Abstract

A number of applications (e.g., AI bot tournaments, sports, peer grading, crowdsourcing) use pairwise comparison data and the Bradley-Terry-Luce (BTL) model to evaluate a given collection of items (e.g., bots, teams, students, search results). Past work has shown that under the BTL model, the widely-used maximum-likelihood estimator (MLE) is minimax-optimal in estimating the item parameters, in terms of the mean squared error. However, another important desideratum for designing estimators is fairness. In this work, we consider one specific type of fairness, which is the notion of bias in statistics. We show that the MLE incurs a suboptimal rate in terms of bias. We then propose a simple modification to the MLE, which “stretches” the bounding box of the maximum-likelihood optimizer by a small constant factor from the underlying ground truth domain. We show that this simple modification leads to an improved rate in bias, while maintaining minimax-optimality in the mean squared error. In this manner, our proposed class of estimators provably improves fairness in the sense of bias without loss in accuracy.

1 Introduction

A number of applications involve data in the form of pairwise comparisons among a collection of items, and entail an evaluation of the individual items from this data. An application gaining increasing popularity is competition between pairs of AI bots (e.g., Ontan et al., 2013). Here a number of AI bots compete with each other in pairwise matchups for a certain task, where each bot plays every other bot a certain number of times in a round robin fashion, with the goal of evaluating the quality of each bot. A second example is the evaluation of self-play of AI algorithms in their training phase (Silver et al., 2017), where again, different copies of an AI bot play against each other a number of times. Applications involving humans include sports and online games such as the English Premier League of football (Király and Qian, 2017; SinceAWin.com, 2019) (unofficial ratings), and official world rankings for chess such as FIDE (International Chess Federation, 2017) and USCF (Glickman and Doan, 2017) ratings.

A common method of evaluating the items based on pairwise comparisons is to assume that the probability of an item beating another equals the logistic function of the difference in the true quality of the two items, and then infer the true quality from the observed outcomes of the comparisons (e.g., the Elo rating system). Various applications employ such an approach to rating from pairwise comparisons, with some modifications tailored to that specific application. Our goal is not to study the application-specific versions, but the foundational underpinnings of such rating systems.

In this paper, we study the pairwise-comparison model that underlies (Glickman and Jones, 1999; Aldous, 2017) these rating systems, namely the Bradley-Terry-Luce (BTL) model (Bradley and Terry, 1952; Luce, 1959). The BTL model assumes that each item is associated to an unknown real-valued parameter representing the quality of that item, and assumes that the probability of an item beating another is the logistic function applied to the difference of the parameters of these two items. The BTL model is also employed in the applications of peer grading (Shah et al., 2013; Lamon et al., 2016) (where the grades of the students are set as the BTL parameters to be estimated), crowdsourcing (Chen et al., 2016; Ponce-López et al., 2016), and understanding consumer choice in marketing (Green et al., 1981).
1.1 BTL model and maximum likelihood estimation

Now we present a formal definition of the BTL model. Let $d \geq 2$ denote the number of items. The $d$ items are associated to an unknown parameter vector $\theta^* \in \mathbb{R}^d$ whose $i$th entry represents the underlying quality of item $i \in [d]$. When any item $i \in [d]$ is compared with any item $j \in [d]$ in the BTL model, then item $i$ beats item $j$ with probability

$$
\frac{1}{1 + e^{-(\theta_i^* - \theta_j^*)}},
$$

independent of all other comparisons. The probability of item $j$ beating $i$ is one minus the expression (1) above. We consider the “league format” (Aldous, 2017) of comparisons where every pair of items is compared $k$ times.

We follow the usual assumption (Hajek et al., 2014; Shah et al., 2016) under the BTL model that the true parameter vector $\theta^*$ lies in the set $\Theta_B$ parameterized by a known constant $B > 0$, defined as:

$$
\Theta_B = \{ \theta \in \mathbb{R}^d \mid \|\theta\|_\infty \leq B \text{ and } \sum_{i=1}^d \theta_i = 0 \}. 
$$

The first constraint requires that the magnitude of the parameters is bounded by some constant $B$. We call this constraint the “box constraint”. It has been shown that a box constraint is necessary, because otherwise the estimation error can diverge to infinity (Shah et al., 2016, Appendix G). The second constraint requires that the parameters sum to 0. This is without loss of generality due to the shift-invariance property of the BTL model.

A large amount of both theoretical (Hunter, 2004; Hajek et al., 2014; Szörényi et al., 2015; Negahban et al., 2016; Shah et al., 2016) and applied (Stigler, 1994; Sham and Curtis, 1995; Chen et al., 2016; Ponce-López et al., 2016) literature focuses on the goal of estimating the parameter vector $\theta^*$ of the BTL model. A standard and widely-studied estimator is the maximum-likelihood estimator (MLE):

$$
\hat{\theta}^{(B)} = \arg\min_{\theta \in \Theta_B} \ell(\theta),
$$

where $\ell$ is the negative log-likelihood function. Letting $W_{ij}$ denote a random variable representing the number of times that item $i \in [d]$ beats item $j \in [d]$, the log-likelihood function $\ell$ is given by:

$$
\ell(\theta) := \ell(W_{ij}; \theta) = -\sum_{1 \leq i < j \leq d} \left[ W_{ij} \log \left( \frac{1}{1 + e^{-(\theta_i - \theta_j)}} \right) + W_{ji} \log \left( \frac{1}{1 + e^{-(\theta_i - \theta_j)}} \right) \right].
$$

1.2 Metrics

**Accuracy.** A common metric used in the literature on estimating the BTL model is the accuracy of the estimate, measured in terms of the mean squared error. Formally, the (worst-case) accuracy of any estimator $\hat{\theta}$ is defined as:

$$
\alpha(\hat{\theta}) := \sup_{\theta^* \in \Theta_B} \mathbb{E}[\|\hat{\theta} - \theta^*\|_2^2].
$$

Importantly, past work (Hajek et al., 2014; Shah et al., 2016) has shown that the MLE (3) has the appealing property of being minimax-optimal in terms of the accuracy defined in (4).

**Bias.** Another important desideratum for designing and evaluating estimators is fairness. For example, in sports or online games, we do not want to assign scores in such a way that it systematically gives certain players higher scores than their true quality, but at the same time gives certain other players lower scores than their true quality. In this paper, we adopt the standard definition of bias in statistics as our notion of fairness.

For any estimator, the bias incurred by this estimator on a parameter is defined as the difference between the expected value of the estimator and the true value of the parameter.

$$
\beta(\hat{\theta}) := \sup_{\theta^* \in \Theta_B} \mathbb{E}[\hat{\theta} - \theta^*] = \mathbb{E}[\hat{\theta}].
$$

1.3 Contribution I: Performance of MLE

Our first contribution is to analyze the widely-used MLE (3) in terms of its bias. Let us begin with a visual illustration through simulation. Consider $d = 25$ items with parameter values equally spaced in the interval $[-1, 1]$ (here we have $B = 1$), where $k = 5$ pairwise comparisons are observed between each pair of items under the BTL model. We estimate the parameters using the MLE, and plot the bias on each item across 5000 iterations of the simulation in Fig. 1 (striped red). The MLE shows a systematic bias: it induces a negative bias (under-estimation) on the large positive parameters, and a positive bias (over-estimation) on the large negative parameters. In the applications of interest, the MLE thus systematically underestimates the abilities of the top players/students/items and overestimates the abilities of those at the bottom.

In this paper, we theoretically quantify the bias incurred by the MLE as follows.
As shown by our results to follow, this bias is suboptimal. Our proof for this result indicates that the bias is incurred because the MLE operates under the accurately specified model with the box constraint at $B$. That is, the MLE “clips” the estimate to lie within the set $\Theta_B$. This issue is visible in the simulation of Fig. 1 where the bias of the MLE is the largest when the true values of the parameters are near the boundaries $\pm B$. For example, consider a true parameter whose value equals $B$. The estimate of this parameter sometimes equals $B$ (due to the box constraint), and sometimes is smaller than $B$ (due to the randomness of the data). Therefore, in expectation, the estimate of this parameter incurs a negative bias. An analogous argument explains the positive bias when the true parameter equals or is close to $-B$.

**Theorem 1.1** (MLE bias lower bound; Informal). The MLE (3) incurs a bias $\beta(\hat{\theta}(B))$ lower bounded as $\Omega(\frac{1}{\sqrt{d}})$.

Consequently, in this work, we propose the following simple modification to the MLE which is a middle ground between the MLE and the unconstrained MLE. Specifically, we consider a “stretched-MLE” associated to a parameter $A$ such that $A > B$. Given the parameter $A$, the stretched-MLE is identical to the MLE (3) but “stretches” the box constraint to $A$:

$$\hat{\theta}(A) = \arg\min_{\theta \in \Theta_A} \ell(\theta),$$

where $\Theta_A := \{\theta \in \mathbb{R}^d \mid \|\theta\|_\infty \leq A\}$ and $\sum_{i=1}^d \theta_i = 0$. That is, $\Theta_A$ simply replaces the box constraint $\|\theta\|_\infty \leq B$ in (2) by the “stretched” box constraint $\|\theta\|_\infty \leq A$.

The bias induced by the stretched-MLE (with $A = 2$) in the previous experiment is also shown in Fig. 1 (solid blue). Note that the maximum bias (at the leftmost item with the largest negative parameter, or the rightmost item with the largest positive parameter) is significantly reduced compared to the MLE. Moreover, the bias induced by the stretched-MLE looks qualitatively more evened out across the items.

Our second main theoretical result proves that the stretched-MLE indeed incurs a significantly lower bias.
The logistic nature (1) of the BTL model relates our concept of proper vs. improper learning in learning theory. The MLE can be considered a proper estimator that requires output an estimate from true set \( \Theta_B \). The stretched-MLE is an improper estimator that outputs an estimate in the “stretched” set \( \Theta_A \), and the unconstrained MLE is a (fully) improper estimator that is allowed to output any arbitrary estimate.

1.5 Related work

The logistic nature (1) of the BTL model relates our work to studies of logistic regression (e.g., Portnoy, 1988; He and Shao, 2000; Fan et al., 2019), among which the work on high-dimensional logistic regression (Sur and Candès, 2019; Salehi et al., 2019; Zhao et al., 2020) is the most closely related to ours. The papers (Sur and Candès, 2019; Zhao et al., 2020) consider an unconstrained MLE in logistic regression, and shows its bias in the opposite direction as compared to our results on the standard MLE (constrained) in the BTL model. Specifically, the papers (Sur and Candès, 2019; Zhao et al., 2020) show that the large positive coefficients are overestimated, and the large negative coefficients are underestimated. There are several additional key differences between the results in Sur and Candès (2019) as compared to the present paper. The paper (Sur and Candès, 2019) studies the asymptotic bias of the unconstrained MLE, showing that the unconstrained MLE is not consistent. On the other hand, we operate in a regime where the MLE is still consistent, and study finite-sample bounds. Moreover, the paper (Sur and Candès, 2019) assumes that the predictor variables are i.i.d. Gaussian. On the other hand, in the BTL model the probability that item \( i \) beats item \( j \) can be written as \( \frac{1-1/(1+e^{x_{ij}^T \theta^*})}{1+e^{x_{ij}^T \theta^*}} \), where each predictor variable \( x_{ij} \in \mathbb{R}^d \) has entry \( i \) equal to 1, entry \( j \) equal to −1, and the remaining entries equal to 0. A subsequent paper (Salehi et al., 2019) considers a regularized MLE in logistic regression where the regularizer can reduce the asymptotic bias.

A common way to achieve bias reduction is to employ finite-sample correction, such as Jackknife (Quenouille, 1949) and other methods (Cox and Snell, 1968; Anderson and Richardson, 1979; Firth, 1993) to the MLE (or other estimators). These methods operate in a low-dimensional regime (small \( d \)) where the MLE is asymptotically unbiased. Informally, these methods use a Taylor expansion, write the expression for the bias as an infinite sum, and modify the estimator in a variety of ways to eliminate the lower-order terms in this bias expression. However, since the expression is an infinite sum, eliminating the first term does not guarantee a low rate of the bias. Moreover, since the Taylor expansion terms in the infinite sum are implicit functions of \( \theta^* \), eliminating lower-order terms does not directly translate to explicit worst-case guarantees.

Returning to the pairwise-comparison setting, in addition to the mean squared error, past work has also considered accuracy in terms of the \( \ell_1 \) norm error (Agarwal et al., 2018) and the \( \ell_\infty \) norm error (Chen and Suh, 2015; Jang et al., 2017; Chen et al., 2019). The \( \ell_\infty \) bound for a regularized MLE is analyzed in Chen et al. (2019). Our proof for bounding the bias of the standard MLE (unregularized) relies on a high-probability \( \ell_\infty \) bound for the unconstrained MLE (unregularized). It is important to note that the bound for the regularized MLE from Chen et al. (2019) does not carry to the unconstrained MLE, because the proof from Chen et al. (2019) relies on the strong convexity of the regularizer. On the other hand, our intermediate result provides a partial answer to the open question in Chen et al. (2019) about the \( \ell_\infty \) norm for the unregularized MLE (Lemma A.5 in Appendix A): We establish an \( \ell_\infty \) bound for the unregularized MLE when \( p_{obs} = 1 \), which has the same rate as that of the regularized MLE in (Chen et al., 2019).

Another common occurrence of bias is regression towards the mean (Stigler, 1997). It is the phenomenon that random variables taking large (or small) values in one measurement are likely to take more moderate
(closer to average) values in subsequent measurements. On the contrary, we consider items whose indices are fixed (and are not order statistics). For fixed indices, our results suggest that under the BTL model, the bias (under-estimation of large true values) is in the opposite direction as that in regression towards the mean (over-estimation of large observed values).

Finally, there is a vast literature on different notions of fairness in various problems such as classification and resource allocation (e.g. Barocas et al., 2019, Chapter 3; Marsh and Schilling, 1994 and references therein). While other notions of fairness are of interest for future research, in this work we focus on the classical statistical notion of bias to capture our intuition of minimizing the maximum disparity in estimation.

2 Main results

In this section, we formally provide our main theoretical results on bias and on the mean squared error.

2.1 Bias

Recall that \( d \) denotes the number of items and \( k \) denotes the number of comparisons per pair of items. The true parameter vector is \( \theta^* \in \Theta_B \) for some prespecified constant \( B > 0 \). The following theorem provides bounds on the bias of the standard MLE \( \hat{\theta}^{(B)} \) and that of our stretched-MLE \( \hat{\theta}^{(A)} \) with parameter \( A \). Specifically, it shows that if \( A \) is a finite constant strictly greater than \( B \), then our stretched-MLE has a much smaller bias than the MLE when \( d \) and \( k \) are sufficiently large.

**Theorem 2.1.** (a) There exists a constant \( c > 0 \) that depends only on the constant \( B \), such that

\[
\beta(\hat{\theta}^{(B)}) \geq \frac{c}{\sqrt{d}} \tag{7a}
\]

for all \( d \geq d_0 \) and all \( k \geq k_0 \), where \( d_0 \) and \( k_0 \) are constants that depend only on the constant \( B \).

(b) Let \( A \) be any finite constant such that \( A > B \). There exists a constant \( c > 0 \) that depends only on the constants \( A \) and \( B \), such that

\[
\beta(\hat{\theta}^{(A)}) \leq c \frac{\log d + \log k}{d_0 k_0} \tag{7b}
\]

for all \( d \geq d_0 \) and all \( k \geq k_0 \), where \( d_0 \) and \( k_0 \) are constants that depend only on the constants \( A \) and \( B \).

We note that in Theorem 2.1(b), we allow \( A \) to be any positive constant as long as \( A > B \). Therefore, the difference between \( A \) and \( B \) can be any arbitrarily small constant. It is perhaps surprising that stretching the box constraint only by a small constant yields such a significant improvement in the bias. We provide intuition behind this result in Section 2.1.1. The complete proof is provided in Appendix A.

2.1.1 Intuition for Theorem 2.1

In this section, we provide intuition why stretching the box constraint from \( B \) to \( A \) significantly reduces the bias. Specifically, we consider a simplified setting with \( d = 2 \) items. Due to the centering constraint, we have \( \theta_2^* = -\theta_1^* \) for the true parameters, and we have \( \hat{\theta}_2 = -\hat{\theta}_1 \) for any estimator \( \hat{\theta} \) that satisfies the centering constraint. Therefore, it suffices to focus only on item 1. Denote \( \mu \) as the random variable representing the fraction of times that item 1 beats item 2, and denote the true probability that item 1 beats item 2 as \( \mu^* \). The standard MLE \( \hat{\theta}^{(B)} \), the stretched-MLE \( \hat{\theta}^{(A)} \) and the unconstrained MLE \( \hat{\theta}^{(\infty)} \) can be solved in closed form:

\[
\hat{\theta}_1^{(B)}(\mu) = \begin{cases} 
-B & \text{if } \mu \in [0, \mu_-] \\
-\frac{1}{2} \log \left( \frac{1}{\mu} - 1 \right) & \text{if } \mu \in (\mu_-, \mu_+) \\
B & \text{if } \mu \in [\mu_+, 1].
\end{cases} 
\]

\[
\hat{\theta}_1^{(A)}(\mu) = \begin{cases} 
-A & \text{if } \mu \in [0, 1/e] \\
-\frac{1}{2} \log \left( \frac{1}{\mu} - 1 \right) & \text{if } \mu \in \left(\frac{1}{1+e-2\pi}, \frac{1}{1+e-2\pi}, 1\right) \\
A & \text{if } \mu \in \left[1, \frac{1}{1+e-2\pi}\right].
\end{cases} 
\]

\[
\hat{\theta}_1^{(\infty)}(\mu) = -\frac{1}{2} \log \left( \frac{1}{\mu} - 1 \right).
\]

See Fig. 2a for a comparison of these three estimators. Now we consider the bias incurred by these three estimators. For intuition, let us consider the case \( \theta_1^* = B \), which incurs the largest bias in our simulation of Fig. 1. If the observation \( \mu \) were noiseless (and thus equals the true probability \( \mu^+ \)), then all three estimators would output the true parameter \( B \). However, the observation \( \mu \) is noisy, and only concentrates around \( \mu_+ \). To investigate how these three estimators behave differently under this noise, we zoom in to the region around \( \mu = \mu_+ \) indicated by the grey box in Fig. 2a. (Note that the observation \( \mu \) can lie outside the grey box, but for intuition we ignore this low-probability event due to concentration.)

The behaviors of the three estimators in the grey box are shown in Fig. 2b, Fig. 2c and Fig. 2d, respectively. For each of these estimators, the blue dots on the x-axis denotes the noisy observation of \( \mu \) across different iterations, and the blue dots on the estimator function denotes the corresponding noisy estimates. The ex-


Figure 2: Intuition on the sources of bias. (a) The estimators standard MLE $\hat{\theta}^{(B)}$, stretched-MLE $\hat{\theta}^{(A)}$ and unconstrained MLE $\hat{\theta}^{(\infty)}$ (on item 1), as a function of $\mu$ when there are $d = 2$ items. We consider $\theta^* = [B, -B]$, under which the true probability that item 1 beats item 2 is $\mu_+$. We zoom in to the region around $\mu = \mu_+$ indicated by the grey box. (b) The standard MLE $\hat{\theta}^{(B)}$ incurs a negative bias, because the estimate is required to be at most $B$. (c) The unconstrained MLE $\hat{\theta}^{(\infty)}$ incurs a positive bias by Jensen’s inequality, because the estimator function is convex on $\mu \in (0.5, 1)$. (d) Our estimator balances out the negative bias and the positive bias.

expected value of the estimator is a mean over the blue dots on the estimator function. For the standard MLE $\hat{\theta}^{(B)}$ (Fig. 2b), the box constraint requires that the estimate shall never exceed $B$. We call this phenomenon the “clipping” effect, which introduces a negative bias. For the unconstrained MLE $\hat{\theta}^{(\infty)}$ (Fig. 2c), since the estimator function is convex, by Jensen’s inequality, the unconstrained MLE $\hat{\theta}^{(\infty)}$ introduces a positive bias. Our proposed stretched-MLE $\hat{\theta}^{(A)}$ (Fig. 2d) lies in the middle between the standard MLE and the unconstrained MLE. Therefore, the stretched-MLE balances out the negative bias from the “clipping” effect and the positive bias from the convexity of the estimator function, thereby yielding a smaller bias on the item parameter. In practice, one can numerically tune the parameter $A$ to minimize the bias across all possible parameter vector $\theta^* \in \Theta_B$. Simulation results on different values of $A$ are included in Section 3.

2.2 Accuracy

Given the result of Theorem 2.1 on the bias reduction of the estimator $\hat{\theta}^{(A)}$, we revisit the mean squared error. Past work (Hajek et al., 2014; Shah et al., 2016) has shown that the standard MLE $\hat{\theta}^{(B)}$ is minimax-optimal in terms of the mean squared error. The following theorem shows that this minimax-optimality also holds for our proposed stretched-MLE $\hat{\theta}^{(A)}$, where $A$ is any constant such that $A > B$. The theorem statement and its proof follows Theorem 2 from (Shah et al., 2016), after some modification to accommodate the new bounding box parameter $A$.

**Theorem 2.2.** (a) [Theorem 2(a) from Shah et al. (2016)] There exists a constant $c > 0$ that depends only on the constant $B$, such that any estimator $\hat{\theta}$ has a mean squared error lower bounded as

$$\alpha(\hat{\theta}) \geq \frac{c}{k},$$

for all $k \geq k_0$, where $k_0$ is a constant that depends only on the constant $B$.

(b) Let $A$ be any finite constant such that $A > B$. There exists a constant $c > 0$ that depends only on the constants $A$ and $B$, such that

$$\alpha(\hat{\theta}^{(A)}) \leq \frac{c}{k}.$$

Theorem 2.2 shows that using the estimator $\hat{\theta}^{(A)}$ retains the minimax-optimality achieved by $\hat{\theta}^{(B)}$ in terms of the mean squared error. Combining Theorem 2.1 and Theorem 2.2 shows the Pareto improvement of our estimator $\hat{\theta}^{(A)}$: the estimator $\hat{\theta}^{(A)}$ decreases the rate of the bias, while still performing optimally on the mean squared error.

The proof of Theorem 2.2 closely mimics the proof of Theorem 2(b) from Shah et al. (2016), replacing the steps involving the domain $\Theta_B$ by the stretched domain $\Theta_A$. The details are provided in Appendix B.

3 Simulations

In this section, we explore our problem space and compare the standard MLE and our proposed stretched-MLE by simulations. Both estimators are solutions to convex optimization problems, so we use the CVXPY package for ease of implementation (for faster methods such as Minorization Maximization (MM), see Hunter, 2004; Hajek et al., 2014). In what follows, we set $B = 1$, and unless specified otherwise we set $A = 2$. 

---

**Figure 2:** Intuition on the sources of bias. (a) The estimators standard MLE $\hat{\theta}^{(B)}$, stretched-MLE $\hat{\theta}^{(A)}$ and unconstrained MLE $\hat{\theta}^{(\infty)}$ (on item 1), as a function of $\mu$ when there are $d = 2$ items. We consider $\theta^* = [B, -B]$, under which the true probability that item 1 beats item 2 is $\mu_+$. We zoom in to the region around $\mu = \mu_+$ indicated by the grey box. (b) The standard MLE $\hat{\theta}^{(B)}$ incurs a negative bias, because the estimate is required to be at most $B$. (c) The unconstrained MLE $\hat{\theta}^{(\infty)}$ incurs a positive bias by Jensen’s inequality, because the estimator function is convex on $\mu \in (0.5, 1)$. (d) Our estimator balances out the negative bias and the positive bias.

---

**Theorem 2.2.** (a) [Theorem 2(a) from Shah et al. (2016)] There exists a constant $c > 0$ that depends only on the constant $B$, such that any estimator $\hat{\theta}$ has a mean squared error lower bounded as

$$\alpha(\hat{\theta}) \geq \frac{c}{k},$$

for all $k \geq k_0$, where $k_0$ is a constant that depends only on the constant $B$.

(b) Let $A$ be any finite constant such that $A > B$. There exists a constant $c > 0$ that depends only on the constants $A$ and $B$, such that

$$\alpha(\hat{\theta}^{(A)}) \leq \frac{c}{k}.$$
and \( \theta^* = [1, -1, 1, -1, \ldots, -1] \). We also evaluate the performance of other values of \( \theta^* \) subsequently. Error bars in all the plots represent the standard error of the mean.

(i) **Dependence on \( d \):** We vary the number of items \( d \), while fixing \( k = 5 \). The results are shown in Fig. 3. Observe that the stretched-MLE has a significantly smaller bias, and performs on par with the MLE in terms of the mean squared error when \( d \) is large. Moreover, the simulations also suggest the rate of bias as of order \( \frac{1}{\sqrt{d}} \) for the MLE and \( \frac{1}{d} \) for the stretched-MLE, as predicted by our theoretical results.

(ii) **Dependence on \( k \):** We vary the number of comparisons \( k \) per pair of items, while fixing \( d = 10 \). The results are shown in Fig. 4. As in the simulation (i) with varying \( d \), we observe that the stretched-MLE has a significantly smaller bias, and performs on par with the MLE in terms of the mean squared error. Moreover, the simulations also suggest the rate of bias as of order \( \frac{1}{\sqrt{k}} \) for the MLE and \( \frac{1}{k} \) for the stretched-MLE, as predicted by our theoretical results.

(iii) **Different settings of the true parameter \( \theta^* \):**

Our theoretical result considers the worst-case bias and accuracy. In this simulation, we empirically compare the performance of the stretched-MLE under different settings of the true parameter vector \( \theta^* \) (recall that setting \( A = 1 \) is equivalent to the standard MLE). Specifically, we consider the following values of \( \theta^* \):

- **Worst case:** \( \theta^* = [1, -1, \ldots, -1] \).
- **Worst case \( (0.5) \):** \( \theta^* = [0.5, -0.5, \ldots, -0.5] \).
- **Bipolar:** half of the values are 1, and the other half are \(-1\).
- **Linear:** the values are equally spaced in the interval \([-1,1]\).
- **All zeros:** all parameters are 0.

We fix \( d = 10 \) and \( k = 5 \), varying \( A \in [0.5, 3] \) under different settings of the true parameter vector \( \theta^* \). The results are shown in Fig. 5. Two high-level takeaways from the empirical evaluations are that the bias generally reduces with an increase in \( A \) till past \( B \), and that the mean squared error remains relatively constant beyond \( A = 1 \) in the plotted range. In more detail, for the bias, we observe that the performance primarily depends on the largest magnitude of the items (that is, \( ||\theta^*||_\infty \)). For the settings worst case, bipolar and linear (where \( ||\theta^*||_\infty = 1 \)), the bias keeps decreasing when \( A \) is past \( B = 1 \). For the setting worst-case \( (0.5) \) (where \( ||\theta^*||_\infty = 0.5 \)), the bias keeps decreasing when \( A \) is past 0.5. This makes sense since in this case we effectively have \( B = 0.5 \) (although the algorithm would not know this in practice). The bias for the setting all zeros stays small across values of \( A \). For the mean squared error, the increase when \( A \) is past 1 is relatively small under most of the settings of the true parameter vector \( \theta^* \). The bipolar setting has the largest increase in the mean squared error. Under this setting, all parameters \( \theta^*_i \) take values at the boundaries \pm B, and therefore the estimates of all parameters are
affected by the box constraint.

(iv) Sparse observations: So far we have considered a league format where \( k \) comparisons are observed between any pair of items. Now we consider a random-design setup, where \( k \) comparisons are observed between any pair of items independently with probability \( p_{\text{obs}} \in (0, 1) \), and none otherwise (Negahban et al., 2016; Chen et al., 2019). In our simulations, we set \( p_{\text{obs}} = \frac{1}{\sqrt{d}} \) and \( k = 5 \). We discard an iteration if the graph is not connected, since the problem is not identifiable under such a graph. The results are shown in Fig. 6. We observe that the stretched-MLE continues to outperform MLE in terms of bias, and perform on par in terms of the mean squared error.

4 Conclusions and discussions

In this work, we show that the widely-used MLE is suboptimal in terms of bias, and propose a class of estimators called the “stretched-MLE”, which provably reduces the bias while maintaining the minimax-optimality in terms of accuracy. These results on the performance of the MLE and the stretched-MLE are of both theoretical and practical interest. From the theoretical point of view, our analysis and proofs provide insights on the cause of the bias, explain why stretching the box alleviates this cause, and prove theoretical guarantees in bias reduction by stretching the box. Our results on the benefits of the stretched-MLE thus suggest theoreticians to consider the stretched-MLE for analysis instead of the standard MLE.

From the practical point of view, when the constant \( B \) is unknown, practitioners often estimate the value of \( B \) by fitting the data or from past experience. Our results thus suggest that one should estimate \( B \) leniently, as an estimation smaller than or equal to the true \( B \) causes significant bias. Moreover, our proposed estimator is a simple modification to the MLE, which can be incorporated into any existing implementation at ease.

Our results lead to several open problems. First, it is of interest to extend our theoretical analysis to settings with sparse observations. For example, one may consider a random-design setup, where \( k \) comparisons are observed between any pair independently with probability \( p_{\text{obs}} = \frac{1}{\sqrt{d}} \) and none otherwise (Negahban et al., 2016; Chen et al., 2019) (also see simulation (iv) in Section 3). In terms of the bias under this random-design setup, we think that the lower-bound for MLE and the upper-bound for our stretched-MLE depend on \( d \) and \( k \) also as \( \Omega(\frac{1}{\sqrt{dK}}) \) and \( \tilde{\Omega}(\frac{1}{dK}) \) respectively; we think that the dependence of the stretched-MLE on \( p_{\text{obs}} \) is no worse than that of the standard MLE. Second, it is of interest to extend our results to other parametric models such as the Thurstone model (Thurstone, 1927), and other notions of fairness. Finally, in applications where the estimated parameters need to lie in a certain range (such as exam scores in between 0 and 100), it is of interest to consider how to map the estimates by the stretched-MLE to the required range.
Acknowledgements

The work of JW and NBS was supported in part by NSF grants 1755656 and 1763734. The work of RR was supported in part by U. S. Office of Naval Research award N00014-18-1-2099.

References


