A PROOFS

Proof for Definition 1 We write a proof in the case where $E = \{\mathbb{R}^d\}$. If $K > 1$, the proof can be applied independently on each set of the partition.

Let $(f_n)_{n \in \mathbb{N}}$ be such that $f_n(0) = 0$ for all $n \in \mathbb{N}$ and

$$W_2[(\nabla f_n)_\sharp \mu, \nu] \leq \frac{1}{n + 1} + \inf_{f \in F_{L^2}} W_2[(\nabla f)_\sharp \mu, \nu].$$

Let $x_0 \in \text{supp}(\mu)$. Then there exists $C > 0$ such that for all $n \in \mathbb{N}$, $\|\nabla f_n(x_0)\| \leq C$. Indeed, suppose this is not true. Take $r > 0$ such that $V := \mu[B(x_0, r)] > 0$. By Prokhorov’s theorem, there exists $R > 0$ such that $\nu [B(0, R)] \geq 1 - \frac{1}{2}$. Then for $C > 0$ large enough, there exists an $n \in \mathbb{N}$ such that:

$$W_2^2[(\nabla f_n)_\sharp \mu, \nu] = \min_{\pi \in H(\mu, \nu)} \int \|\nabla f_n(x) - y\|^2 d\pi(x, y)$$

$$\geq \int \|\nabla f_n(x) - \text{proj}_{B(0, R)} [\nabla f_n(x)]\|^2 d\mu(x)$$

$$\geq \frac{1}{2} V \min_{x \in B(x_0, r), y \in B(0, R)} \|\nabla f_n(x) - y\|^2$$

$$\geq \frac{1}{2} V(C - Lr - R)$$

which contradicts the definition of $f_n$ when $C$ is sufficiently large.

Then for $x \in \mathbb{R}^d$,

$$\|\nabla f_n(x)\| \leq L\|x - x_0\| + \|\nabla f_n(x_0)\| \leq L\|x - x_0\| + C.$$

Since $(\nabla f_n)_{n \in \mathbb{N}}$ is equi-Lipschitz, it converges uniformly (up to a subsequence) to some function $g$ by the Arzelà–Ascoli theorem. Note that $g$ is $L$-Lipschitz. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \|\nabla f_n - g\| \leq \epsilon$. Then for $n \geq N$ and $x \in \mathbb{R}^d$,

$$\|f_n(x)\| = \int_0^1 \langle \nabla f_n(tx), x \rangle dt \leq \|x\|(\|g\| + \epsilon)$$

so that $(f_n(x))$ converges up to a subsequence. Let $\phi, \psi$ be two extractions and $\alpha, \beta$ such that $f_{\phi(n)}(x) \to \alpha$ and $f_{\psi(n)}(x) \to \beta$. Then

$$|\alpha - \beta| = \lim_{n \to \infty} \int_0^1 \langle \nabla f_{\phi(n)}(tx) - \nabla f_{\psi(n)}(tx), x \rangle dt$$

$$\leq \lim_{n \to \infty} \|x\| \|\nabla f_{\phi(n)} - \nabla f_{\psi(n)}\| = 0.$$

This shows that $(f_n)_{n \in \mathbb{N}}$ converges pointwise to some function $f_*$. In particular, $f_*$ is convex. For $x \in \mathbb{R}^d$, using Lebesgue’s dominated convergence theorem,

$$f_*(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \int_0^1 \langle \nabla f_n(tx), x \rangle dt$$

$$= \int_0^1 \langle \lim_{n \to \infty} \nabla f_n(tx), x \rangle dt = \int_0^1 \langle g(tx), x \rangle dt$$

so $f_*$ is differentiable and $\nabla f_* = g$. Using Lebesgue’s dominated convergence theorem, uniform (hence pointwise) convergence of $(\nabla f_n)_{n \in \mathbb{N}}$ to $\nabla f_*$ shows that $(\nabla f_n)_\sharp \mu \to (\nabla f_*)_\sharp \mu$. Then classical optimal transport stability theorems e.g. (Villani, 2009, Theorem 5.19) show that

$$W_2[(\nabla f_*)_\sharp \mu, \nu] = \lim_{n \to \infty} W_2[(\nabla f_n)_\sharp \mu, \nu]$$

$$= \inf_{f \in F_{L^2}} W_2[(\nabla f)_\sharp \mu, \nu],$$

i.e. $f_*$ is a minimizer.

Proof of Theorem 1 For $f \in F_{L^2, E}$, $\nabla f_\mu = \sum_{i=1}^n a_i \delta_{f(x_i)}$. Writing $z_i = \nabla f(x_i)$, we wish to minimize $W_2^2(\sum_{i=1}^n a_i \delta_{z_i}, \nu)$ over all the points $z_1, \ldots, z_n \in \mathbb{R}^d$ such that there exists $f \in F_{L^2, E}$ with $\nabla f(x_i) = z_i$ for all $i \in \{n\}$. Following (Taylor, 2017, Theorem 3.8), there exists such a $f$ if, and only if, there exists $u \in \mathbb{R}^n$ such that for all $k \in [K]$ and for all $i, j \in I_k$,

$$u_i \geq u_j + (z_j, x_j - x_i) + \frac{1}{2} \frac{1}{(1 - \ell/L)} (\frac{1}{L} \|z_i - z_j\|^2$$

$$+ \ell \|x_i - x_j\|^2 - 2 \ell \|z_i - z_j, x_j - x_i\|).$$

Then minimizing over $f \in F_{L^2, E}$ is equivalent to minimizing over $(z_1, \ldots, z_n, u)$ under these interpolation constraints.

The second part of the theorem is a direct application of (Taylor, 2017, Theorem 3.14).

Proof of Proposition 1 Let $f : \mathbb{R} \to \mathbb{R}$. Then $f \in F_{L^2, E}$ if and only if it is convex and $L$-smooth on each set $E_k$, $k \in [K]$, i.e. if and only if for any $k \in [K]$, $0 \leq f'|_{E_k} \leq L$.

For a measure $\mu$, let us write $F_\mu$ and $Q_\mu$, the cumulative distribution function and the quantile function (i.e. the generalized inverse of the cumulative distribution function). Then $Q_{\nabla f_\mu} = \nabla f \circ Q_\mu$.

Using the closed-form formula for the Wasserstein distance in dimension 1, the objective we wish to minimize (over $f \in F_{L^2, E}$) is:

$$W_2^2(f_\mu, \nu) = \int_0^1 [f' \circ Q_\mu(t) - Q_{\nu}(t)]^2 dt.$$ 

Suppose $\mu$ has a density w.r.t the Lebesgue measure. Then by a change of variable, the objective becomes

$$\int_{-\infty}^{+\infty} [f'(x) - Q_{\nu} \circ F_\mu(x)]^2 d\mu(x) = \|f' - \pi\|_{L^2(\mu)}^2.$$ 

Indeed, $Q_{\nu} \circ F_\mu$ is the optimal transport map from $\mu$ to $\nu$, hence its own barycentric projection. The result
We now successively upper bound terms (4), (5), (6).

Since almost surely, \( \mu_n \rightarrow \mu \) and \( \mu \) has compact support, cf. (Santambrogio, 2015, Theorem 5.10). For the same reason, almost surely:

\[
(6) = W_2(\hat{\nu}_n, \nu) \xrightarrow{n \to \infty} 0.
\]

Finally, since \( f_\ast \in \mathcal{F}_L, L, \mathcal{E} \) and \( \nabla \hat{f}_n \) is an optimal SSNB potential, it almost surely holds:

\[
(5) = W_2(\nabla \hat{f}_n \mu_n, \hat{\nu}_n) \leq W_2(\nabla f_\ast \mu_n, \hat{\nu}_n) \xrightarrow{n \to \infty} W_2(\nabla f_\ast \mu, \nu) = 0
\]

because \( (\hat{\mu}_n, \hat{\nu}_n) \xrightarrow{} (\mu, \nu) \), and by definition of \( f_\ast \), \( \nabla f_\ast \mu = \nu \).

Proof of Proposition 2 Since \( \mathcal{E} = \{\mathbb{R}^d\} \), and using the triangular inequality for the Wasserstein distance,

\[
W_2(\mu, \nu) - \hat{W}_2(\mu, \nu) = |W_2(\mu, \nu) - W_2(\mu, \nabla \hat{f}_n \mu)|
\]

\[
\leq W_2(\nabla \hat{f}_n \mu, \nu)
\]

\[
\leq W_2(\nabla \hat{f}_n \mu, \nabla \hat{f}_n \hat{\nu}_n) \quad (4)
\]

\[
+ W_2(\nabla \hat{f}_n \hat{\mu}_n, \hat{\nu}_n) \quad (5)
\]

\[
+ W_2(\hat{\nu}_n, \nu) \quad (6)
\]

We now successively upper bound terms (4), (5), (6).

Since \( \nabla \hat{f}_n \) is \( L \)-Lipschitz, almost surely:

\[
(4) = W_2(\nabla \hat{f}_n \mu, \nabla \hat{f}_n \hat{\mu}_n) \leq L W_2(\mu, \hat{\mu}_n) \xrightarrow{n \to \infty} 0
\]