq L\|x - x_0\| + C.$$

Since $(\nabla f_n)_{n\in\mathbb{N}}$ is equi-Lipschitz, it converges uniformly (up to a subsequence) to some function g by Arzelà–Ascoli theorem. Note that g is L-Lipschitz.
Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \|\nabla f_n - g\|_\infty \leq \epsilon$. Then for $n \geq N$ and $x \in \mathbb{R}^d$,

$$|f_n(x)| = \left| \int_0^1 \langle \nabla f_n(tx), x \rangle \, dt \right| \leq \|x\|(\|g\|_\infty + \epsilon)$$

so that $(f_n(x))$ converges up to a subsequence. Let ϕ, ψ be two extractions and α, β such that $f_{\phi(n)}(x) \to \alpha$ and $f_{\psi(n)}(x) \to \beta$. Then

$$\begin{aligned} |\alpha - \beta| &= \lim_{n\to\infty} \left| \int_0^1 \langle \nabla f_{\phi(n)}(tx) - \nabla f_{\psi(n)}(tx), x \rangle \, dt \right| \\ &\leq \lim_{n\to\infty} \|x\| \|\nabla f_{\phi(n)} - \nabla f_{\psi(n)}\|_\infty = 0. \end{aligned}$$

This shows that $(f_n)_{n\in N}$ converges pointwise to some function $f_\star$. In particular, $f_\star$ is convex. For $x \in \mathbb{R}^d$, using Lebesgue's dominated convergence theorem,

$$\begin{aligned} f_\star(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^1 \langle \nabla f_n(tx), x \rangle \, dt \\ &= \int_0^1 \left\langle \lim_{n\to\infty} \nabla f_n(tx), x \right\rangle dt = \int_0^1 \langle g(tx), x \rangle \, dt \end{aligned}$$

so $f_\star$ is differentiable and $\nabla f_\star = g$. Using Lebesgue's dominated convergence theorem, uniform (hence pointwise) convergence of $(\nabla f_n)_{n\in\mathbb{N}}$ to $\nabla f_\star$ shows that $(\nabla f_n)_\sharp\mu \rightharpoonup (\nabla f_\star)_\sharp\mu$. Then classical optimal transport stability theorems *e.g.* (Villani, 2009, Theorem 5.19) show that

$$\begin{aligned} W_2\left[(\nabla f_\star)_\sharp\mu, \nu\right] &= \lim_{n\to\infty} W_2\left[(\nabla f_n)_\sharp\mu, \nu\right] \\ &= \inf_{f\in\mathcal{F}_{\ell,L}} W_2\left[(\nabla f)_\sharp\mu, \nu\right], \end{aligned}$$

i.e. $f_\star$ is a minimizer.

Proof of Theorem 1 For $f \in \mathcal{F}_{\ell,L,\mathcal{E}}$, $\nabla f_\sharp\mu = \sum_{i=1}^n a_i \delta_{\nabla f(x_i)}$. Writing $z_i = \nabla f(x_i)$, we wish to minimize $W_2^2\left(\sum_{i=1}^n a_i\delta_{z_i}, \nu\right)$ over all the points $z_1, \ldots, z_n \in \mathbb{R}^d$ such that there exists $f \in \mathcal{F}_{\ell,L,\mathcal{E}}$ with $\nabla f(x_i) = z_i$ for all $i \in [\![n]\!]$. Following (Taylor, 2017, Theorem 3.8), there exists such a f if, and only if, there exists $u \in \mathbb{R}^n$ such that for all $k \in [\![K]\!]$ and for all $i, j \in I_k$,

$$\begin{aligned} u_i \geq u_j + \langle z_j, x_i - x_j \rangle + \frac{1}{2(1 - \ell/L)} \bigg(&\frac{1}{L}\|z_i - z_j\|^2 \\ &+ \ell\|x_i - x_j\|^2 - 2\frac{\ell}{L}\langle z_j - z_i, x_j - x_i \rangle \bigg). \end{aligned}$$

Then minimizing over $f \in \mathcal{F}_{\ell,L,\mathcal{E}}$ is equivalent to minimizing over $(z_1, \ldots, z_n, u)$ under these interpolation constraints.

The second part of the theorem is a direct application of (Taylor, 2017, Theorem 3.14).

Proof of Proposition 1 Let $f : \mathbb{R} \to \mathbb{R}$. Then $f \in \mathcal{F}_{\ell,L,\mathcal{E}}$ if and only if it is convex and L-smooth on each set E_k, $k \in [\![K]\!]$, *i.e.* if and only if for any $k \in [\![K]\!]$, $0 \leq f''|_{E_k} \leq L$.

For a measure ρ, let us write F_ρ and Q_ρ the cumulative distribution function and the quantile function (*i.e.* the generalized inverse of the cumulative distribution function). Then $Q_{\nabla f_\sharp\mu} = \nabla f \circ Q_\mu$.

Using the closed-form formula for the Wasserstein distance in dimension 1, the objective we wish to minimize (over $f \in \mathcal{F}_{\ell,L,\mathcal{E}}$) is:

$$W_2^2(f'_\sharp\mu, \nu) = \int_0^1 \left[f' \circ Q_\mu(t) - Q_\nu(t)\right]^2 dt.$$

Suppose μ has a density w.r.t the Lebesgue measure. Then by a change of variable, the objective becomes

$$\int_{-\infty}^{+\infty} \left[f'(x) - Q_\nu \circ F_\mu(x)\right]^2 d\mu(x) = \|f' - \overline{\pi}\|^2_{L^2(\mu)}.$$

Indeed, $Q_\nu \circ F_\mu$ is the optimal transport map from μ to ν, hence its own barycentric projection. The result follows.

Suppose now that μ is purely atomic, and write $\mu = \sum_{i=1}^n a_i \delta_{x_i}$ with $x_1 \leq \ldots \leq x_n$. For $0 \leq i \leq n$, put $\alpha_i = \sum_{k=1}^i a_k$ with $\alpha_0 = 0$. Then

$$\begin{aligned} W_2^2(f'_\sharp \mu, \nu) &= \sum_{i=1}^n \int_{\alpha_{i-1}}^{\alpha_i} (f'(x_i) - Q_\nu(t))^2 dt \\ &= \sum_{i=1}^n a_i \left[f'(x_i) - \frac{1}{a_i} \left(\int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t) dt \right) \right]^2 \\ &+ \int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t)^2 dt - \frac{1}{a_i} \left(\int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t) dt \right)^2 . \end{aligned}$$

Since $\sum_{i=1}^n \int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t)^2 dt - \frac{1}{a_i} \left(\int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t) dt \right)^2$ does not depend on f, minimizing $W_2^2(f'_\sharp \mu, \nu)$ over $f \in \mathcal{F}_{\ell, L, \mathcal{E}}$ is equivalent to solve

$$\min_{f \in \mathcal{F}_{\ell,L,\mathcal{E}}} \sum_{i=1}^n a_i \left[f'(x_i) - \frac{1}{a_i} \left(\int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t) dt \right) \right]^2 .$$

There only remains to show that $\overline{\pi}(x_i) = \frac{1}{a_i} \int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t)\, dt$. Using the definition of the conditional expectation and the definition of π:

$$\begin{aligned} \overline{\pi}(x_i) &= \frac{1}{a_i} \int_{-\infty}^{+\infty} y\, \mathbf{1}\{x = x_i\}\, d\pi(x,y) \\ &= \frac{1}{a_i} \int_{-\infty}^{+\infty} y\, \mathbf{1}\{x = x_i\}\, d(Q_\mu, Q_\nu)_\sharp \mathscr{L}^1|_{[0,1]} \\ &= \frac{1}{a_i} \int_0^1 Q_\nu(t)\, \mathbf{1}\{Q_\mu(t) = x_i\}\, dt \\ &= \frac{1}{a_i} \int_{\alpha_{i-1}}^{\alpha_i} Q_\nu(t)\, dt. \end{aligned}$$

Proof of Proposition 2 Since $\mathcal{E} = \{\mathbb{R}^d\}$, and using the triangular inequality for the Wasserstein distance,

$$\begin{aligned} \left| W_2(\mu,\nu) - \widehat{W}_2(\mu,\nu) \right| &= \left| W_2(\mu,\nu) - W_2(\mu, \nabla \hat{f}_{n\sharp}\mu) \right| \\ &\leq W_2 \left(\nabla \hat{f}_{n\sharp}\mu, \nu \right) \\ &\leq W_2 \left(\nabla \hat{f}_{n\sharp}\mu, \nabla \hat{f}_{n\sharp}\hat{\mu}_n \right) \quad (4) \\ &\quad + W_2 \left(\nabla \hat{f}_{n\sharp}\hat{\mu}_n, \hat{\nu}_n \right) \quad (5) \\ &\quad + W_2 \left(\hat{\nu}_n, \nu \right). \quad (6) \end{aligned}$$

We now successively upper bound terms (4), (5), (6).

Since $\nabla \hat{f}_n$ is L-Lispchitz, almost surely:

$$(4) = W_2 \left(\nabla \hat{f}_{n\sharp}\mu, \nabla \hat{f}_{n\sharp}\hat{\mu}_n \right) \leq L\, W_2 \left(\mu, \hat{\mu}_n \right) \underset{n \to \infty}{\longrightarrow} 0$$

since almost surely, $\hat{\mu}_n \rightharpoonup \mu$ and μ has compact support, cf. (Santambrogio, 2015, Theorem 5.10). For the same reason, almost surely:

$$(6) = W_2 \left(\hat{\nu}_n, \nu \right) \underset{n \to \infty}{\longrightarrow} 0.$$

Finally, since $f_\star \in \mathcal{F}_{\ell,L,\mathcal{E}}$ and $\nabla \hat{f}_n$ is an optimal SSNB potential, it almost surely holds:

$$\begin{aligned} (5) = W_2 \left(\nabla \hat{f}_{n\sharp}\hat{\mu}_n, \hat{\nu}_n \right) &\leq W_2 \left(\nabla f_{\star\sharp}\hat{\mu}_n, \hat{\nu}_n \right) \\ &\underset{n \to \infty}{\longrightarrow} W_2 \left(\nabla f_{\star\sharp}\mu, \nu \right) = 0 \end{aligned}$$

because $(\hat{\mu}_n, \hat{\nu}_n) \rightharpoonup (\mu, \nu)$, and by definition of $f_\star$, $\nabla f_{\star\sharp}\mu = \nu$.