Appendices

Other related work

Bandits with Knapsacks (BwK). This problem was introduced by Badanidiyuru et al. (2018) and its framework is as follows: there is a fixed set of n arms denoted by $\mathcal{X}$ and there are d resources being consumed. In each round $t \in [T]$, where T is a finite and known time horizon, an algorithm picks an arm $x_t \in \mathcal{X}$ and upon committing to this action, it receives a reward $r_t \in [0, 1]$ and consumes an amount $c_{t,i} \in [0, 1]$ of resource $i \in [d]$. There is a hard budget constraint $B_i \in \mathbb{R}_+$ for each resource $i \in [d]$ and the algorithm stops as soon as one or more budget constraint is violated. The overall reward of the algorithm equals the sum of rewards in all the rounds preceding the stopping time. The goal of the algorithm is to maximize the expected total reward. Depending on the input model for rewards and costs of the arms, the BwK has been classified into stochastic BwK and adversarial BwK categories. Our problem framework considers the adversarial input model for the case with one resource, i.e., $d = 1$. Nonetheless, the BwK problem is different from our setting in the following several important respects:

- The action set in the BwK problem is discrete and finite whereas we consider convex and compact domains $\mathcal{X} \subset \mathbb{R}^n_+$.
- The rewards in the BwK problem are linear in the arms. On the other hand, we consider more general class of DR-submodular utility functions.
- The BwK problem only observes bandit feedback for the reward and resource consumption while we consider the full feedback setting.
- In the BwK problem, the budget constraints are strict and the algorithm stops as soon as one of the budget constraints is violated. However, in our setting, we allow budget violations as long as the total budget violation is sub-linear in the time horizon T.

A Motivating applications

In the following, we present a number of other applications that could be cast into our framework. These applications along with the online ad placement problem introduced in the Introduction section of the paper show that the online continuous DR-submodular maximization problem with long-term budget constraints is indeed well-motivated.

Crowdsourcing markets. In this problem, there exists a requester with a limited budget B_T that submit jobs and benefits from them being completed. There are n types of jobs available to be assigned to workers arriving online. At each step $t \in [T]$, a worker arrives and the requester has to assign a bundle $x_t \in \mathcal{X} = \{x \in \mathbb{R}^n_+ : 0 \preceq x \preceq 1\}$ of the jobs to the worker. The worker has an unknown private cost $[p_t]_i \ \forall i \in [n]$ for performing one unit of the i-th job where $[p_t]_i$ denotes the i-th entry of vector p_t. Therefore, the total cost of the job assignments to this worker equals $\langle p_t, x_t \rangle$. The rewards obtained by the requester from this job assignment is a DR-submodular function $f_t(x_t)$. The DR property of the utility function captures the diminishing returns of assigning more jobs to the worker, i.e., as the number of assigned jobs to the worker increases, she has less time to devote to each fixed job $i \in [n]$ and therefore, the reward (quality of the completed task) obtained from the worker performing one unit of job i decreases. In other words, if $x \preceq y$, $\nabla_i f(x) \geq \nabla_i f(y) \ \forall i \in [n]$ holds. The goal is to maximize the overall rewards obtained by the requester while the budget constraint is not violated as well. Note that if the jobs are indivisible, for all $t \in [T]$, the utility function f_t corresponds to the multilinear extension of the monotone submodular set function $F_t : 2^n \to \mathbb{R}$ and using the lossless pipage rounding technique of Calinescu et al. (2011), we allocate an integral bundle of jobs to the workers at each step.

Welfare maximization with production cost. In this problem, there is a seller who has n types of products for sale that may be produced on demand using a fixed limited budget B_T. At each step $t \in [T]$, an agent (customer) arrives online and the seller has to assign a bundle $x_t \in \mathcal{X} = \{x \in \mathbb{R}^n_+ : 0 \preceq x \preceq 1\}$ of products to the agent. Producing each unit of each product $i \in [n]$ costs an unknown amount $[p_t]_i$ and the production cost of the item may change over time $\{1, \ldots, T\}$ because of the fluctuations of the prices of primitive resources. Therefore, the total production cost at step $t \in [T]$ equals $\langle p_t, x_t \rangle$. The agent has an unknown private DR-submodular valuation function f_t over the items where the DR property characterizes the diversity of the assigned bundle. Therefore, the utility obtained by assigning the bundle x_t equals $f_t(x_t)$. The goal is to maximize the overall valuation of the agents while satisfying the budget constraint of the seller. Note that if the products are indivisible, for all $t \in [T]$, the utility function f_t corresponds to the multilinear extension of the monotone submodular set function $F_t : 2^n \to \mathbb{R}$ and using the lossless pipage rounding technique of Calinescu et al. (2011), we allocate an integral bundle of products to the agents at each step.

B Proof of Lemma 4.1

Since $\lambda_{t+1} = [\lambda_t - \mu \nabla_\lambda \mathcal{L}_t(x_t, \lambda_t)]_+ = [(1-\delta\mu^2)\lambda_t + \mu g_t(x_t)]_+$ and $\lambda_1 = 0$, we have:

$$\begin{aligned}
\lambda_{t+1} &\geq (1-\delta\mu^2)\lambda_t + \mu g_t(x_t) \\
&\geq (1-\delta\mu^2)^2\lambda_{t-1} + \mu g_t(x_t) + (1-\delta\mu^2)\mu g_{t-1}(x_{t-1}) \\
&\geq \mu \sum_{s=1}^{t} (1-\delta\mu^2)^{t-s} g_s(x_s) + (1-\delta\mu^2)^t \underbrace{\lambda_1}_{=0} \\
&= \mu \sum_{s=1}^{t} (1-\delta\mu^2)^{t-s} g_s(x_s).
\end{aligned}$$

Similarly, we can derive the other inequality as follows:

$$\begin{aligned}
\lambda_{t+1} &\leq |(1-\delta\mu^2)\lambda_t + \mu g_t(x_t)| \\
&\leq (1-\delta\mu^2)\lambda_t + \mu |g_t(x_t)| \\
&\leq (1-\delta\mu^2)^2\lambda_{t-1} + \mu |g_t(x_t)| + (1-\delta\mu^2)\mu |g_{t-1}(x_{t-1})| \\
&\leq \mu \sum_{s=1}^{t} (1-\delta\mu^2)^{t-s} |g_s(x_s)| + (1-\delta\mu^2)^t \underbrace{\lambda_1}_{=0} \\
&= \mu \sum_{s=1}^{t} (1-\delta\mu^2)^{t-s} |g_s(x_s)|.
\end{aligned}$$

C Proof of Lemma 4.2

We have:

$$\begin{aligned}
&\sum_{\tau=0}^{W-1} \lambda_{t+\tau} g_{t+\tau}(x_W^*) = \\
&\sum_{\tau=0}^{W-1} \lambda_t g_{t+\tau}(x_W^*) + \sum_{\tau=0}^{W-1} \big(\lambda_{t+\tau} - \lambda_t\big) g_{t+\tau}(x_W^*) \\
&\leq \lambda_t \sum_{\tau=0}^{W-1} g_{t+\tau}(x_W^*) + \sum_{\tau=0}^{W-1} \big(|\lambda_{t+\tau} - \lambda_t|\big) |g_{t+\tau}(x_W^*)| \\
&\overset{(a)}{\leq} \lambda_t \sum_{\tau=0}^{W-1} g_{t+\tau}(x_W^*) + \sum_{\tau=0}^{W-1} \Big(\sum_{s=1}^{\tau} \underbrace{|\lambda_{t+s} - \lambda_{t+s-1}|}_{(b)}\Big) \underbrace{|g_{t+\tau}(x_W^*)|}_{\leq G},
\end{aligned}$$

where (a) is due to the triangle inequality.
Using the result of Lemma 4.1, for all non-negative integers $r \geq 1$, we can write:

$$\begin{aligned}
\lambda_r &\leq \mu \sum_{u=1}^{r-1} (1-\delta\mu^2)^{r-1-u} \underbrace{|g_u(x_u)|}_{\leq G} \\
&\leq \mu G \sum_{u=1}^{r-1} (1-\delta\mu^2)^{r-1-u} \\
&\leq \mu G \sum_{u=0}^{\infty} (1-\delta\mu^2)^{u} \\
&= \frac{\mu G}{1-(1-\delta\mu^2)} \\
&= \frac{G}{\delta\mu}.
\end{aligned}$$

Consider the term (b). If $(1-\delta\mu^2)\lambda_{t+s-1} + \mu g_{t+s-1}(x_{t+s-1}) < 0$ (equivalently, $-\lambda_{t+s-1} \geq \frac{\mu}{1-\delta\mu^2} g_{t+s-1}(x_{t+s-1})$), we have:

$$\begin{aligned}
(b) &= |-\lambda_{t+s-1}| \\
&\leq \frac{\mu}{1-\delta\mu^2} |g_{t+s-1}(x_{t+s-1})| \\
&\leq \frac{\mu}{1-\delta\mu^2} G \\
&\leq 2\mu G.
\end{aligned}$$

We will choose parameters δ and μ in our algorithm such that $\delta\mu^2 \ll 1$ holds.
Otherwise, if $(1-\delta\mu^2)\lambda_{t+s-1} + \mu g_{t+s-1}(x_{t+s-1}) \geq 0$, we have:

$$\begin{aligned}
(b) &= |(1-\delta\mu^2)\lambda_{t+s-1} + \mu g_{t+s-1}(x_{t+s-1}) - \lambda_{t+s-1}| \\
&= |-\delta\mu^2\lambda_{t+s-1} + \mu g_{t+s-1}(x_{t+s-1})| \\
&\leq \delta\mu^2\lambda_{t+s-1} + \mu |g_{t+s-1}(x_{t+s-1})| \\
&\leq \delta\mu^2 \frac{G}{\delta\mu} + \mu G \\
&= 2\mu G.
\end{aligned}$$

Therefore, we can write:

$$\begin{aligned}
&\sum_{\tau=0}^{W-1} \lambda_{t+\tau} g_{t+\tau}(x_W^*) \leq \\
&\lambda_t \sum_{\tau=0}^{W-1} g_{t+\tau}(x_W^*) + \sum_{\tau=0}^{W-1} \Big(\sum_{s=1}^{\tau} 2\mu G\Big) G \\
&= \lambda_t \sum_{\tau=0}^{W-1} g_{t+\tau}(x_W^*) + \sum_{\tau=0}^{W-1} 2\mu G^2 \tau \\
&= \lambda_t \sum_{\tau=0}^{W-1} g_{t+\tau}(x_W^*) + \mu G^2 W(W-1).
\end{aligned}$$

D Proof of Lemma 4.3

Fix $k \in [K]$. Using L-smoothness of the function $\mathcal{L}_t$, we have:

$$\begin{aligned}
&\mathcal{L}_t(x_t^{(k+1)}, \lambda_t) \geq \\
&\mathcal{L}_t(x_t^{(k)}, \lambda_t) + \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)}\rangle - \frac{L}{2K^2}\|v_t^{(k)}\|_2^2 \\
&\overset{(a)}{\geq} \mathcal{L}_t(x_t^{(k)}, \lambda_t) + \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)}\rangle - \frac{LR^2}{2K^2} \\
&= \mathcal{L}_t(x_t^{(k)}, \lambda_t) + \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)} - x_W^*\rangle \\
&+ \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), x_W^*\rangle - \frac{LR^2}{2K^2} \\
&= \mathcal{L}_t(x_t^{(k)}, \lambda_t) + \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)} - x_W^*\rangle \\
&+ \frac{1}{K}\langle \nabla f_t(x_t^{(k)}), x_W^*\rangle - \frac{1}{K}\lambda_t\langle \nabla g_t(x_t^{(k)}), x_W^*\rangle - \frac{LR^2}{2K^2} \\
&\overset{(b)}{=} \mathcal{L}_t(x_t^{(k)}, \lambda_t) + \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)} - x_W^*\rangle \\
&+ \frac{1}{K}\langle \nabla f_t(x_t^{(k)}), x_W^*\rangle - \frac{1}{K}\lambda_t g_t(x_W^*) - \frac{1}{K}\lambda_t\frac{B_T}{T} - \frac{LR^2}{2K^2},
\end{aligned}$$

where (a) is due to the assumption that $\text{diam}(\mathcal{X}) \leq R$. Note that in order to obtain (b), we have used linearity of the budget functions for all $t \in [T]$ to write $\langle \nabla g_t(x_t^{(k)}), x_W^*\rangle = \langle p_t, x_W^*\rangle = g_t(x_W^*) + \frac{B_T}{T}$. More general assumptions such as convexity would not be enough for the proof to go through.

Considering that $f_t(x)$ is monotone DR-submodular for all $t \in [T]$, we can write:

$$\begin{aligned}
f_t(x_W^*) - f_t(x_t^{(k)}) &\overset{(c)}{\leq} f_t(x_W^* \vee x_t^{(k)}) - f_t(x_t^{(k)}) \\
&\overset{(d)}{\leq} \langle \nabla f_t(x_t^{(k)}), (x_W^* \vee x_t^{(k)}) - x_t^{(k)}\rangle \\
&= \langle \nabla f_t(x_t^{(k)}), (x_W^* - x_t^{(k)}) \vee 0\rangle \\
&\overset{(e)}{\leq} \langle \nabla f_t(x_t^{(k)}), x_W^*\rangle,
\end{aligned}$$

where for $a, b \in \mathbb{R}^n$, $a \vee b$ denotes the entry-wise maximum of vectors a and b, (c) and (e) are due to monotonocity of f_t and (d) uses concavity of f_t along non-negative directions.

Therefore, we conclude:

$$\begin{aligned}
\mathcal{L}_t(x_t^{(k+1)}, \lambda_t) \geq\ & \mathcal{L}_t(x_t^{(k)}, \lambda_t) \\
&+ \frac{1}{K}\langle \nabla_x \mathcal{L}_t(x_t^{(k)}, \lambda_t), v_t^{(k)} - x_W^*\rangle \\
&+ \frac{1}{K}\big(f_t(x_W^*) - f_t(x_t^{(k)})\big) - \frac{1}{K}\lambda_t g_t(x_W^*) \\
&- \frac{1}{K}\lambda_t\frac{B_T}{T} - \frac{LR^2}{2K^2}.
\end{aligned}$$

Equivalently, we can write:

$$\begin{aligned}
&f_t(x_W^*) - f_t(x_t^{(k+1)}) \leq (1 - \frac{1}{K})\big(f_t(x_W^*) - f_t(x_t^{(k)})\big) \\
&- \lambda_t\big(g_t(x_t^{(k+1)}) - g_t(x_t^{(k)})\big) + \frac{1}{K}\lambda_t g_t(x_W^*) + \frac{1}{K}\lambda_t\frac{B_T}{T} \\
&+ \frac{LR^2}{2K^2} + \frac{1}{K}\langle \nabla \mathcal{L}_t(x_t^{(k)}, \lambda_t), x_W^* - v_t^{(k)}\rangle \\
&= (1 - \frac{1}{K})\big(f_t(x_W^*) - f_t(x_t^{(k)})\big) + \frac{1}{K}\big[\lambda_t\frac{B_T}{T} - \lambda_t\langle p_t, v_t^{(k)}\rangle \\
&+ \lambda_t g_t(x_W^*) + \frac{LR^2}{2K} + \langle \nabla \mathcal{L}_t(x_t^{(k)}, \lambda_t), x_W^* - v_t^{(k)}\rangle\big].
\end{aligned} \tag{1}$$

Replacing t by $t+\tau$ in inequality (1) and taking the sum over $\tau \in \{0, \dots, W-1\}$ and $t \in \{1, \dots, T-W+1\}$, we obtain:

$$\begin{aligned}
&\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*) - f_{t+\tau}(x_{t+\tau}^{(k+1)})\big) \leq \\
&(1 - \frac{1}{K})\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*) - f_{t+\tau}(x_{t+\tau}^{(k)})\big) \\
&+ \frac{1}{K}\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big[-\lambda_{t+\tau}\langle p_{t+\tau}, v_{t+\tau}^{(k)}\rangle + \lambda_{t+\tau}g_{t+\tau}(x_W^*) \\
&+ \lambda_{t+\tau}\frac{B_T}{T} + \frac{LR^2}{2K} + \langle \nabla \mathcal{L}_{t+\tau}(x_{t+\tau}^{(k)}, \lambda_{t+\tau}), x_W^* - v_{t+\tau}^{(k)}\rangle\big].
\end{aligned} \tag{2}$$

Applying inequality (2) recursively for all $k \in \{1, \dots, K\}$, we obtain:

$$\begin{aligned}
&\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*) - f_{t+\tau}(\underbrace{x_{t+\tau}^{(K+1)}}_{=x_{t+\tau}})\big) \leq \\
&\Pi_{k=0}^{K-1}(1 - \frac{1}{K})\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*) - f_{t+\tau}(x_{t+\tau}^{(0)})\big) \\
&+ \sum_{k=0}^{K-1}\frac{1}{K}\Pi_{j=k+1}^{K-1}(1 - \frac{1}{K})\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big[-\lambda_{t+\tau}\langle p_{t+\tau}, v_{t+\tau}^{(k)}\rangle \\
&+ \lambda_{t+\tau}g_{t+\tau}(x_W^*) + \lambda_{t+\tau}\frac{B_T}{T} + \frac{LR^2}{2K} \\
&+ \langle \nabla \mathcal{L}_{t+\tau}(x_{t+\tau}^{(k)}, \lambda_{t+\tau}), x_W^* - v_{t+\tau}^{(k)}\rangle\big].
\end{aligned} \tag{3}$$

Using the regret bound of Online Gradient Ascent instance $\mathcal{E}_k$ $\forall k \in [K]$, the following holds (Theorem

3.1. of Hazan et al., 2016):

$$
\begin{aligned}
&\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t),x_W^*-v_t^{(k)}\rangle = \\
&\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t),x_W^*\rangle-\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t),v_t^{(k)}\rangle \\
&\leq \max_x\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t),x\rangle-\sum_{t=1}^{T}\langle\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t),v_t^{(k)}\rangle \\
&\leq \frac{R^2}{\mu}+\frac{\mu}{2}\sum_{t=1}^{T}\|\nabla_x\mathcal{L}_t(x_t^{(k)},\lambda_t)\|^2 \\
&= \frac{R^2}{\mu}+\frac{\mu}{2}\sum_{t=1}^{T}\|\nabla_x f_t(x_t^{(k)})-\lambda_t p_t\|^2 \\
&\overset{(a)}{\leq} \frac{R^2}{\mu}+\frac{\mu}{2}\sum_{t=1}^{T}\big(2\|\nabla_x f_t(x_t^{(k)})\|^2+2\lambda_t^2\|p_t\|^2\big) \\
&\overset{(b)}{\leq} \frac{R^2}{\mu}+\beta^2\mu T+\beta^2\mu\sum_{t=1}^{T}\lambda_t^2,
\end{aligned}
$$

where (a) uses the inequality $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2\ \forall a,b\in\mathbb{R}^n$ and (b) is due to β-Lipschitzness of functions f_t, g_t for all $t\in[T]$.
Using the inequality $(1-\frac{1}{K})^K \leq \frac{1}{e}$ in (3), we have:

$$
\begin{aligned}
&\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*)-f_{t+\tau}(x_{t+\tau})\big) \leq \\
&\frac{1}{e}\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*)-f_{t+\tau}(x_{t+\tau}^{(0)})\big) \\
&+\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\sum_{k=0}^{K-1}\frac{1}{K}\big[-\lambda_{t+\tau}\langle p_{t+\tau},v_{t+\tau}^{(k)}\rangle+\lambda_{t+\tau}g_{t+\tau}(x_W^*) \\
&+\lambda_{t+\tau}\frac{B_T}{T}+\frac{LR^2}{2K}+\langle\nabla\mathcal{L}_{t+\tau}(x_{t+\tau}^{(k)},\lambda_{t+\tau}),x_W^*-v_{t+\tau}^{(k)}\rangle\big] \\
&=\frac{1}{e}\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big(f_{t+\tau}(x_W^*)-\underbrace{f_{t+\tau}(0)}_{=0}\big) \\
&+\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big[-\lambda_{t+\tau}g_{t+\tau}(x_{t+\tau})-\lambda_{t+\tau}\frac{B_T}{T} \\
&+\lambda_{t+\tau}g_{t+\tau}(x_W^*)+\lambda_{t+\tau}\frac{B_T}{T}+\frac{LR^2}{2K} \\
&+\sum_{k=0}^{K-1}\frac{1}{K}\langle\nabla\mathcal{L}_{t+\tau}(x_{t+\tau}^{(k)},\lambda_{t+\tau}),x_W^*-v_{t+\tau}^{(k)}\rangle\big]. \qquad (4)
\end{aligned}
$$

Rearranging the terms in (4), we obtain:

$$
\begin{aligned}
&\underbrace{\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\big((1-\frac{1}{e})f_{t+\tau}(x_W^*)-f_{t+\tau}(x_{t+\tau})\big)}_{\text{(a)}} \\
&+\underbrace{\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\lambda_{t+\tau}g_{t+\tau}(x_{t+\tau})}_{\text{(b)}} \leq \\
&\frac{LR^2}{2K}W(T-W+1)+\underbrace{\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\lambda_{t+\tau}g_{t+\tau}(x_W^*)}_{\text{(c)}} \\
&+\sum_{k=0}^{K-1}\frac{1}{K}\underbrace{\sum_{t=1}^{T-W+1}\sum_{\tau=0}^{W-1}\langle\nabla L_{t+\tau}(x_{t+\tau}^{(k)},\lambda_{t+\tau}),x_W^*-v_{t+\tau}^{(k)}\rangle}_{\text{(d)}}.
\end{aligned}
\tag{5}
$$

(a) could be lower bounded as follows:

$$
\begin{aligned}
\text{(a)} &= WR_T-\sum_{i=1}^{W-1}(W-i)\big([(1-\frac{1}{e})f_i(x_W^*)-f_i(x_i)] \\
&\quad+[(1-\frac{1}{e})f_{T-i+1}(x_W^*)-f_{T-i+1}(x_{T-i+1})]\big) \\
&\geq WR_T-2F\sum_{i=1}^{W-1}(W-i) \\
&= WR_T-FW(W-1).
\end{aligned}
\tag{6}
$$

Using Lemma 4.1 with $(1-\delta\mu^2)\leq 1$, we have:

$$
\begin{aligned}
\text{(b)} &= W\sum_{t=1}^{T}\lambda_t g_t(x_t) \\
&\quad-\sum_{i=1}^{W-1}(W-i)\big(\lambda_i g_i(x_i)+\lambda_{T-i+1}g_{T-i+1}(x_{T-i+1})\big) \\
&\geq W\sum_{t=1}^{T}\lambda_t g_t(x_t) \\
&\quad-\sum_{i=1}^{W-1}(W-i)\big(\mu(i-1)G^2+\mu(T-i)G^2\big) \\
&\geq W\sum_{t=1}^{T}\lambda_t g_t(x_t)-\frac{G^2}{2}\mu W(W-1)(T-1).
\end{aligned}
\tag{7}
$$

In order to bound (c), we use Lemma 4.2 and write:

$$
\begin{aligned}
\text{(c)} &\leq \sum_{t=1}^{T-W+1}\big(\lambda_t\underbrace{\sum_{\tau=0}^{W-1}g_{t+\tau}(x_W^*)}_{\leq 0}+G^2\mu W(W-1)\big) \\
&\leq \mu G^2 W(W-1)(T-W+1).
\end{aligned}
\tag{8}
$$

Finally, for a fixed $k \in [K]$, we can bound (d) as follows:

$$
\begin{aligned}
(\mathrm{d}) = & W\sum_{t=1}^{T}\langle \nabla \mathcal{L}_t(x_t^{(k)}), x_W^* - v_t^{(k)}\rangle \\
& - \sum_{i=1}^{W-1}(W-i)\big(\underbrace{[\langle \nabla \mathcal{L}_i(x_i^{(k)}), x_W^* - v_i^{(k)}\rangle]}_{\geq -\beta R(1+\lambda_i)} \\
& + \underbrace{[\langle \nabla \mathcal{L}_{T-i+1}(x_{T-i+1}^{(k)}), x_W^* - v_{T-i+1}^{(k)}\rangle]}_{\geq -\beta R(1+\lambda_{T-i+1})}\big) \\
& \leq \frac{R^2 W}{\mu} + \beta^2\mu T W + \beta^2 \mu W \sum_{t=1}^{T}\lambda_t^2 \\
& + \sum_{i=1}^{W-1}(W-i)\big(2\beta R + \beta R \underbrace{\lambda_i}_{\leq (i-1)\mu G} + \beta R \underbrace{\lambda_{T-i+1}}_{\leq (T-i)\mu G}\big) \\
& = \frac{R^2 W}{\mu} + \beta^2\mu T W + \beta^2 \mu W \sum_{t=1}^{T}\lambda_t^2 + \beta R W(W-1) \\
& + \frac{\beta R G}{2}\mu W(W-1)(T-1). \qquad (9)
\end{aligned}
$$

Using the regret bound for Online Gradient Descent (Theorem 3.1. of Hazan et al., 2016), we have:

$$
\begin{aligned}
& \sum_{t=1}^{T}\big(\mathcal{L}_t(x_t,\lambda_t) - \mathcal{L}_t(x_t,\lambda)\big) = \\
& \sum_{t=1}^{T}\big(-\lambda_t g_t(x_t) + \frac{\delta\mu}{2}\lambda_t^2 + \lambda g_t(x_t) - \frac{\delta\mu}{2}\lambda^2\big) \\
& \leq \frac{\lambda^2}{\mu} + \frac{\mu}{2}\sum_{t=1}^{T}\|\nabla_\lambda \mathcal{L}_t(x_t,\lambda_t)\|^2 \\
& \leq \frac{\lambda^2}{\mu} + \frac{\mu}{2}\sum_{t=1}^{T}\big(-g_t(x_t) + \delta\mu\lambda_t\big)^2 \\
& \overset{(a)}{\leq} \frac{\lambda^2}{\mu} + \frac{\mu}{2}\sum_{t=1}^{T}(2g_t^2(x_t) + 2\delta^2\mu^2\lambda_t^2) \\
& \leq \frac{\lambda^2}{\mu} + G^2\mu T + \delta^2\mu^3\sum_{t=1}^{T}\lambda_t^2, \qquad (10)
\end{aligned}
$$

where we use $(a+b)^2 \leq 2a^2 + 2b^2 \ \forall a, b \in \mathbb{R}$ to derive inequality (a).
Combining (5), (6), (7), (8), (9) and (10), dividing both sides by W and rearranging the terms, we conclude:

$$
\begin{aligned}
& R_T + C_T\lambda + \frac{\delta\mu}{2}\sum_{t=1}^{T}\lambda_t^2 - \frac{\delta\mu}{2}T\lambda^2 - \frac{\lambda^2}{\mu} \leq \\
& (F+\beta R)(W-1) + \frac{G}{2}(G+\beta R)\mu(W-1)(T-1) \\
& + \frac{R^2}{\mu} + (G^2+\beta^2)\mu T + G^2\mu(W-1)(T-W+1) \\
& + \frac{LR^2}{2K}(T-W+1) + (\delta^2\mu^3 + \beta^2\mu)\sum_{t=1}^{T}\lambda_t^2.
\end{aligned}
$$

Note that if T is large enough such that $WT \geq 16R^2$ holds, we can write:

$$
\begin{aligned}
\delta^2\mu^2 + \beta^2 &= 16\beta^4 . \frac{R^2}{\beta^2 WT} + \beta^2 \\
&= \frac{16R^2}{WT}\beta^2 + \beta^2 \\
&\leq 2\beta^2 \\
&= \frac{\delta}{2}.
\end{aligned}
$$

Therefore, we can remove the terms $\sum_{t=1}^{T}\lambda_t^2$ from both sides of the inequality. Ignoring these terms, we obtain the desired result.

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