Supplementary Materials for “Auditing ML Models for Individual Bias and Unfairness”

A Proofs

A.1 Proof of Proposition in Section 2

Proof of Proposition 2.2. For the simplicity of notations, we drop the subscript of the loss function picked by the auditor, that is, we denote $\ell_h$ by $\ell$. Furthermore, let

$$\ell_x^\ast(z) = \ell_x^\ast(x,y) \triangleq \sup_{x \in \mathcal{X}} \{ \ell(x,y) - \lambda c((x,y),(x_2,y)) \}.$$  

By the duality result of Blanchet and Murthy (2019), for any $\varepsilon > 0$, we have

$$\sup_{P:W(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] = \inf_{\lambda \geq 0} \{ \lambda \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] \}$$

and

$$\sup_{P:W_\ast(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] = \inf_{\lambda \geq 0} \{ \lambda \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] \}.$$  

Let $\lambda_\ast \in \text{arg min}_{\lambda \geq 0} \{ \lambda \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] \}$. Then we have

$$\sup_{P:W(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] - \sup_{P:W_\ast(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] = \inf_{\lambda \geq 0} \{ \lambda \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] \} - \lambda_\ast \varepsilon - \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)]$$

$$\leq \lambda_\ast \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] - \lambda_\ast \varepsilon - \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)]$$

$$= \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z) - \ell_x^\ast(Z)].$$

By Assumption A3, we have

$$\ell_x^\ast(z) - \ell_x^\ast(z) = \sup_{x \in \mathcal{X}} \{ \ell(x_2,y) - \lambda_\ast c((x,y),(x_2,y)) \} - \sup_{x \in \mathcal{X}} \{ \ell(x_2,y) - \lambda_\ast c((x,y),(x_2,y)) \}$$

$$\leq \lambda_\ast \sup_{x \in \mathcal{X}} |c((x,y),(x_2,y)) - c((x,y),(x_2,y))|$$

$$\leq \lambda_\ast \eta D^2.$$  

Thus, we conclude that

$$\sup_{P:W(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] - \sup_{P:W_\ast(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] \leq \lambda_\ast \eta D^2.$$

Similarly, we have

$$\sup_{P:W_\ast(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] - \sup_{P:W(P,P_n) \leq \varepsilon} \mathbb{E}_{Z \sim P}[\ell(Z)] \leq \lambda_\ast \eta D^2,$$

where $\lambda_\ast \in \text{arg min}_{\lambda \geq 0} \{ \lambda \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z)] \}$.  

Now, it suffices to show that $\lambda_\ast \leq \frac{\lambda_\ast \varepsilon}{\sqrt{\varepsilon}}$ (and similarly $\lambda_\ast \leq \frac{\lambda_\ast \varepsilon}{\sqrt{\varepsilon}}$). By the optimality of $\lambda_\ast$,

$$\lambda_\ast \varepsilon \leq \lambda_\ast \varepsilon + \mathbb{E}_{Z \sim P_n}[\sup_{x \in \mathcal{X}} \{ \ell(x_2,Y) - \lambda_\ast d_{x_2}^2(X,x_2) \} - \ell(X,Y)]$$

$$= \lambda_\ast \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z) - \ell(Z)]$$

$$\leq \lambda_\ast \varepsilon + \mathbb{E}_{Z \sim P_n}[\ell_x^\ast(Z) - \ell(Z)]$$

$$= \lambda_\ast \varepsilon + \mathbb{E}_{Z \sim P_n}[\sup_{x \in \mathcal{X}} \{ \ell(x_2,Y) - \ell(X,Y) - \lambda d_{x_2}^2(X,x_2) \}].$$
for any $\lambda \geq 0$. By Assumption A2, the right-hand side is at most
\[
\lambda \varepsilon \leq \varepsilon + \mathbb{E}_{Z \sim P_n} \left[ \sup_{x_2 \in \mathbb{X}} \left\{ Ld_{x_2}(X, x_2) - \lambda d_{x_2}^2(X, x_2) \right\} \right] \\
\leq \varepsilon + \sup_{t \geq 0} \{ L t - \lambda t^2 \}.
\]
We minimize the right-hand side with respect to $t$ (set $t = \frac{L}{2\lambda}$) and $\lambda$ (set $\lambda = \frac{L}{2\sqrt{\varepsilon}}$) to obtain $\lambda \varepsilon \leq L\sqrt{\varepsilon}$, or equivalently $\lambda \leq \frac{L}{\sqrt{\varepsilon}}$.

A.2 Proofs of Theorems in Section 3

Proof of Theorem 3.1. We are working with Euclidean space $\mathbb{D} = \mathbb{R}^K$ and $\mathbb{E} = \mathbb{R}$.

By Theorem 3.4, $\psi : \mathbb{R}^K \to \mathbb{R}$ is Hadamard directionally differentiable at $f_*$ (tangentially to $\mathbb{R}^K$).

Since $f_n$ is the empirical version of $f_*$, by central limit theorem, we have
\[
\sqrt{n} (f_n - f_*) \overset{d}{\to} N(0, \Sigma(f_*)) \overset{d}{\sim} Z,
\]
which is tight and supported in $\mathbb{R}^K$.

Via delta method (Theorem 3.3) with $\psi(\cdot)$ and the derivative formula given by Theorem 3.4, we conclude
\[
\sqrt{n} (\psi(f_n) - \psi(f_*)) \overset{d}{\to} \psi'(f_*) (Z) = \inf \{ (\lambda + l)^T Z : (\nu, \mu, \lambda) \in \Lambda \}.
\]
Hence we complete the proof of Theorem 3.1. 

The next theorem adapted from from Bonnans and Shapiro (2000) will turn out to be useful.

Theorem A.1 (Proposition 4.27 in Bonnans and Shapiro (2000)). $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{V}$ are Banach spaces. $f : \mathbb{A} \to \mathbb{R}$ is continuously differentiable. $G + \bullet : \mathbb{A} \times \mathbb{V} \to \mathbb{B}$ is continuously differentiable. $\mathbb{K}$ is a closed convex subset of $\mathbb{B}$. Consider a class of problems
\[
(P_v) : \min_{x \in \mathbb{A}} f(x) \\
\text{subject to} \quad G(x) + v \in \mathbb{K}
\]
parameterized by $v \in \mathbb{V}$. Let $\varphi(v)$ be the optimal value of the problem $P_v$. Suppose that

1. for $v = 0$, the problem $P_0$ is convex;
2. $\varphi(0)$ is finite;
3. $0 \in \text{int}\{G(\mathbb{A}) - \mathbb{K}\}$.

Then the optimal value function $\varphi(v)$ is Hadamard directionally differentiable at $v = 0$. Furthermore,
\[
\lim_{h' \to h, t \to 0^+} \frac{\varphi(t h') - \varphi(0)}{t} = \sup\{ \lambda^T h : \lambda \in \Gamma \}
\]
for any $h \in \mathbb{V}$, where $\Gamma$ is the set of optimal solutions of the dual problem of $P_0$.

Proof of Theorem 3.4. We first prove the theorem without constraint $(D, \Pi) = 0$. In order to employ Theorem A.1, the result of canonical perturbation, we introduce a parameter $t \in \mathbb{R}$, and the optimization problem $\psi(f_*)$ can be equivalently rewritten as
\[
(P1) : \max_{t \in \mathbb{R}, \Pi \in \mathbb{R}^K_{+} \times \mathbb{K}} \left\{ t^T (\Pi^T \mathbf{1}_K - f_*) + t \right\} \\
\text{subject to} \quad \langle C, \Pi \rangle \leq \varepsilon : \nu \\
\Pi K = f_* : \lambda \\
t = 0 : \eta
\]
Thus, we have

\[ Specifically, the dual problem of \( \nabla = \nabla \) where \( \nabla \) is the set of optimal solutions of the dual problem of (P1).

Now applying Theorem (iii) which implies that (ii) Item 2 in Theorem (iii) optimization problem.

(i) To check item 1 in Theorem A.1, we note that \( Q_{0,0,K} \) is a problem of linear programming, and thus a convex optimization problem.

(ii) Item 2 in Theorem A.1 is guaranteed by

\[ \varepsilon \geq 0 = \min\{ (C, \Pi) : \Pi \in \mathbb{R}^{K+K}, \Pi_1 = f_* \}, \]

which implies that \( Q_{0,0,K} \) has a solution, and thus \( \phi(0,0,K) \) is finite.

(iii) \( f_* \in \mathbb{R}^K \) ensures that item 3 in Theorem A.1 holds.

Now applying Theorem A.1 to \( Q_{u,v,w} \), we conclude that \( \phi \) is Hadamard directionally differentiable at the origin. Note that \( \varphi = -\phi \), we can further conclude that \( \varphi \) is also Hadamard directionally differentiable at the origin, and

\[ \lim_{\xi \to 0^+} \frac{\varphi(0, t\xi') - \varphi(0, 0_{K+1})}{t} = - \lim_{\xi \to 0^+} \frac{\phi(0, t\xi') - \phi(0, 0_{K+1})}{t} = - \sup \{ \langle \lambda^\top, w \rangle^\top, \xi \} : (\nu, \lambda, w) \in \Gamma \}, \]

where \( \Gamma \) is the set of optimal solutions of the dual problem of (P1).

Furthermore, one can check that \( \Gamma = \Lambda \times \{-1\} \), where \( \Lambda \) is the set of optimal solutions of the dual problem of \( \psi(f_*) \).

Specifically, the dual problem of \( \psi(f_*) \) is given by

\[ \min_{\nu \geq 0, \lambda_1, \cdots, \lambda_K} -\varphi - \sum_{k=1}^{K} f^{(k)}_* \lambda_k \]

subject to \( c_{ij} \nu + \lambda_i \leq -l_j, \text{ for } 1 \leq i, j \leq K. \)

Thus, we have

\[ \Lambda = \arg \max_{\nu \geq 0, \lambda \in \mathbb{R}^K} \{ \varphi + f^{(k)}_* \lambda : c_{ij} \nu + \lambda_i \leq -l_j, 1 \leq i, j \leq K \} \]
Note that $\psi(f) = \varphi(0, f - f^*, f^T(f - f^*))$, we conclude that $\psi(f)$ is Hadamard directionally differentiable at $f^*$, and the derivative formula is given by

$$
\psi_{f^*}(h) = \lim_{h'\to h, t \to 0^+} \frac{\psi(f^* + th') - \psi(f^*)}{t} \\
= \lim_{h'\to h, t \to 0^+} \frac{\varphi(0, -th', t^Th') - \varphi(0, 0_K, 0)}{t} \\
= \lim_{\xi'\to \xi, t \to 0^+} \frac{\varphi(0, \xi') - \varphi(0, 0_{K+1})}{t} [\text{where } \xi = (-h^T, t^Th)^T] \\
= -\sup\{\langle \lambda^T, w \rangle \cdot \xi : (\nu, \lambda, w) \in \Gamma\} \\
= -\sup\{\langle (\lambda^T, -1)^T, (-h^T, t^Th)^T \rangle : (\nu, \lambda) \in \Lambda\} \\
= -\sup\{-\langle \lambda + l, h \rangle : (\nu, \lambda) \in \Lambda\} \\
= \inf\{\langle \lambda + l, h \rangle : (\nu, \lambda) \in \Lambda\}.
$$

For the case with constraint $\langle D, \Pi \rangle = 0$, note that the dual problem of $\psi(f^*)$ changes slightly into

$$
\min_{\nu, \mu \geq 0, \lambda_1, \ldots, \lambda_K} -\epsilon\nu - \sum_{k=1}^K f^{(k)}_* \lambda_k \\
\text{subject to } c_{ij}\nu + d_{ij}\mu + \lambda_i \leq -l_j, \text{ for } 1 \leq i, j \leq K,
$$

and

$$
\Lambda = \arg\max_{\nu, \mu \geq 0, \lambda \in \mathbb{R}^K} \{\epsilon\nu + f_*^T \lambda : c_{ij}\nu + d_{ij}\mu + \lambda_i \leq -l_j, 1 \leq i, j \leq K\}.
$$

Hence we complete the proof of Theorem 3.4. \qed

A.3 Proofs of Theorems in Section 4

The following lemma adapted from Hong and Li (2018) provides a general recipe for the consistency of our two bootstrap strategies.

**Lemma A.2** (Theorem 3.1 in Hong and Li (2018)). Suppose $\mathcal{D}$ and $\mathcal{E}$ are Banach Spaces and $\phi : \mathcal{D}_0 \subseteq \mathcal{D} \mapsto \mathcal{E}$ is Hadamard directionally differentiable at $\theta_0$ tangentially to $\mathcal{D}_0$. Let $\bar{\theta}_n : \{X_i\}_{i=1}^n \mapsto \mathcal{D}_0$ be such that for some $r_n \uparrow \infty$, $r_n \left\{\bar{\theta}_n - \theta_0\right\} \Rightarrow \mathcal{G}_0$ in $\mathcal{D}$, where $\mathcal{G}_0$ is tight and its support is included in $\mathcal{D}_0$. Then

$$
r_n \left(\phi\left(\bar{\theta}_n\right) - \phi\left(\theta_0\right)\right) \Rightarrow \phi'_{\theta_0}(\mathcal{G}_0).
$$

Let $\mathbb{Z}^n_0 \Rightarrow \mathcal{G}_0$ satisfy regularity of measurability $^1$. Then for $\epsilon_n \to 0$, $r_n\epsilon_n \to \infty$,

$$
\hat{\phi}^*_n\left(\mathbb{Z}_n^*\right) \overset{\text{def}}{=} \phi\left(\bar{\theta}_n + \epsilon_n\mathbb{Z}_n^*\right) - \phi\left(\bar{\theta}_n\right) \epsilon_n \Rightarrow \phi'_{\theta_0}(\mathcal{G}_0).
$$

**Proof of Theorem 4.1.** Hereafter, $\mathcal{G}$ refers to $\mathcal{N}(f_*, \Sigma(f_*))$. By central limit theorem, we have

$$
\sqrt{n}\left\{f_n - f_*\right\} \Rightarrow \mathcal{G}_0 \text{ and } \sqrt{m}\left\{f_{n,m}^* - f_*\right\} \Rightarrow \mathcal{G}_0.
$$

Since $m/n \to 0$, we have

$$
\sqrt{m}\left\{f_{n,m}^* - f_*\right\} = \sqrt{m}\left\{f_{n,m}^* - f_*\right\} - \sqrt{\frac{m}{n}}\sqrt{n}\left\{f_n - f_*\right\} \Rightarrow \mathcal{G}_0.
$$

$^1 \mathbb{Z}_0^*$ is asymptotically measurable jointly in the data and the bootstrap weights; $g(\mathbb{Z}_0^*)$ is a measurable function of the bootstrap weights outer almost surely in the data for every bounded, continuous map $g : \mathcal{D} \mapsto \mathcal{E}$; $\mathcal{G}_0$ is Borel measurable and separable.
Let \( r_n = \sqrt{n}, \epsilon_n = 1/\sqrt{m} \) and \( Z_n^* = \sqrt{m}\{f_{n,m}^* - f_n\} \). Then \( \epsilon_n \to 0, r_n \epsilon_n \to \infty, \) and \( Z_n^* \sim \mathbb{G}_0 \). Applying Lemma A.2, we conclude
\[
\sqrt{m} \left\{ \psi(f_{n,m}^*) - \psi(f_n) \right\} = \frac{\psi\left( f_n + \frac{1}{\sqrt{m}} \sqrt{m}\{f_{n,m}^* - f_n\} \right) - \psi(f_n)}{1/\sqrt{m}} = \frac{\psi(f_n + \epsilon_n Z_n^*) - \psi(f_n)}{\epsilon_n} \sim \psi'_{f_n}(\mathbb{G}_0).
\]

Finally, note that \( \sqrt{m}\{\psi(f_n) - \psi(f_n)\} \sim \psi'_{f_n}(\mathbb{G}_0) \), we have
\[
\sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\left( \sqrt{m} \left\{ \psi(f_{n,m}^*) - \psi(f_n) \right\} \right) \right] - \mathbb{E} \left[ g\left( \sqrt{m} \{\psi(f_n) - \psi(f_n)\} \right) \right] \right| \leq \sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\left( \sqrt{m} \{\psi(f_{n,m}^*) - \psi(f_n)\} \right) \right] - \mathbb{E} \left[ g\psi'_{f_n}(\mathbb{G}_0) \right] \right| + \sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\psi'_{f_n}(\mathbb{G}_0) \right] - \mathbb{E} \left[ g\left( \sqrt{m} \{\psi(f_n) - \psi(f_n)\} \right) \right] \right| = o_p(1) + o_p(1) = o_p(1)
\]
by triangle inequality. Hence we complete the proof of Theorem 4.1.

**Proof of Theorem 4.2.** By central limit theorem, we have
\[
\sqrt{m}\{f_n - f_*\} \sim \mathbb{G}_0 \sim \mathcal{N}(0_k, \Sigma(f_*))
\]
As \( \epsilon \to 0, n \to \infty, \) we have
\[
T(f_n, \epsilon) \to \mathbb{R}^K \text{ and } Z_n^* = N(0, \Sigma(f_*); T) \sim \mathcal{N}(0_k, \Sigma(f_*)) \sim \mathbb{G}_0.
\]
Let \( r_n = \sqrt{n}, \epsilon_n = \epsilon, \) and \( Z_n^* = Z_n^* \). Then \( \epsilon_n \to 0, r_n \epsilon_n \to \infty, \) and \( Z_n^* \sim \mathbb{G}_0 \). Applying Lemma A.2, we conclude
\[
\epsilon^{-1}\{\psi(f_n + \epsilon Z_n^*) - \psi(f_n)\} = \frac{\psi(f_n + \epsilon_n Z_n^*) - \psi(f_n)}{\epsilon_n} \sim \psi'_{f_n}(\mathbb{G}_0).
\]

Similar to the previous proof, note that \( \sqrt{m}\{\psi(f_n) - \psi(f_n)\} \sim \psi'_{f_n}(\mathbb{G}_0) \), we have
\[
\sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\left( \epsilon^{-1}\{\psi(f_n + \epsilon Z_n^*) - \psi(f_n)\} \right) \right] - \mathbb{E} \left[ g\left( \sqrt{m}\{\psi(f_n) - \psi(f_n)\} \right) \right] \right| \leq \sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\left( \epsilon^{-1}\{\psi(f_n + \epsilon Z_n^*) - \psi(f_n)\} \right) \right] - \mathbb{E} \left[ g\psi'_{f_n}(\mathbb{G}_0) \right] \right| + \sup_{g \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ g\psi'_{f_n}(\mathbb{G}_0) \right] - \mathbb{E} \left[ g\left( \sqrt{m}\{\psi(f_n) - \psi(f_n)\} \right) \right] \right| = o_p(1) + o_p(1) = o_p(1)
\]
by triangle inequality. Hence we complete the proof of Theorem 4.2.

**Proof of Theorem 4.3.** By standard results in Politis et al. (1999), under bootstrap consistency, we have
\[
\liminf_{n \to \infty} \mathbb{P}\left( \psi(f_n) \in \left[ \psi(f_n) - \frac{c_{1-\alpha/2}}{\sqrt{n}}, \psi(f_n) - \frac{c_{\alpha/2}}{\sqrt{n}} \right] \right) = 1 - \alpha
\]
if the limiting distribution is continuous at the boundary of quantiles;
\[
\liminf_{n \to \infty} \mathbb{P}\left( \psi(f_n) \in \left[ \psi(f_n) - \frac{c_{1-\alpha/2}}{\sqrt{n}}, \psi(f_n) - \frac{c_{\alpha/2}}{\sqrt{n}} \right] \right) > 1 - \alpha
\]
if the limiting distribution is discontinuous at the boundary of quantiles.

Proof of Theorem 4.5. For any \( f \in \Delta_K \) such that \( \psi(f) \leq \delta \),

\[
\mathbb{P} \left( \sqrt{n} \psi(f_n) > \sqrt{n} \delta + c_{1-\alpha} \right) \\
= 1 - \mathbb{P} \left( \sqrt{n} \psi(f_n) \leq \sqrt{n} \delta + c_{1-\alpha} \right) \\
= 1 - \mathbb{P} \left( \sqrt{n} \left( \psi(f_n) - \psi(f) \right) \leq c_{1-\alpha} + \sqrt{n} (\delta - \psi(f)) \right) \\
\leq 1 - \mathbb{P} \left( \sqrt{n} \left( \psi(f_n) - \psi(f) \right) \leq c_{1-\alpha} \right) \\
\leq 1 - (1 - \alpha) \\
= \alpha,
\]

where \( c_{1-\alpha} \) is the \((1 - \alpha)\)-th quantile of \( \sqrt{n} \{ \psi(f_n) - \psi(f) \} \). With Bootstrap consistency,

\[
\limsup_{n \to \infty} \sup_{f \in \Delta_K: \psi(f) \leq \delta} \mathbb{P}_f \left( \sqrt{n} \psi(f_n) > \sqrt{n} \delta + c^{\ast}_{1-\alpha} \right) \\
\leq \limsup_{n \to \infty} \sup_{f \in \Delta_K: \psi(f) \leq \delta} \mathbb{P}_f \left( \sqrt{n} \psi(f_n) > \sqrt{n} \delta + c_{1-\alpha} \right) = \alpha.
\]

For any \( f \in \Delta_K \) such that \( \psi(f) > \delta \),

\[
\mathbb{P} \left( \sqrt{n} \psi(f_n) > \sqrt{n} \delta + c^{\ast}_{1-\alpha} \right) \to 1.
\]

\[\square\]

B Bootstrap methods

Algorithm 1 \( m \)-out-of-\( n \) bootstrap

1: \textbf{require:} \( m \) (rule of thumb: \( 2\sqrt{n} \)), \( B \in \mathbb{N} \)  
2: set \( S = \emptyset \)  
3: \textbf{for} \( i = 1, 2, \cdots, B \) \textbf{do:}  
4: \hspace{1em} draw \( Y^{*} \sim \text{Multinomial}(m; f_n) \)  
5: \hspace{1em} append \( \sqrt{m} \{ \psi(Y^{*}/m) - \psi(f_n) \} \) to \( S \)  
6: \textbf{end for}  
7: \textbf{output:} \( S \)

Algorithm 2 numerical derivative method

1: \textbf{require:} \( \epsilon \) (rule of thumb: \( n^{-1/4} \)), \( B \in \mathbb{N} \)  
2: set \( S = \emptyset \), \( i = 1 \)  
3: \textbf{while} \( i \leq B \) \textbf{do:}  
4: \hspace{1em} draw \( Z^{*} \sim \mathcal{N}(0_K, \Sigma(f_n)) \)  
5: \hspace{1em} if \( f_n + \epsilon Z^{*} \in \mathbb{R}^K \):  
6: \hspace{2em} append \( \epsilon^{-1} \{ \psi(f_n + \epsilon Z^{*}) - \psi(f_n) \} \) to \( S \)  
7: \hspace{2em} \( i \leftarrow i + 1 \)  
8: \hspace{1em} else:  
9: \hspace{2em} continue  
10: \textbf{output:} \( S \)