A Proof of Theorem 4.3, lower bound in $\Omega(dt_{\mathsf{mix}})$.

Let us recall the construction of Wolfer and Kontorovich (2019). Taking $0 < \varepsilon \leq 1/8$ and $d = 6k$, $k \geq 2$ fixed, $0 < \eta < 1/48$ and $\boldsymbol{\tau} \in \{0,1\}^{d/3}$, we define the block matrix

$$\boldsymbol{M}_{\eta,\boldsymbol{\tau}} = \begin{pmatrix} C_\eta & R_{\boldsymbol{\tau}} \\ R_{\boldsymbol{\tau}}^\intercal & L_{\boldsymbol{\tau}} \end{pmatrix},$$

where $C_\eta \in \mathbb{R}^{d/3 \times d/3}$, $L_{\boldsymbol{\tau}} \in \mathbb{R}^{2d/3 \times 2d/3}$, and $R_{\boldsymbol{\tau}} \in \mathbb{R}^{d/3 \times 2d/3}$ are given by

$$L_{\boldsymbol{\tau}} = \frac{1}{8} \operatorname{diag}\left(7 - 4\tau_1 \varepsilon, 7 + 4\tau_1 \varepsilon, \dots, 7 - 4\tau_{d/3}\varepsilon, 7 + 4\tau_{d/3}\varepsilon\right),$$

$$C_\eta = \begin{pmatrix} \frac{3}{4} - \eta & \frac{\eta}{d/3-1} & \cdots & \frac{\eta}{d/3-1} \\ \frac{\eta}{d/3-1} & \frac{3}{4} - \eta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\eta}{d/3-1} \\ \frac{\eta}{d/3-1} & \cdots & \frac{\eta}{d/3-1} & \frac{3}{4} - \eta \end{pmatrix},$$

$$R_{\boldsymbol{\tau}} = \frac{1}{8} \begin{pmatrix} 1 + 4\tau_1\varepsilon & 1 - 4\tau_1\varepsilon & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 + 4\tau_2\varepsilon & 1 - 4\tau_2\varepsilon & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 + 4\tau_{d/3}\varepsilon & 1 - 4\tau_{d/3}\varepsilon \end{pmatrix}.$$

Holding η fixed, define the collection

$$\mathcal{H}_\eta = \left\{ \boldsymbol{M}_{\eta,\boldsymbol{\tau}} : \boldsymbol{\tau} \in \{0,1\}^{d/3} \right\} \tag{A.1}$$

of ergodic and symmetric stochastic matrices. Suppose that $\boldsymbol{X} = (X_1, \dots, X_m) \sim (\boldsymbol{M}, \boldsymbol{\mu})$, where $\boldsymbol{M} \in \mathcal{H}_\eta$, and $\boldsymbol{\mu}$ is the uniform distribution over the inner clique nodes, indexed by $\{1, \dots d/3\}$. Define the random variable $T_{\text{CLIQ}/2}$ to be the first time some half of the states in the inner clique were visited,

$$T_{\text{CLIQ}/2} = \inf \left\{ t \geq 1 : \left| \{X_1, \dots, X_t\} \cap [d/3] \right| = d/6 \right\}. \tag{A.2}$$

Lemma B.5 lower bounds the half cover time:

$$m \leq \frac{d}{120\eta} \implies \mathbf{P}\left(T_{\text{CLIQ}/2} > m\right) \geq \frac{1}{5}, \tag{A.3}$$

while Wolfer and Kontorovich (2019, Lemma 6) establishes the key property that any element $\boldsymbol{M}$ of $\mathcal{H}_\eta$ satisfies

$$t_{\mathsf{mix}}(\boldsymbol{M}) = \tilde{\Theta}(1/\eta). \tag{A.4}$$

Let us fix some $i_\star \in [d]$, choose as reference $\overline{\boldsymbol{M}} \doteq \boldsymbol{M}_{\eta,\boldsymbol{0}}$ and as an alternative hypothesis $\boldsymbol{M} \doteq \boldsymbol{M}_{\eta,\boldsymbol{\tau}}$, with $\tau_i = \mathbf{1}\{i = i_\star\}$. Take both chains to have the uniform distribution $\boldsymbol{\mu}$ over the clique nodes as their initial one. It is easily verified that $\left|\!\left|\!\left| \overline{\boldsymbol{M}} - \boldsymbol{M} \right|\!\right|\!\right| = \varepsilon$, so that

$$\mathcal{R}_m \geq \inf_{\mathcal{T}} \left[\mathbf{P}_0\left(\mathcal{T} = 1 | T_{\text{CLIQ}/2} > m\right) \mathbf{P}_0\left(T_{\text{CLIQ}/2} > m\right) + \mathbf{P}_1\left(\mathcal{T} = 0 | T_{\text{CLIQ}/2} > m\right) \mathbf{P}_1\left(T_{\text{CLIQ}/2} > m\right) \right]. \tag{A.5}$$

Further, for $m < \frac{d}{120\eta}$, we have

$$\mathcal{R}_m \geq \frac{1}{5} \inf_{\mathcal{T}} \left[\mathbf{P}_0\left(\mathcal{T} = 1 | T_{\text{CLIQ}/2} > m\right) + \mathbf{P}_1\left(\mathcal{T} = 0 | T_{\text{CLIQ}/2} > m\right) \right]. \tag{A.6}$$

Since $\mathbf{P}(X|Y) \geq \mathbf{P}(X|Y,Z)\,\mathbf{P}(Z|Y)$, we have

$$\mathbf{P}_0\left(\mathcal{T} = 1 | T_{\text{CLIQ}/2} > m\right) \geq \mathbf{P}_0\left(\mathcal{T} = 1 | T_{\text{CLIQ}/2} > m, N_{i_\star} = 0\right) \mathbf{P}_0\left(N_{i_\star} = 0 | T_{\text{CLIQ}/2} > m\right). \tag{A.7}$$

Additionally, the symmetry of our reference chain implies that $\mathbf{P}_0\left(N_{i_\star} = 0 | T_{\mathrm{CLIQ}/2} > m\right) \geq 1/2$. It follows, via an analogous argument that $\mathbf{P}_1\left(N_{i_\star} = 0 | T_{\mathrm{CLIQ}/2} > m\right) \geq 1/2$, so that

$$\mathcal{R}_m \geq \frac{1}{10} \inf_{\mathcal{T}} \left[\mathbf{P}_0\left(\mathcal{T} = 1 | T_{\mathrm{CLIQ}/2} > m, N_{i_\star} = 0\right) + \mathbf{P}_1\left(\mathcal{T} = 0 | T_{\mathrm{CLIQ}/2} > m, N_{i_\star} = 0\right)\right]. \tag{A.8}$$

By Le Cam's theorem (Le Cam, 2012, Chapter 16, Section 4),

$$\mathcal{R}_m \geq \frac{1}{10}\left[1 - \left\| \mathbf{P}_0\left(\boldsymbol{X} | T_{\mathrm{CLIQ}/2} > m, N_{i_\star} = 0\right) - \mathbf{P}_1\left(\boldsymbol{X} | T_{\mathrm{CLIQ}/2} > m, N_{i_\star} = 0\right)\right\|_{\mathsf{TV}}\right]. \tag{A.9}$$

Other than state $i_\star$ and its connected outer nodes, the reference chain $\boldsymbol{M}_{\eta,\mathbf{0}}$ and the alternative chain $\boldsymbol{M}_{\eta,\boldsymbol{\tau}}$ are identical. Conditional on $N_{i_\star} = 0$, the outer states connected to $i_\star$ were never visited, since these are only connected to the rest of the chain via $i_\star$ and our choice of the initial distribution $\boldsymbol{\mu}$ constrains the initial state to the inner clique. Thus, the two distributions over sequences conditioned on $N_{i_\star} = 0$ are identical, causing the term $\|\mathbf{P}_0(\cdot) - \mathbf{P}_1(\cdot)\|_{\mathsf{TV}}$ in (A.9) to vanish:

$$\mathcal{R}_m \geq \frac{1}{10}, \tag{A.10}$$

which proves a sample complexity lower bound of $\tilde{\Omega}(d t_{\mathsf{mix}})$. Since our family of Markov chains has uniform stationary distribution ($\pi_\star = 1/d$), this further proves that the dependence on $\pi_\star$ in our bound is in general not improvable. □

B Auxiliary lemmas

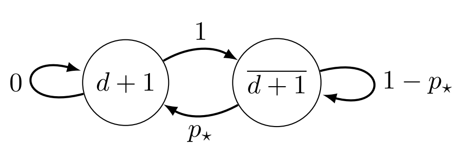
The following standard combinatorial fact will be useful.

Lemma B.1 *Let $(m, n) \in \mathbb{N}^2$ such that $m + 1 \geq 2n$. Then there are $\binom{m-n+1}{n}$ ways of selecting n non-consecutive integers from $[m]$.*

Lemma B.2 *For any $\boldsymbol{M} \in \mathcal{G}_{p_\star}$ defined in (6.3), started with initial distribution $\boldsymbol{p}$ defined in (6.5),*

$$P(m, n, p_\star) \doteq \mathbf{P}_{\boldsymbol{M},\boldsymbol{p}}\left(N_{d+1} = n\right) = \begin{cases} (1-p_\star)^n & \text{if } n = 0 \\ p_\star^n (1-p_\star)^{m-2n}\left[\binom{m-n+1}{n} - \binom{m-n}{n-1} p_\star\right] & \text{if } 1 \leq n \leq \frac{m+1}{2} \\ 0 & \text{if } n > \frac{m+1}{2} \end{cases}.$$

Proof: For any $\boldsymbol{M} \in \mathcal{G}_{p_\star}$, we construct the following associated two-state Markov chain, with initial distribution $(1 - \boldsymbol{p}_\star, \boldsymbol{p}_\star)$, where all states $i \in [d]$ are merged into a single state, which we call $\overline{d+1}$, while state $d+1$ is kept distinct. Observe that this two-state Markov chain is the same for all $\boldsymbol{M} \in \mathcal{G}_{p_\star}$, regardless of $\boldsymbol{\eta}$, and that the probability distribution of the number of visits to state $d+1$, when sampling from $\boldsymbol{M}$, is the same as when sampling from this newly constructed chain.



Let $m \geq 1$. The case where $n = 0$ is trivial as it corresponds to n failures to reach the state $d+1$, and there is only one such path. When $n > \frac{m+1}{2}$, there is no path of length m that contains n visits to state $d+1$, as any visit to this state almost surely cannot be directly followed by another visit to this same state. It remains to analyze the final case where $1 \leq n \leq \frac{m+1}{2}$. Take $(x_1, \ldots, x_m)$ to be a sample path in which the state $d+1$ was visited n times. We consider two sub-cases.

The last state in the sample path is $d+1$: In the case where $x_m = d+1$, note that also necessarily $x_{m-1} = \overline{d+1}$. The $n-1$ previous visits to state $d+1$ were followed by a probability 1 transition to state $\overline{d+1}$, and the remaining transitions have value $1 - p_\star$, so that $\mathbf{P}\left(\boldsymbol{X} = \boldsymbol{x}\right) = p_\star^{n-1} 1^{n-1} (1-p_\star)^{m-(n-1)-(n-1)-1} p_\star =$

$p_\star^n(1-p_\star)^{m-2n+1}$. Since the last two states in the sequence are fixed and known, $x_m = d+1$, $x_{m-1} = \overline{d+1}$, counting the number of such paths amounts to counting the number of subsets of $m-2$ of size $n-1$ such that no two elements are consecutive, i.e. $\binom{(m-2)-(n-1)+1}{n-1} = \binom{m-n}{n-1}$ (Lemma B.1).

The last state in the sample path is $\overline{d+1}$: By reasoning similar to above, such paths have probability $p_\star^n(1-p_\star)^{m-2n}$. To count such paths, consider all possible subsets of m of size n such that no two elements are consecutive, and subtract the count of paths in the other case where the last state was $d+1$. There are then $\binom{m-n+1}{n} - \binom{m-n}{n-1} = \binom{m-n}{n}$ such paths.

It follows that

$$\begin{aligned} P(m,n,p_\star) &= \binom{m-n}{n-1} p_\star^n (1-p_\star)^{m-2n+1} + \binom{m-n}{n} p_\star^n (1-p_\star)^{m-2n} \\ &= p_\star^n (1-p_\star)^{m-2n} \left[\binom{m-n+1}{n} - \binom{m-n}{n-1} p_\star \right]. \end{aligned} \tag{B.1}$$

□

Lemma B.3 *Let $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \mathcal{G}_{p_\star}$, defined in* (6.3)*, and start both chains with initial distribution $\boldsymbol{p}$ defined in* (6.5)*. For arbitrary $(m,n) \in \mathbb{N}^2$ such that $m+1 \geq 2n$, let $N_{d+1} = \sum_{t=1}^m \mathbf{1}\{X_t = d+1\}$ be the number of visits to state $(d+1)$. Then, for trajectories $\boldsymbol{X} = (X_1, \dots, X_m)$ sampled from either chain, we have*

$$\begin{aligned} &\left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} \leq n) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} \leq n) \right\|_{\mathsf{TV}} \\ &\leq \left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = n) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} = n) \right\|_{\mathsf{TV}}. \end{aligned} \tag{B.2}$$

Proof: Partitioning over all possible number of visits to $d+1$ for $\boldsymbol{M}_1$,

$$\begin{aligned} \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} \leq n) &= \sum_{k=0}^{\infty} \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} \leq n, N_{d+1} = k)\, \mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} = k \mid N_{d+1} \leq n) \\ &= \sum_{k=0}^{n} \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = k)\, \mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} = k \mid N_{d+1} \leq n) \\ &= \sum_{k=0}^{n} \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = k) \frac{\overbrace{\mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} \leq n \mid N_{d+1} = k)}^{=1} \mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} = k)}{\mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} \leq n)} \end{aligned} \tag{B.3}$$

From Lemma B.2, we have

$$\begin{aligned} \mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} = k) = \mathbf{P}_{\boldsymbol{M}_2}(N_{d+1} = k) &= P(m,k,p_\star) \\ \mathbf{P}_{\boldsymbol{M}_1}(N_{d+1} \leq n) = \mathbf{P}_{\boldsymbol{M}_2}(N_{d+1} \leq n) &= \sum_{s=0}^{n} P(m,s,p_\star), \end{aligned} \tag{B.4}$$

and subsequently,

$$\begin{aligned} &\left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} \leq n) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} \leq n) \right\|_{\mathsf{TV}} \\ &= \left\| \sum_{k=0}^{n} \Big(\mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = k) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} = k) \Big) \frac{P(m,k,p_\star)}{\sum_{s=0}^n P(m,s,p_\star)} \right\|_{\mathsf{TV}} \\ &\leq \sum_{k=0}^{n} \left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = k) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} = k) \right\|_{\mathsf{TV}} \frac{P(m,k,p_\star)}{\sum_{s=0}^n P(m,s,p_\star)} \\ &\leq \max_{k \in \{0,\dots,n\}} \left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = k) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} = k) \right\|_{\mathsf{TV}} \\ &\leq \left\| \mathbf{P}_{\boldsymbol{M}_1}(\boldsymbol{X} \mid N_{d+1} = n) - \mathbf{P}_{\boldsymbol{M}_2}(\boldsymbol{X} \mid N_{d+1} = n) \right\|_{\mathsf{TV}}. \end{aligned} \tag{B.5}$$

□

The following lemma shows that for the family $\mathcal{G}_{p_\star}$ of chains constructed in (6.3), conditioned on the number of visits to state $d+1$, it is possible to control the total variation between two trajectories drawn from two chains of the class in terms of the total variation between product distributions.

Lemma B.4 *Let $\boldsymbol{M}_{\boldsymbol{\eta}_1}, \boldsymbol{M}_{\boldsymbol{\eta}_2} \in \mathcal{G}_{p_\star}$ defined in* (6.3)*, both started with initial distribution $\boldsymbol{p}$ defined in* (6.5)*. Then, for $1 \leq n \leq \frac{m+1}{2}$,*

$$\left\| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} | N_{d+1} = n \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} | N_{d+1} = n \right) \right\|_{\mathsf{TV}} \leq \left\| \boldsymbol{\eta}_1^{\otimes n} - \boldsymbol{\eta}_2^{\otimes n} \right\|_{\mathsf{TV}}. \tag{B.6}$$

Proof: Total variation and ℓ_1 norm are equal up to a conventional factor of 2,

$$\begin{aligned} &2 \left\| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} | N_{d+1} = n \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} | N_{d+1} = n \right) \right\|_{\mathsf{TV}} \\ &= \sum_{\boldsymbol{x} = (x_1, \dots, x_m) \in [d+1]^m} \left| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} | N_{d+1} = n \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} = \boldsymbol{x} | N_{d+1} = n \right) \right|. \end{aligned} \tag{B.7}$$

Notice now that

$$\begin{aligned} \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} | N_{d+1} = n \right) &= \frac{\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(N_{d+1} = n | \boldsymbol{X} = \boldsymbol{x} \right) \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right)}{\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(N_{d+1} = n \right)} \\ &= \frac{\mathbf{1} \left\{ n_{d+1} = n \right\} \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right)}{\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(N_{d+1} = n \right)}, \end{aligned} \tag{B.8}$$

and similarly for $\boldsymbol{M}_{\boldsymbol{\eta}_2}$, so that

$$\begin{aligned} &2 \left\| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} | N_{d+1} = n \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} | N_{d+1} = n \right) \right\|_{\mathsf{TV}} \\ &= \sum_{\boldsymbol{x} \in [d+1]^m} \left| \frac{\mathbf{1} \left\{ n_{d+1} = n \right\} \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right)}{\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(N_{d+1} = n \right)} - \frac{\mathbf{1} \left\{ n_{d+1} = n \right\} \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} = \boldsymbol{x} \right)}{\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(N_{d+1} = n \right)} \right|. \end{aligned} \tag{B.9}$$

Invoking Lemma B.2, write

$$P(m, n, p_\star) = \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(N_{d+1} = n \right) = \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(N_{d+1} = n \right).$$

For $1 \leq n \leq \frac{m+1}{2}$,

$$\begin{aligned} &2 \left\| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} | N_{d+1} = n \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} | N_{d+1} = n \right) \right\|_{\mathsf{TV}} \\ &= \frac{1}{P(m, n, p_\star)} \sum_{\boldsymbol{x} \in [d+1]^m} \mathbf{1} \left\{ n_{d+1} = n \right\} \left| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} = \boldsymbol{x} \right) \right| \\ &= \frac{1}{P(m, n, p_\star)} \left(\sum_{\substack{\boldsymbol{x} \in [d+1]^m \\ n_{d+1} = n \\ x_m = d+1}} \left| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} = \boldsymbol{x} \right) \right| \right. \\ &\left. + \sum_{\substack{\boldsymbol{x} \in [d+1]^m \\ n_{d+1} = n \\ x_m \neq d+1}} \left| \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}} \left(\boldsymbol{X} = \boldsymbol{x} \right) - \mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}} \left(\boldsymbol{X} = \boldsymbol{x} \right) \right| \right). \end{aligned} \tag{B.10}$$

Recall that it is impossible to visit state $d+1$ twice in a row. Computing the first sum,

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{x}\in[d+1]^m \\ n_{d+1}=n \\ x_m=d+1}} \left|\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}}\left(\boldsymbol{X}=\boldsymbol{x}\right)-\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}}\left(\boldsymbol{X}=\boldsymbol{x}\right)\right| \\
&= \sum_{\substack{S=(s_1,\dots,s_n) \\ S\subset[m] \\ s_n=m \\ i\neq j \implies |s_i-s_j|>1}} \sum_{(x_{s_1},\dots,x_{s_{n-1}})\in[d+1]^{n-1}} d^{m-2n+1} p_\star^n \left(\frac{1-p_\star}{d}\right)^{m-2n+1} \left|\prod_{k=1}^{n-1}\boldsymbol{\eta}_1(x_{s_k})-\prod_{k=1}^{n-1}\boldsymbol{\eta}_2(x_{s_k})\right| \\
&= \binom{(m-2)-(n-1)+1}{n-1} p_\star^n \left(1-p_\star\right)^{m-2n+1} \sum_{(x_{s_1},\dots,x_{s_{n-1}})\in[d]^{n-1}} \left|\prod_{k=1}^{n-1}\boldsymbol{\eta}_1(x_{s_k})-\prod_{k=1}^{n-1}\boldsymbol{\eta}_2(x_{s_k})\right| \\
&= \binom{m-n}{n-1} p_\star^n \left(1-p_\star\right)^{m-2n+1} 2\left\|\boldsymbol{\eta}_1^{\otimes n-1}-\boldsymbol{\eta}_2^{\otimes n-1}\right\|_{\mathsf{TV}},
\end{aligned} \tag{B.11}$$

where the second inequality is from Lemma B.1. Similarly for the second sum,

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{x}\in[d+1]^m \\ n_{d+1}=n \\ x_m\neq d+1}} \left|\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}}\left(\boldsymbol{X}=\boldsymbol{x}\right)-\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}}\left(\boldsymbol{X}=\boldsymbol{x}\right)\right| \\
&= \sum_{\substack{S=(s_1,\dots,s_n) \\ S\subset[m] \\ s_n\neq m \\ i\neq j \implies |s_i-s_j|>1}} \sum_{(x_{s_1},\dots,x_{s_n})\in[d+1]^{n}} d^{m-2n} p_\star^n \left(\frac{1-p_\star}{d}\right)^{m-2n} \left|\prod_{k=1}^{n}\boldsymbol{\eta}_1(x_{s_k})-\prod_{k=1}^{n}\boldsymbol{\eta}_2(x_{s_k})\right| \\
&= \left(\binom{m-n+1}{n}-\binom{m-n}{n-1}\right) p_\star^n \left(1-p_\star\right)^{m-2n} \sum_{(x_{s_1},\dots,x_{s_n})\in[d]^{n}} \left|\prod_{k=1}^{n}\boldsymbol{\eta}_1(x_{s_k})-\prod_{k=1}^{n}\boldsymbol{\eta}_2(x_{s_k})\right| \\
&= \binom{m-n}{n} p_\star^n \left(1-p_\star\right)^{m-2n} 2\left\|\boldsymbol{\eta}_1^{\otimes n}-\boldsymbol{\eta}_2^{\otimes n}\right\|_{\mathsf{TV}}.
\end{aligned} \tag{B.12}$$

Hence,

$$\begin{aligned}
&2\left\|\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_1}}\left(\boldsymbol{X}|N_{d+1}=n\right)-\mathbf{P}_{\boldsymbol{M}_{\boldsymbol{\eta}_2}}\left(\boldsymbol{X}|N_{d+1}=n\right)\right\|_{\mathsf{TV}} P(m,n,p_\star) \\
&= 2p_\star^n\left(1-p_\star\right)^{m-2n}\left[\binom{m-n}{n-1}\left(1-p_\star\right)\left\|\boldsymbol{\eta}_1^{\otimes n-1}-\boldsymbol{\eta}_2^{\otimes n-1}\right\|_{\mathsf{TV}}+\binom{m-n}{n}\left\|\boldsymbol{\eta}_1^{\otimes n}-\boldsymbol{\eta}_2^{\otimes n}\right\|_{\mathsf{TV}}\right] \\
&\leq 2p_\star^n\left\|\boldsymbol{\eta}_1^{\otimes n}-\boldsymbol{\eta}_2^{\otimes n}\right\|_{\mathsf{TV}}\left(1-p_\star\right)^{m-2n}\left[\binom{m-n}{n-1}\left(1-p_\star\right)+\binom{m-n}{n}\right] \\
&= 2\left\|\boldsymbol{\eta}_1^{\otimes n}-\boldsymbol{\eta}_2^{\otimes n}\right\|_{\mathsf{TV}} P(m,n,p_\star) \text{ (Lemma B.1).}
\end{aligned} \tag{B.13}$$

□

Lemma B.5 (Cover time) *For $\boldsymbol{M}\in\mathcal{H}_\eta$ [defined in (A.1)], the "half cover time" random variable $T_{\mathrm{CLIQ}/2}$ [defined in (A.2)] satisfies*

$$m \leq \frac{d}{120\eta} \implies \mathbf{P}\left(T_{\mathrm{CLIQ}/2} > m\right) \geq \frac{1}{5}. \tag{B.14}$$

Proof: The proof pursues a strategy similar to Wolfer and Kontorovich (2019), which is adapted to "half" rather than "full" coverings. Let $\boldsymbol{M}\in\mathcal{H}_\eta$ and $\boldsymbol{M}_I\in\mathcal{M}_{d/3}$ be such that $\boldsymbol{M}_I$ consists only in the inner clique of $\boldsymbol{M}$, and each outer rim state got absorbed into its unique inner clique neighbor:

$$\boldsymbol{M}_I = \begin{pmatrix} 1-\eta & \frac{\eta}{d/3-1} & \dots & \frac{\eta}{d/3-1} \\ \frac{\eta}{d/3-1} & 1-\eta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\eta}{d/3-1} \\ \frac{\eta}{d/3-1} & \dots & \frac{\eta}{d/3-1} & 1-\eta \end{pmatrix}.$$

By construction, it is clear that $T_{\text{CLIQ}/2}$ is almost surely greater than the half cover time of $\boldsymbol{M}_I$. The latter corresponds to a generalized coupon half collection time $U_{\text{COVER}/2} = 1 + \sum_{i=1}^{d/6-1} U_i$ where U_i is the time increment between the ith and the $(i+1)$th unique visited state. Formally, if $\boldsymbol{X}$ is a random walk according to $\boldsymbol{M}_I$ (started from any state), then $U_1 = \min\{t > 1 : X_t \neq X_1\}$ and for $i > 1$,

$$U_i = \min\{t > 1 : X_t \notin \{X_1, \dots, X_{U_{i-1}}\}\} - U_{i-1}. \tag{B.15}$$

The random variables $U_1, U_2, \dots, U_{d/6-1}$ are independent and $U_i \sim \text{Geometric}\left(\eta - \frac{(i-1)\eta}{d/3}\right)$, whence

$$\mathbf{E}\left[U_i\right] = \frac{d/3}{\eta(d/3 - i + 1)}, \qquad \mathbf{Var}\left[U_i\right] = \frac{1 - \left(\eta - \frac{(i-1)\eta}{d/3}\right)}{\left(\eta - \frac{(i-1)\eta}{d/3}\right)^2} \tag{B.16}$$

and

$$\mathbf{E}\left[U_{\text{COVER}/2}\right] \geq 1 + \frac{d/3}{\eta}(\sigma_{d/3} - \sigma_{d/6}), \qquad \mathbf{Var}\left[U_{\text{COVER}/2}\right] \leq \frac{(d/3)^2}{\eta^2}\frac{\pi^2}{6}, \tag{B.17}$$

where $\sigma_d = \sum_{i=1}^d \frac{1}{i}$, and $\pi = 3.1416\dots$. Since $\ln(d+1) \leq \sigma_d \leq 1 + \ln d$, and for $d = 6k, k \geq 2$, we have $\sigma_d - \sigma_{d/2} \geq \ln 2$ it follows that

$$\mathbf{E}\left[U_{\text{COVER}/2}\right] \geq \frac{d}{\eta}\frac{\ln 2}{3}, \qquad \mathbf{Var}\left[U_{\text{COVER}/2}\right] \leq \frac{d^2}{\eta^2}\frac{\pi^2}{54}. \tag{B.18}$$

Invoking the Paley-Zygmund inequality with $\theta = 1 - \frac{\sqrt{15}}{6\ln 2}$, yields

$$\mathbf{P}\left(U_{\text{COVER}/2} > \theta\mathbf{E}\left[U_{\text{COVER}/2}\right]\right) \geq \left(1 + \frac{\mathbf{Var}\left[U_{\text{COVER}/2}\right]}{(1-\theta)^2(\mathbf{E}\left[U_{\text{COVER}/2}\right])^2}\right)^{-1} \geq \frac{1}{5}, \tag{B.19}$$

so that for $m \leq \frac{d}{120\eta}$ we have $\mathbf{P}\left(T_{\text{CLIQ}/2} > m\right) \geq \frac{1}{5}$. □