

The Nonstochastic Control Problem

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Abstract

We consider the problem of controlling an unknown linear dynamical system in the presence of (nonstochastic) adversarial perturbations and adversarial convex loss functions. In contrast to classical control, the a priori determination of an optimal controller here is hindered by the latter’s dependence on the yet unknown perturbations and costs. Instead, we measure regret against an optimal linear policy in hindsight, and give the first efficient algorithm that guarantees a sublinear regret bound, scaling as $O(T^{2/3})$, in this setting.

Keywords: Control Theory, Robust Control, Online Learning, System Identification.

1. Introduction

Classical control theory assumes that nature evolves according to well-specified dynamics that is perturbed by i.i.d. noise. While this approximation has proven very useful for controlling some real world systems, it does not allow for construction of truly robust controllers. The focus of this paper is the construction of truly robust controllers **even when the underlying system is unknown and the perturbations are adversarially chosen**. For this purpose we describe the nonstochastic control problem and study efficient algorithms to solve it for linear dynamical systems.

Specifically, we consider the case in which the underlying system is linear, but has potentially adversarial perturbations (that can model deviations from linearity), i.e.

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad (1)$$

where x_t is the (observed) dynamical state, u_t is a learner-chosen control and w_t is an adversarial disturbance. The goal of the controller is to minimize a sum of sequentially revealed adversarial cost functions $c_t(x_t, u_t)$ over the state-control pairs that it visits.

The adversarial nature of w_t prohibits an a priori computation of an optimal policy that is the hallmark of classical optimal control. Instead, we consider algorithms for online control that iteratively produce a control u_t based on previous observations. The goal in this game-theoretic setting is to minimize policy regret, or the regret compared to the best controller from a class Π , chosen

with complete foreknowledge of the system dynamics, the cost sequence, and all the disturbances:

$$\text{Regret} = \sum_{t=1}^T c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^T c_t(x_t^\pi, u_t^\pi).$$

Notice that the cost of the benchmark is measured on the counterfactual state-action sequence (x_t^π, u_t^π) that the benchmark policy in consideration visits, as opposed to the state-sequence visited by the learner. This implies instance-wise near-optimality for every perturbation sequence.

We give a formal definition of the nonstochastic control problem henceforth. Informally, non-stochastic control for linear dynamical systems (LDS) can be stated as follows:

Nonstochastic Control for LDS: Without knowledge of the underlying system A, B , or the perturbations w_t , iteratively generate controls u_t to minimize regret, over sequentially revealed adversarial convex costs $c_t(x_t, u_t)$, against the class of all linear policies.

Our main result is an efficient algorithm for the nonstochastic control problem which attains the following guarantee:

Theorem 1 (Informal Statement) *For an unknown linear dynamical system where the perturbations w_t (and convex costs c_t) are bounded and chosen by an adversary, there exists an efficient algorithm that generates an adaptive sequence of controls $\{u_t\}$ for which*

$$\text{Regret} = O(\text{poly}(\text{natural-parameters})T^{2/3}).$$

1.1. Technical Outline

Our starting point is the recent work of [Agarwal et al. \(2019a\)](#), where the authors proposed a novel class of policies choosing actions as a linear combination of past perturbations, $u_t = \sum_{i=1}^k M_i w_{t-i}$. They demonstrate that learning the coefficients M_i , via online convex optimization, allows their controller to compete with the class of all linear state-feedback policies. This latter class is important, since it is known to be optimal for the standard setting of normal i.i.d noise and quadratic loss functions, also known as the Linear Quadratic Regulator (LQR), and associated robust control settings (see [Başar and Bernhard \(2008\)](#) for examples).

The caveat in [Agarwal et al. \(2019a\)](#) is that the system matrices (A, B) need to be known. In the case of a known system, the disturbances can be simply computed via observations of the state, ie. $w_t = x_{t+1} - Ax_t - Bu_t$. However, if the system is unknown, it is not clear how to generalize their approach. Fundamentally, the important component that is difficulty in identifying the system, or the matrices A, B , from the observations. This is non-trivial since the noise is assumed to be adversarial, and was posed as a question in [Tu \(2019\)](#).

In this paper we show how to overcome this difficulty and obtain sublinear regret for controlling an unknown system in the presence of adversarial noise and adversarial loss functions. The regret notion we adopt is policy regret against linear policies, exactly as in [Agarwal et al. \(2019a\)](#). An important component that we use is *adversarial sys-id*: an efficient method for uncovering the underlying system even in the presence of adversarial perturbations. This method is **not** based on naive least squares method of regressing (x_t, u_t) on x_{t+1} . In particular, without independent, zero-mean w_t 's, the latter approach can produce inconsistent estimates of the system matrices.

1.2. Related Work

Robust Control: The classical control literature deals with adversarial perturbations in the dynamics in a framework known as H_∞ control, see e.g. [Stengel \(1994\)](#); [Zhou et al. \(1996\)](#). In this setting, the controller solves for the best linear controller assuming worst case noise to come, i.e.,

$$\min_{K_1} \max_{w_1} \min_{K_2} \dots \min_{K_T} \max_{w_T} \sum_t c_t(x_t, u_t).$$

This approach is overly pessimistic as it optimizes over the worst-case noise. In contrast, the metric in nonstochastic control (NSC) is regret, which adapts to the per-instance perturbations.

Learning to control stochastic LDS: There has been a resurgence of literature on control of linear dynamical systems in the recent machine learning venues. The case of known systems was extensively studied in the control literature, see the survey [Stengel \(1994\)](#). Sample complexity and regret bounds for control (under Gaussian noise) were obtained in [Abbasi-Yadkori and Szepesvári \(2011\)](#); [Dean et al. \(2018\)](#); [Abbasi-Yadkori et al. \(2019\)](#); [Mania et al. \(2019\)](#); [Cohen et al. \(2019\)](#). The works of [Abbasi-Yadkori et al. \(2014\)](#), [Cohen et al. \(2018\)](#) and [Agarwal et al. \(2019b\)](#) allow for control in LDS with adversarial loss functions. Provable control in the Gaussian noise setting via the policy gradient method was studied in [Fazel et al. \(2018\)](#). These works operate in the absence of perturbations or assume that the same are i.i.d., as opposed to our adversarial.

Control with adversarial perturbations: The most relevant reformulation of the control problem that enables our result is the recent work of [Agarwal et al. \(2019a\)](#), who use online learning techniques and convex relaxation to obtain provable bounds for controlling LDS with adversarial perturbations. However, the result and the algorithm make extensive use of the availability of the system matrices. [Tu \(2019\)](#) ask if the latter result can be extended to unknown systems, a question that we answer in the affirmative.

System identification. For the stochastic setting, several works ([Faradonbeh et al., 2018](#); [Simchowitz et al., 2018](#); [Sarkar and Rakhlin, 2019](#)) propose to use the least-squares procedure for parameter identification. In the adversarial setting, least-squares can lead to inconsistent estimates. For the partially observed stochastic setting, [Oymak and Ozay \(2019\)](#); [Sarkar et al. \(2019\)](#); [Simchowitz et al. \(2018\)](#) give results guaranteeing parameter recovery using Gaussian inputs. Of these, the results in [Simchowitz et al. \(2019\)](#) also apply to the adversarial setting. We offer a simpler analysis for parameter identification, and develop rigorous perturbation bounds for the control algorithm necessary to make guarantees on the quality of the control solution. Other relevant work from the machine learning literature includes spectral filtering techniques for learning and open-loop control of partially observable systems ([Hazan et al., 2017](#); [Arora et al., 2018](#); [Hazan et al., 2018](#)).

2. Problem Definition

2.1. Nontochastic Control: The General Case

Many dynamical systems admit the following dynamical description. These may be seen as discrete-time analogues of controlled diffusion processes ([Krylov, 2008](#)).

$$x_{t+1} = f(x_t, u_t) + w_t$$

Here $f(x, u)$ is a transition function, and w_t are perturbations or deviations from nominal dynamics. We consider an online control problem during the course of which a controller must iteratively choose a control input $u_t \in \mathbb{R}^n$, and suffers a loss $c_t(x_t, u_t)$, where $x_t \in \mathbb{R}^m$ is the state of the system. The controller is thereafter presented some output in the form of a resultant information set \mathcal{I}_t . For example, the controller may observe the state x_t , and cost function c_t , but the transition function f might be unknown. A policy $\pi = (\pi_1, \dots, \pi_t : \pi_t :: \mathcal{I}_t \rightarrow \mathbb{R}^n)$ is a mapping from observed information to control. We denote a set of policies by Π . We measure the performance of a control algorithm through the metric of policy regret: the difference between the aggregate cost of the controller and that of the best policy in hindsight from a certain class.

Definition 2 (Nonstochastic Control) *A Nonstochastic Control problem instance is given by tuple $(f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \{w_t \in \mathbb{R}^m\}, \{c_t : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}\}, \Pi)$, where for any sequence of controls u_1, \dots, u_T , the states are produced as $x_{t+1} = f(x_t, u_t) + w_t$. The goal of the learner is to choose an adaptive sequence of controls to minimize regret against the policy class Π , defined as:*

$$\text{Regret} = \sum_{t=1}^T c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^T c_t(x_t^\pi, u_t^\pi),$$

where (x_t^π, u_t^π) is the state-control pair visited by the benchmark policy $\pi \in \Pi$ in consideration.

We specialize the above definition to linear dynamical systems below.

2.2. Nonstochastic Control for Linear Dynamical Systems

We consider the setting of linear dynamical systems with time-invariant dynamics, i.e.

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^m$ and $u_t \in \mathbb{R}^n$. The perturbation sequence w_t may be adversarially chosen at the beginning of the interaction, and is unknown to the learner. Likewise, the system is augmented with time-varying convex cost functions $c_t(x, u)$. The total cost associated with a sequence of (random) controls, derived through an algorithm \mathcal{A} , is

$$J(\mathcal{A}) = \sum_{t=1}^T c_t(x_t, u_t).$$

With some abuse of notation, we will denote by $J(K)$ the cost associated with the execution of controls as a linear controller K would suggest, ie. $u_t = -Kx_t$. The following conditions are assumed on the cost and the perturbations ¹.

Assumption 3 *The perturbation sequence is bounded, ie. $\|w_t\| \leq W$, and chosen at the start of the interaction, implying that this sequence w_t does not depend on the choice of u_t .*

Assumption 4 *As long as $\|x_t\|, \|u_t\| \leq D$, the convex costs admit $\|\nabla_{(x,u)} c_t(x, u)\| \leq GD$.*

The fundamental Linear Quadratic Regulator problem is a specialization of the above to the case when the perturbations are i.i.d. Gaussian and the cost functions are positive quadratics, ie.

$$c_t(x, u) = x^\top Qx + u^\top Ru.$$

1. Without loss of generality we shall assume that $G, D, W, \kappa \geq 1$ holds, since these are upper bounds.

Objective We consider the setting where the learner has no knowledge of A, B and the perturbation sequence w_t . In this case, any inference of these quantities may only take place indirectly through the observation of the state x_t . Furthermore, the learner is made aware of the cost function c_t only once the choice of u_t has been made.

Under such constraints, the objective of the algorithm is to choose an (adaptive) sequence of controls that ensure that the cost suffered in this manner is comparable to that of the best choice of a linear controller with complete knowledge of system dynamics A, B and the foreknowledge of the cost and perturbation sequences $\{c_t, w_t\}$. Formally, we measure regret as

$$\text{Regret} = J(\mathcal{A}) - \min_{K \in \mathcal{K}} J(K).$$

\mathcal{K} is the set of (κ, γ) -strongly stable linear controllers defined below. The notion of strong stability, introduced in [Cohen et al. \(2018\)](#), offers a quantification of the classical notion of a stable controller in manner that permits a discussion on non-asymptotic regret bounds.

Definition 5 (Strong Stability) *A linear controller K is (κ, γ) -strongly stable for a linear dynamical system specified via (A, B) if there exists a decomposition of $A - BK = QLQ^{-1}$ with $\|L\| \leq 1 - \gamma$, and $\|A\|, \|B\|, \|K\|, \|Q\|, \|Q^{-1}\| \leq \kappa$.*

We also assume the learner has access to a fixed stabilizing controller \mathbb{K} . When operating under unknown transition matrices, the knowledge of a stabilizing controller permits the learner to prevent an inflation of the size of the state beyond reasonable bounds.

Assumption 6 *The learner knows a linear controller \mathbb{K} that is (κ, γ) -strongly stable for the true, but unknown, transition matrices (A, B) defining the dynamical system.*

The non-triviality of the regret guarantee rests on the benchmark set not being empty. As noted in [Cohen et al. \(2018\)](#), a sufficient condition to ensure the existence of a strongly stable controller is the controllability of the linear system (A, B) . Informally, controllability for a linear system is characterized by the ability to drive the system to any desired state through appropriate control inputs in the presence of deterministic dynamics, ie. $x_{t+1} = Ax_t + Bu_t$.

Definition 7 (Strong Controllability) *For a linear dynamical system (A, B) , define, for $k \geq 1$, a matrix $C_k \in \mathbb{R}^{n \times km}$ as*

$$C_k = [B, AB, A^2B \dots A^{k-1}B].$$

A linear dynamical system (A, B) is controllable with controllability index k if C_k has full row-rank. In addition, such a system is also (k, κ) -strongly controllable if $\|(C_k C_k^\top)^{-1}\| \leq \kappa$.

As with stability, a quantitative analog of controllability first suggested in [Cohen et al. \(2018\)](#) is presented above. It is useful to note that, as a consequence of the Cayley-Hamilton theorem, for a controllable system the controllability index is always at most the dimension of the state space. We adopt the assumption that the system $(A - B\mathbb{K}, B)$ is (k, κ) strongly controllable.

Assumption 8 *The linear dynamical system $(A - B\mathbb{K}, B)$ is (k, κ) -strongly controllable.*

3. Preliminaries

This section sets up the concepts that aid the algorithmic description and the analysis.

3.1. Parameterization of the Controller

The total cost objective of a linear controller is non-convex in the canonical parameterization (Fazel et al., 2018), ie. $J(K)$ is not convex in K . To remedy this, we use an alternative perturbation-based parameterization for controller, recently proposed in Agarwal et al. (2019a), where the advised control is linear in the past perturbations (as opposed to the state). This permits that the offline search for an optimal controller may be posed as a convex program.

Definition 9 A perturbation-based policy $M = (M^{[0]}, \dots, M^{[H-1]})$ chooses control u_t at state x_t ,

$$u_t = -\mathbb{K}x_t + \sum_{i=1}^H M^{[i-1]}w_{t-i}.$$

3.2. State Evolution

Under the execution of a stationary policy M , the state may be expressed as a linear transformation Ψ (defined below) of the perturbations, where the transformation Ψ 's is linear in the matrices M 's.

$$x_{t+1} = (A - B\mathbb{K})^{H+1}x_{t-H} + \sum_{i=0}^{2H} \Psi_i(M|A, B)w_{t-i}$$

Definition 10 For a matrix pair (A, B) , define the state-perturbation transfer matrix:

$$\Psi_i(M|A, B) = (A - B\mathbb{K})^i \mathbf{1}_{i \leq H} + \sum_{j=0}^H (A - B\mathbb{K})^j B M^{[i-j-1]} \mathbf{1}_{i-j \in [1, H]}.$$

Definition 11 Define the surrogate state y_{t+1} and the surrogate action v_t as stated below. The surrogate cost f_t as chosen to be the specialization of the t -th cost function with the surrogate state-action pair as the argument.

$$\begin{aligned} y_{t+1}(M|A, B, \{w\}) &= \sum_{i=0}^{2H} \Psi_i(M|A, B)w_{t-i} \\ v_t(M|A, B, \{w\}) &= -\mathbb{K}y_t(M|A, B, \{w\}) + \sum_{i=1}^H M^{[i-1]}w_{t-i} \\ f_t(M|A, B, \{w\}) &= c_t(y_t(M|A, B, \{w\}), v_t(M|A, B, \{w\})) \end{aligned}$$

4. The Algorithm

Our approach follows the explore-then-commit paradigm, identifying the underlying the deterministic-equivalent dynamics to within some accuracy using random inputs in the exploration phase. Such an approximate recovery of parameters permits an approximate recovery of the perturbations, thus facilitating the execution of the perturbation-based controller on the approximated perturbations.

Algorithm 1 Adversarial control via system identification.

Input: learning rate η , horizon H , number of iterations T , rounds of exploration T_0 .

Phase 1: System Identification.

Call Algorithm 2 with a budget of T_0 rounds to obtain system estimates \hat{A}, \hat{B} .

Phase 2: Robust Control.

Define the constraint set \mathcal{M} as $\mathcal{M} = \{M = (M^{[0]}, \dots, M^{[H-1]}) : \|M^{[i-1]}\| \leq \kappa^4(1 - \gamma)^i\}$.

Initialize $\hat{w}_{T_0} = x_{T_0+1}$ and $\hat{w}_t = 0$ for $t < T_0$.

for $t = T_0 + 1, \dots, T$ **do**

 Choose the action

$$u_t = -\mathbb{K}x_t + \sum_{i=1}^H M_t^{[i-1]} \hat{w}_{t-i}.$$

 Observe the new state x_t , the cost function $c_t(x, u)$.

 Record an estimate $\hat{w}_t = x_{t+1} - \hat{A}x_t - \hat{B}u_t$.

 Update $M_{t+1} = \Pi_{\mathcal{M}}(M_t - \eta \nabla f_t(M_t | \hat{A}, \hat{B}, \{\hat{w}\}))$.

end for

Algorithm 2 System identification via random inputs.

Input: number of iterations T_0 .

for $t = 0, \dots, T_0$ **do**

 Execute the control $u_t = -\mathbb{K}x_t + \eta_t$ with $\eta_t \sim_{i.i.d.} \{\pm 1\}^n$.

 Record the observed state x_t .

end for

Declare $N_j = \frac{1}{T_0 - k} \sum_{t=0}^{T_0 - k - 1} x_{t+j+1} \eta_t^\top$, for all $j \in [k]$.

Define $C_0 = (N_0, \dots, N_{k-1})$, $C_1 = (N_1, \dots, N_k)$, and return \hat{A}, \hat{B} as

$$\hat{B} = N_0, \quad \hat{A}' = C_1 C_0^\top (C_0 C_0^\top)^{-1}, \quad \hat{A} = \hat{A}' + \hat{B} \mathbb{K}.$$

Theorem 12 *Under the assumptions 3, 4, 8, 6, when $H = \Theta(\gamma^{-1} \log(\kappa^2 T))$, $\eta = \Theta(GW\sqrt{T})^{-1}$, $T_0 = \Theta(T^{2/3} \log \delta^{-1})$, the regret incurred by Algorithm 1 for controlling an unknown linear dynamical system admits the upper bound stated below with probability at least $1 - \delta$.*

$$\text{Regret} = O(\text{poly}(\kappa, \gamma^{-1}, k, m, n, W) T^{2/3} \log \delta^{-1})$$

5. Regret Analysis

To present the proof concisely, we set up a few articles of use. For a generic algorithm \mathcal{A} operating on a generic linear dynamical system specified via a matrix pair (A, B) and perturbations $\{w\}$, let

1. $J(\mathcal{A}|A, B, \{w\})$ be the cost of executing \mathcal{A} , as incurred on the last $T - T_0$ time steps,
2. $x_t(\mathcal{A}|A, B, \{w\})$ be the state achieved at time step t , and
3. $u_t(\mathcal{A}|A, B, \{w\})$ be the control executed at time step t .

We also note that following result from [Agarwal et al. \(2019a\)](#) that applies to the case when the matrices (A, B) that govern the underlying dynamics are made known to the algorithm.

Theorem 13 (Known System; Agarwal et al. (2019a)) *Let \mathbb{K} be a (κ, γ) -strong stable controller for a system (A, B) , $\{w_t\}$ be a (possibly adaptive) perturbation sequence with $\|w_t\| \leq W$, and c_t be costs satisfying Assumption 4. Then there exists an algorithm \mathcal{A} (Algorithm 1.2 with a learning rate of $\eta = \Theta(GW\sqrt{T})^{-1}$ and $H = \Theta(\gamma^{-1} \log(\kappa^2 T))$), utilizing $\mathbb{K}, (A, B)$, that guarantees*

$$J(\mathcal{A}|A, B, \{w\}) - \min_{K \in \mathcal{K}} J(K|A, B, \{w\}) \leq O(\text{poly}(m, n, \kappa, \gamma^{-1})GW^2\sqrt{T} \log T).$$

Proof of Theorem 12. Define $K = \operatorname{argmin}_{K \in \mathcal{K}} J(K)$. Let J_0 be the contribution to the regret associated with the first T_0 rounds of exploration. By Lemma 20, we have that $J_0 \leq 16T_0 Gn\kappa^8\gamma^{-2}W^2$.

Let \mathcal{A} refer to the algorithm, from [Agarwal et al. \(2019a\)](#), executed in Phase 2. By Lemma 14,

$$\begin{aligned} \text{Regret} &\leq J_0 + \sum_{t=T_0+1}^T c_t(x_t, u_t) - J(K|A, B, \{w\}), \\ &\leq J_0 + (J(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) - J(K|\hat{A}, \hat{B}, \{\hat{w}\})) + (J(K|\hat{A}, \hat{B}, \{\hat{w}\}) - J(K|A, B, w)). \end{aligned}$$

Let $\|A - \hat{A}\|, \|B - \hat{B}\| \leq \varepsilon_{A,B}$ and $\varepsilon_{A,B} \leq 10^{-3}\kappa^{-10}\gamma^2$ in the arguments below. The middle term above can be upper bounded by the regret of algorithm \mathcal{A} on the fictitious system (\hat{A}, \hat{B}) and the perturbation sequence $\{\hat{w}\}$. Before we can invoke Theorem 13, observe that

1. By Lemma 15, \mathbb{K} is $(2\kappa, 0.5\gamma)$ -strongly stable on (\hat{A}, \hat{B}) , as long as $\varepsilon_{A,B} \leq 0.25\kappa^{-3}\gamma$,
2. Lemma 18 ensures $\|\hat{w}_t\| \leq 2\sqrt{n}\kappa^3\gamma^{-1}W$, as long as $\varepsilon_{A,B} \leq 10^{-3}\kappa^{-10}\gamma^2$.

With the above observations in place, Theorem 13 guarantees

$$J(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) - J(K|\hat{A}, \hat{B}, \{\hat{w}\}) \leq \text{poly}(m, n, \kappa, \gamma^{-1})GW^2\sqrt{T} \log T.$$

The last expression in the preceding line can be bound by Lemma 16, since Lemma 18 also bounds $\|w_t - \hat{w}_t\| \leq 20\sqrt{n}\kappa^{11}\gamma^{-3}W\varepsilon_{A,B}$ whenever $\varepsilon_{A,B} \leq 10^{-3}\kappa^{-10}\gamma^2$ for $t \geq T_0 + 1$.

$$|J(K|\hat{A}, \hat{B}, \{\hat{w}\}) - J(K|A, B, w)| \leq 32Gn\kappa^{11}\gamma^{-3} + 20^3TGn\kappa^{22}\gamma^{-7}W^2\varepsilon_{A,B}$$

Set $\varepsilon_{A,B} = \min(10^{-3}\kappa^{-10}\gamma^2, T^{-\frac{1}{3}})$ in Theorem 19 to conclude. ■

The regret minimizing algorithm, Phase 2 of Algorithm 1, chooses M_t so as to optimize for the cost of the perturbation-based controller on a *fictitious* linear dynamical system (\hat{A}, \hat{B}) subject to the perturbation sequence $\{\hat{w}\}$. The following lemma shows that the definition of \hat{w}_t ensures that state-control sequence visited by Algorithm 1 coincides with the sequence visited by the regret-minimizing algorithm on the *fictitious* system.

Lemma 14 (Simulation Lemma) *Let \mathcal{A} be the algorithm, from [Agarwal et al. \(2019a\)](#), executed in Phase 2, and (x_t, u_t) be the state-control iterates produced by Algorithm 1. Then for $t \geq T_0 + 1$,*

$$x_t = x_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}), \quad u_t = u_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}), \quad \text{and} \quad \sum_{t=T_0+1}^T c_t(x_t, u_t) = J(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}).$$

Proof This proof follows by induction on $x_t = x_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\})$. Note that at the start of \mathcal{A} , it is fed the initial state $x_{T_0+1}(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) = x_{T_0+1}$ by the choice of \hat{w}_{T_0} . Say that for some $t \geq T_0 + 1$, it happens that the inductive hypothesis is true. Consequently,

$$\begin{aligned} u_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) &= -\mathbb{K}x_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) + \sum_{i=1}^H M_t^{[i-1]} \hat{w}_{t-i} = -\mathbb{K}x_t + \sum_{i=1}^H M_t^{[i-1]} \hat{w}_{t-i} = u_t. \\ x_{t+1}(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) &= \hat{A}x_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) + \hat{B}u_t(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) + \hat{w}_t = \hat{A}x_t + \hat{B}u_t + \hat{w}_t = x_{t+1} \end{aligned}$$

This, in turn, implies by choice of \hat{w}_t that the next states produced at the next time steps match. \blacksquare

The lemma stated below guarantees that the strong stability of \mathbb{K} is approximately preserved under small deviations of the system matrices.

Lemma 15 (Preservation of Stability) *If \mathbb{K} is (κ, γ) -strongly stable for a linear system (A, B) , ie. $A - BK = QLQ^{-1}$, then \mathbb{K} is $(\kappa + \varepsilon_{A,B}, \gamma - 2\kappa^3\varepsilon_{A,B})$ -strongly stable for (\hat{A}, \hat{B}) , ie.*

$$\hat{A} - \hat{B}\mathbb{K} = Q\hat{L}Q^{-1}, \quad \|\hat{A}\|, \|\hat{B}\| \leq \kappa + \varepsilon_{A,B}, \quad \|\hat{L}\| \leq 1 - \gamma + 2\kappa^3\varepsilon_{A,B},$$

as long as $\|A - \hat{A}\|, \|B - \hat{B}\| \leq \varepsilon_{A,B}$. Furthermore, in such case, the transforming matrices Q that certify strong stability in both these cases coincide, and it holds $\|\hat{L} - L\| \leq 2\kappa^3\varepsilon_{A,B}$.

Proof Let $A - BK = QLQ^{-1}$ with $\|Q\|, \|Q^{-1}\|, \|\mathbb{K}\|, \|A\|, \|B\| \leq \kappa, \|L\| \leq 1 - \gamma$. Now

$$\begin{aligned} \hat{A} - \hat{B}\mathbb{K} &= QLQ^{-1} + (\hat{A} - A) - (\hat{B} - B)\mathbb{K} \\ &= Q(L + Q^{-1}((\hat{A} - A) - (\hat{B} - B)\mathbb{K})Q)Q^{-1} \end{aligned}$$

It suffices to note that $\|Q^{-1}((\hat{A} - A) - (\hat{B} - B)\mathbb{K})Q\| \leq 2\kappa^3\varepsilon_{A,B}$. \blacksquare

The next lemma establishes that if the same linear state-feedback policy is executed on the actual and the *fictional* linear dynamical system, the difference between the costs incurred in the two scenarios varies proportionally with some measure of distance between the two systems.

Lemma 16 (Stability of Value Function) *Let $\|A - \hat{A}\|, \|B - \hat{B}\| \leq \varepsilon_{A,B} \leq 0.25\kappa^{-3}\gamma$, and K be any (κ, γ) -strongly stable controller with respect to (A, B) . Then, for any perturbation sequence that satisfies $\|w_t - \hat{w}_t\| \leq \varepsilon_w \leq W_0$ except possibly for the first step where $\|\hat{w}_0\| \leq W_0^2$, it holds*

$$|J(K|\hat{A}, \hat{B}, \{\hat{w}\}) - J(K|A, B, w)| \leq 10^3 TG\kappa^8\gamma^{-3}W_0(\varepsilon_W + W_0\varepsilon_{A,B}) + 32G\kappa^5\gamma^{-2}W_0^2.$$

Proof Under the action of a linear controller K , which is (κ, γ) -strongly stable for (A, B) , it holds

$$x_{t+1}(K|A, B, \{w\}) = \sum_{i=0}^T (A - BK)^i w_{t-i}.$$

2. This handles the case for \hat{w}_{T_0} in Algorithm 1. Assume that $W_0 \geq W$.

Consequently, $\|x_t(K|A, B, \{w\})\| \leq \kappa^2 \gamma^{-1} W$, and, since K is $(2\kappa, 2^{-1}\gamma)$ -strongly stable for (\hat{A}, \hat{B}) by Lemma 15, $\|x_t(K|\hat{A}, \hat{B}, \{\hat{w}\})\| \leq 16\kappa^2 \gamma^{-1} W_0$. It follows

$$|J(\mathcal{A}|\hat{A}, \hat{B}, \{\hat{w}\}) - J(\mathcal{A}|A, B, \{w\})| \leq 32G\kappa^3 \gamma^{-1} W_0 \sum_{t=0}^T \|x_{t+1}(K|A, B, \{w\}) - x_{t+1}(K|\hat{A}, \hat{B}, \{\hat{w}\})\|$$

Finally, by using strong stability of the controller, we have

$$\begin{aligned} & \|x_{t+1}(K|A, B, \{w\}) - x_{t+1}(K|\hat{A}, \hat{B}, \{\hat{w}\})\| \leq \sum_{t=0}^T \left\| (A - BK)^i w_{t-i} - (\hat{A} - \hat{B}K)^i \hat{w}_{t-i} \right\| \\ & \leq \sum_{t=0}^T \left(\left\| (A - BK)^i w_{t-i} - (A - BK)^i \hat{w}_{t-i} \right\| + \left\| (A - BK)^i \hat{w}_{t-i} - (\hat{A} - \hat{B}K)^i \hat{w}_{t-i} \right\| \right) \\ & \leq \kappa^2 (1 - \gamma)^t W_0 + \kappa^2 \gamma^{-1} \varepsilon_w + 16W_0 \kappa^5 \gamma^{-2} \varepsilon_{A,B} \end{aligned}$$

The last line follows from invocation of Lemma 17 on L, L' matrices that certify the strongly stability of K on (A, B) and (\hat{A}, \hat{B}) respectively. \blacksquare

Lemma 17 For any matrix pair $L, \Delta L$, such that $\|L\|, \|L + \Delta L\| \leq 1 - \gamma$, we have

$$\sum_{t=0}^{\infty} \|(L + \Delta L)^t - L^t\| \leq \gamma^{-2} \|\Delta L\|$$

Proof We make the inductive claim that $\|(L + \Delta L)^t - L^t\| \leq t(1 - \gamma)^{t-1} \|\Delta L\|$. The truth of this claim for $t = 0, 1$ is easily verifiable. Assuming the inductive claim for some t , observe

$$\|(L + \Delta L)^{t+1} - L^{t+1}\| \leq \|L((L + \Delta L)^t - L^t)\| + \|\Delta L(L + \Delta L)^t\| \leq (t + 1)(1 - \gamma)^t \|\Delta L\|.$$

Finally, observe that $\sum_{t=0}^{\infty} t(1 - \gamma)^{t-1} \leq \gamma^{-2}$. \blacksquare

Lemma 18 During Algorithm 1.2, it holds, as long as $\varepsilon_{A,B} \leq 10^{-3} \kappa^{-10} \gamma^2$, for any $t \geq T_0 + 1$

$$\|x_t\| \leq 4\sqrt{n}\kappa^{10} \gamma^{-3} W, \quad \|w_t - \hat{w}_t\| \leq 20\sqrt{n}\kappa^{11} \gamma^{-3} W \varepsilon_{A,B}, \quad \text{and} \quad \|\hat{w}_{t-1}\| \leq 2\sqrt{n}\kappa^3 \gamma^{-1} W.$$

Proof For a linear system evolving as $x_{t+1} = Ax_t + Bu_t + w_t$, if the control is chosen as $u_t = -\mathbb{K}x_t + \tilde{u}_t$, the following is true, where $A' = A - B\mathbb{K}$.

$$x_{t+1} = \sum_{i=0}^t (A')^{t-i} (w_i + B\tilde{u}_i). \quad (2)$$

In Phase 2, \tilde{u}_t is chosen as $\tilde{u}_t = \sum_{i=1}^H M_t^{[i-1]} \hat{w}_{t-i}$. We put forward the inductive hypothesis that for all $t \in [T_0 + 1, t_0]$, we have that $\|x_t\| \leq X := 4\sqrt{n}\kappa^{10} \gamma^{-3} W$, $\|\hat{w}_t - w_t\| \leq Y := 20\sqrt{n}\kappa^{11} \gamma^{-3} W \varepsilon_{A,B}$. If so, $\|\hat{w}_t\| \leq \max(W + Y, \|\hat{w}_{T_0}\|) = W + Y + \sqrt{n}\kappa^3 \gamma^{-1} W$ for all $t \leq t_0$, by Lemma 20. The following display completes the induction.

$$\begin{aligned} \|x_{t_0+1}\| & \leq \kappa^2 \gamma^{-1} (W + \kappa^5 \gamma^{-1} (W + Y + \sqrt{n}\kappa^3 \gamma^{-1} W)) \leq X \\ \|\hat{w}_{t_0+1} - w_{t_0+1}\| & = \|(A - \hat{A})x_{t_0+1} + (B - \hat{B})u_{t_0+1}\| \\ & \leq \varepsilon_{A,B} (X + (\kappa X + \kappa^4 \gamma^{-1} (W + Y + \sqrt{n}\kappa^3 \gamma^{-1} W))) \leq Y \end{aligned}$$

The base case be verified via computation. \blacksquare

6. System Identification via Random Inputs

This section details the guarantees afforded by the system identification procedure, which attempts to identify the deterministic-equivalent dynamics (A, B) by first identifying matrices of the form $(A')^i B$, where $A' = A - B\mathbb{K}$, and then recovering A by solving a linear system of equations.

Theorem 19 (System Recovery) *Under assumptions 3, 8, 6, when Algorithm 2 is run for T_0 steps, the output pair (\hat{A}, \hat{B}) satisfies, with probability $1 - \delta$, that $\|\hat{A} - A\|_F, \|\hat{B} - B\|_F \leq \varepsilon_{A,B}$, where*

$$T_0 = 10^3 k m n^2 \kappa^{10} \gamma^{-2} W^2 \varepsilon_{A,B}^{-2} \log(k m n \delta^{-1}).$$

Proof Observe that A' is a unique solution of the system of equations (in X) presented below.

$$A C_k = [A' B, (A')^2 B, \dots, (A')^k B] = X [B, A' B, \dots, (A')^{k-1} B] = X C_k$$

Now, if, for all j , $\|N_j - (A')^j B\| \leq \varepsilon$, it follows that $\|C_0 - C_k\|_F, \|C_1 - A C_k\|_F \leq \varepsilon \sqrt{k}$; in addition, $\|\hat{B} - B\| \leq \varepsilon$. By Lemma 22 on rows, we have

$$\|\hat{A}' - A'\|_F \leq \frac{2\varepsilon \sqrt{k m \kappa}}{\sigma_{\min}(C_k) - \varepsilon \sqrt{k}}.$$

So, setting $\varepsilon = \frac{\varepsilon_{A,B}}{10\sqrt{k m \kappa^2}}$ suffices. This may be accomplished by Lemma 21. ■

Lemma 20 *When the control inputs are chosen as $u_t = -\mathbb{K}x + \eta_t$, where $\eta_t \sim \{\pm 1\}^n$, it holds*

$$\begin{aligned} \|x_t\| &\leq \sqrt{n} \kappa^3 \gamma^{-1} W, & \|u_t\| &\leq 2\sqrt{n} \kappa^4 \gamma^{-1} W, \\ c_t(x_t, u_t) - \min_{K \in \mathcal{K}} c_t(x^K, u^K) &\leq 16 G n \kappa^8 \gamma^{-2} W^2. \end{aligned}$$

Proof In conjunction with Equation 2, the strong stability of \mathbb{K} suffices to establish this claim.

$$\|x_t\| \leq (W + \|B\| \sqrt{n}) \|Q\| \|Q^{-1}\| \sum_{i=0}^{t-1} \|L^i\| \leq (W + \kappa \sqrt{n}) \kappa^2 \sum_{i=0}^{t-1} (1 - \gamma)^i$$

In addition to sub-multiplicativity of the norm, we use that $\sum_{i=0}^{t-1} (1 - \gamma)^i \leq \gamma^{-1}$. ■

6.1. Step 1: Moment Recovery

The following lemma promises an approximate recovery of $(A')^i B$'s through an appeal to arguments involving measures of concentration.

Lemma 21 *Algorithm 2 satisfies for all $j \in [k]$, with probability $1 - \delta$ or more*

$$\|N_j - (A')^j B\| \leq n \kappa^3 \gamma^{-1} W \sqrt{\frac{8 \log(m n k \delta^{-1})}{T_0 - k}}.$$

Proof Let $N_{j,t} = x_{t+j+1}\eta_t^\top$. With Equation 2, the fact that η_i is zero-mean with isotropic unit covariance, and that it is chosen independently of $\{w_j\}, \{\eta_j\}_{j \neq i}$ implies $\mathbb{E}[N_j] = \mathbb{E}[N_{j,t}] = (A')^j B$.

$N_{j,t}$'s are bounded as $\|N_{j,t}\| \leq n\kappa^3\gamma^{-1}W$. Involving the same instance of η in multiple terms, they may not be independent across the second index t . To remedy this, define a sequence of centered random variables $\tilde{N}_{j,t} = N_{j,t} - (A')^j B$, and observe that they form a martingale difference sequence with respect to $\{\eta_t\}$, ie.

$$\mathbb{E}[\tilde{N}_{j,t} | \eta_0, \dots, \eta_{t-1}] = \mathbb{E}[x_{t+j+1}\eta_t^\top | \eta_0, \dots, \eta_{t-1}] - (A')^j B = 0.$$

Together with a union bound on choice of $j \in [k]$, the application of Matrix Azuma inequality on a standard symmetric dilation of the sequence concludes the proof of the claim. \blacksquare

6.2. Step 2: Recovery of System Matrices

The following is a standard result on the perturbation analysis for linear systems (See [Bhatia \(2013\)](#), for example). A proof is presented here for completeness.

Lemma 22 *Let x^* be the solution to the linear system $Ax = b$, and \hat{x} be the solution to $(A + \Delta A)x = b + \Delta b$, then as long as $\|\Delta A\| \leq \sigma_{\min}(A)$, it is true that*

$$\|x^* - \hat{x}\| \leq \frac{\|\Delta b\| + \|\Delta A\| \|x^*\|}{\sigma_{\min}(A) - \|\Delta A\|}$$

Proof Observe the (in)equalities that imply the claim.

$$\begin{aligned} (A + \Delta A)(\hat{x} - x^*) &= b + \Delta b - Ax^* - \Delta Ax^* = \Delta b - \Delta Ax^* \\ \|(A + \Delta A)^{-1}\| &\leq \frac{1}{\sigma_{\min}(A) - \|\Delta A\|} \end{aligned}$$

\blacksquare

7. Conclusions

We define a generalization of the control setting to the nonstochastic regime in which the dynamics, in addition to the perturbations, are adversarial and unknown to the learner. In this setting we give the first efficient algorithm with sublinear regret, answering an open question posed in [Tu \(2019\)](#).

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