Supplemental Material

Contents

S1. Simplification of the first-layer kernel

In this section, we get explicit control in spectral norm of the difference between the empirical (i.e. finite-size) NTK and the version in eqn. (22) that arises through the simplification of the first-layer kernel K_1 in eqn. (12). We will use the notation $A_{i:} = (A_{i1}, \dots, A_{in})$, where $A_{:i}$ is defined similarly. Recall from eqns. (8) and (9) that the empirical NTK is given by

$$K := \frac{X^\top X}{n_0} \odot \frac{(F')^\top \operatorname{diag}(W_2)^2 F'}{n_1} + \frac{F^\top F}{n_1} + \gamma I \tag{S1}$$

and from eqn. (22) the simplified kernel is given by

$$K_{\text{simp}} := \zeta \frac{X^\top X}{n_0} + (\eta' - \zeta) I + \frac{F^\top F}{n_1} + \gamma I, \tag{S2}$$

Also, define

$$R := \frac{\zeta'}{n_0} \mathbf{1}\mathbf{1}^\top, \tag{S3}$$

where $\zeta' := \left[\mathbb{E}_{z\sim\mathcal{N}(0,1)} \sigma''(z)\right]^2$. In this section, we show for any $\varepsilon, \delta > 0$

$$\mathbb{P}\left\{ \|K - K_{\text{simp}} - R\| > n_0^{2\varepsilon - 1/4} \right\} < \delta \tag{S4}$$

for sufficiently large n_0.

Let $\mathbb{E}$ be expectation over W_1 and W_2 conditional on X. We note that with high-probability for any $\varepsilon > 0$ that $(X^\top X/n_0)_{ab} = \delta_{ab} + \mathcal{O}\left(n_0^{\varepsilon - 1/2}\right)$ for all a and b, and that $\|X^\top X/n_0\| \le n^\varepsilon$, since X is i.i.d. Gaussian. The use of $\mathcal{O}$ hides uniform constants.

Define

$$\Delta_k := \frac{X^\top X}{n_0} \odot \left((W_2)_k^2 \left(F'_{k:}\right)^\top F'_{k:} - \bar{M} \right) \quad \text{and} \quad \Delta := \frac{1}{n_1} \sum_{k=1}^{n_1} \Delta_k, \tag{S5}$$

where $\bar{M} := \mathbb{E}\left(F'_{k:}\right)^\top F'_{k:}$ (which does not depend on k), so $\mathbb{E}\Delta_k = 0$. Then

$$K - K_{\text{simp}} - R = \Delta + \left[\frac{X^\top X}{n_0} \odot \left(M - \bar{M}\right) - R \right] + \left(\frac{X^\top X}{n_0} \odot M - K_{\text{simp}} \right), \tag{S6}$$

where $M := \zeta \mathbf{1}\mathbf{1}^\top + (\eta' - \zeta)I$. Elementary arguments given in Sec. S1.2 show that, in operator norm, the two rightmost terms in eqn. (S6) are bounded by $\mathcal{O}(n_0^{3\varepsilon - 1/2})$. In Sec. S1.1, we bound Δ by using the fact that, conditional on X, Δ is a sum of independent random matrices to apply the matrix Bernstein inequality (Tropp, 2015).

S1.1. Bounding Δ

We start with a supremum bound on $\|\Delta_k\|$. For any vector $\mathbf{v} = \sum_k v_k \mathbf{e}_k$, we have

$$\|\Delta_k \mathbf{v}\| \le \sum_k |v_k| \|\Delta_k \mathbf{e}_k\| \le n_1^{-1} \sup_{a,b} |(W_2)_k^2 F'_{ka} F'_{kb} - \bar{M}| \cdot \|X^\top X / n_0\| \sqrt{m}, \tag{S7}$$

by the Cauchy-Schwarz inequality. Note that by assumption on X, eqn. (S7) is $\mathcal{O}\left(n_0^{2\varepsilon} m^{1/2} / n_1\right) = \mathcal{O}(n_0^{3\varepsilon - 1/2})$.

Now we bound the variance term. Consider the (a, b) entry of $\mathbb{E}\Delta_k^2$:

$$\begin{aligned} &\frac{1}{n_1^2} \sum_{l=1}^{m} (X^\top X/n_0)_{al} (X^\top X/n_0)_{lb} \mathbb{E}\left[\left((W_2)_k^2 F'_{ka} F'_{kl} - \bar{M}_{al} \right) \left((W_2)_k^2 F'_{kl} F'_{kb} - \bar{M}_{lb} \right) \right] \\ &= \frac{1}{n_1^2} \sum_{l=1}^{m} (X^\top X/n_0)_{al} (X^\top X/n_0)_{lb} \left(3\mathbb{E}\left[F'_{ka} F'_{kl} F'_{kl} F'_{kb}\right] - \mathbb{E}\left[F'_{kl} F'_{kb}\right] \mathbb{E}\left[F'_{ka} F'_{kl}\right] \right), \end{aligned}$$

which we note is the same for all k. We now calculate these 2- and 4-point expectations to leading order.

Since the entries of WX are multivariate Gaussian conditional on X, we find

$$\mathbb{E} F'_{ka} F'_{kb} = \mathbb{E} f'(Z_a) f'(Z_b), \tag{S8}$$

where

$$(Z_a, Z_b) \sim \mathcal{N}\left(\mathbf{0}, \frac{1}{n_0} \begin{pmatrix} X_{:a}^\top X_{:a} & X_{:a}^\top X_{:b} \\ X_{:a}^\top X_{:b} & X_{:b}^\top X_{:b} \end{pmatrix} \right) \tag{S9}$$

$$\equiv \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} 1 + \mathcal{O}\left(n_0^{\varepsilon - 1/2}\right) & \mathcal{O}\left(n_0^{\varepsilon - 1/2}\right) \\ \mathcal{O}\left(n_0^{\varepsilon - 1/2}\right) & 1 + \mathcal{O}\left(n_0^{\varepsilon - 1/2}\right) \end{pmatrix} \right). \tag{S10}$$

Taylor expanding in the covariance term, one can show that, for all a,

$$\mathbb{E} f'(Z_a)^2 = \eta' + \tilde{R} \left(\frac{X_{:a}^\top X_{:a}}{n_0} - 1 \right) + \mathcal{O}(n_0^{2\varepsilon - 1}), \tag{S11}$$

where $\tilde{R} := \mathbb{E}\left[f''(Z)^2 + f'(Z) f'''(Z) \right]$, and for all $a \neq b$,

$$\mathbb{E} f'(Z_a) f'(Z_b) = \zeta + \frac{\xi \xi''}{2} \left(\frac{X_{:a}^\top X_{:a}}{n_0} + \frac{X_{:b}^\top X_{:b}}{n_0} - 2 \right) + \zeta' \frac{X_{:a}^\top X_{:b}}{n_0} + \mathcal{O}(n_0^{2\varepsilon - 1}), \tag{S12}$$

where $\xi := \mathbb{E} f'(Z)$ and $\xi'' := \mathbb{E} f''(Z)$. Using the same argument, we find

$$\mathbb{E}\left(F'_{ka}\right)^2 \left(F'_{kl}\right)^2 = \left(\eta'\right)^2 + \mathcal{O}(n_0^{2\varepsilon - 1/2}); \tag{S13}$$

$$\mathbb{E}\left(F'_{ka}\right)^4 = C_4 + \mathcal{O}(n_0^{2\varepsilon-1/2}); \tag{S14}$$

for l, a, b distinct,

$$\mathbb{E}F'_{ka}F'_{kb}\left(F'_{kl}\right)^2 = \zeta\eta' + \mathcal{O}\left(n_0^{2\varepsilon-1/2}\right); \tag{S15}$$

for $l \neq a$,

$$\mathbb{E}F'_{ka}\left(F'_{kl}\right)^3 = C_3 + \mathcal{O}(n_0^{2\varepsilon-1/2}), \tag{S16}$$

for some constants C_3 and C_4.

Thus, we may write

$$\sum_k \mathbb{E}\Delta_k^2 = \frac{1}{n_1}(X^\top X/n_0)^2 \odot M_2 + E, \tag{S17}$$

where

$$M_2 := (3\zeta\eta' - \zeta^2)\mathbf{1}\mathbf{1}^\top + 3\eta'(\eta' - \zeta)I \tag{S18}$$

and

$$E_{ab} := \frac{1}{n_1}\sum_l (X^\top X/n_0)_{al}(X^\top X/n_0)_{lb}\varepsilon_{abl} \tag{S19}$$

for some $\varepsilon_{abl} = \mathcal{O}(n_0^{2\varepsilon-1/2})$. We find $\|\frac{1}{n_1}(X^\top X/n_0)^2 \odot M_2\| = \mathcal{O}(n_0^{\varepsilon}/n_1)$ and

$$\begin{aligned}
\|E\| &\leq \|E\|_{\mathrm{F}} && \text{(S20)}\\
&= \left(\sum_{a,b}|E_{ab}|^2\right)^{1/2} && \text{(S21)}\\
&= \left(\frac{1}{n_1^2}\sum_{a,b}\left(\sum_l (X^\top X/n_0)_{al}^2 (X^\top X/n_0)_{lb}^2 \sum_l \varepsilon_{abl}^2\right)\right)^{1/2} && \text{(S22)}\\
&= \sqrt{\mathcal{O}(n_0^{6\varepsilon-1}n_1^{-2}m^2\left(m^2 n_0^{-2} + m n_0^{-1} + m n_0^{-2} + 1\right)} && \text{(S23)}\\
&= \mathcal{O}(n_0^{3\varepsilon-1/2}) && \text{(S24)}
\end{aligned}$$

using the Cauchy-Schwarz inequality and that assumption that all dimensions are on the same order.

Thus finally applying the matrix Bernstein inequality with $t = Cn_0^{4\varepsilon-1/4}$ for some sufficiently large constant C, we find for any $\delta > 0$

$$\mathbb{P}\left\{\|\Delta\| > Cn_0^{4\varepsilon-1/4}\right\} < \delta \tag{S25}$$

for sufficiently large n_0. Moreover, eqn. (S25) holds with X random as it is independent of W_1 and W_2, and our assumptions on X hold for any $\delta' > 0$ for sufficiently large n_0.

S1.2. Bounding remaining terms

Using eqns. (S11) and (S12), we have

$$\begin{aligned}
\frac{X^\top X}{n_0} \odot \left(\bar{M} - M\right) - R = \tilde{R}\left(\frac{X^\top X}{n_0} - I\right) \odot I + \frac{\xi\xi''}{2}\frac{X^\top X}{n_0} \odot \left(\mathbf{e}\mathbf{1}^\top + \mathbf{1}\mathbf{e}^\top\right) \odot \left(\mathbf{1}\mathbf{1}^\top - I\right)\\
+ \zeta'\frac{X^\top X}{n_0} \odot \frac{X^\top X}{n_0} \odot \left(\mathbf{1}\mathbf{1}^\top - I\right) - R + E,
\end{aligned} \tag{S26}$$

where E's diagonal entries are $\mathcal{O}(n_0^{2\varepsilon-1})$ and off-diagonal entries are $\mathcal{O}(n_0^{3\varepsilon-3/2})$. Taking the terms one by one, we first bound

$$\left\|\tilde{R}\left(\frac{X^\top X}{n_0} - I\right) \odot I\right\| = \sup_a \left|\tilde{R}\left(\frac{X_{:a}^\top X_{:a}}{n_0} - 1\right)\right| = \mathcal{O}(n^{\varepsilon-1/2}) \tag{S27}$$

Next, we bound

$$\left\|\frac{\xi\xi''}{2}\frac{X^\top X}{n_0}\odot\left(\mathbf{e}\mathbf{1}^\top+\mathbf{1}\mathbf{e}^\top\right)\odot\left(\mathbf{1}\mathbf{1}^\top-I\right)\right\|\leq\mathcal{O}(n_0^{\varepsilon-1/2}).\tag{S28}$$

Eqn. (S28) can be demonstrated by taking the 4th power of the trace as in (El Karoui et al., 2010). This is expected, since the entries are mean zero and have variance order $\mathcal{O}(n_0^{-1})$. Proving the spectral bound is a straightforward calculation using the independence of the entries of X, but we avoid details here. The final term can also be bounded in this way, yielding,

$$\left\|\zeta'\frac{X^\top X}{n_0}\odot\frac{X^\top X}{n_0}\odot\left(\mathbf{1}\mathbf{1}^\top-I\right)-R\right\|=\mathcal{O}(n_0^{\varepsilon-1/2}).\tag{S29}$$

The inclusion of the matrix R is necessary, due to the nonzero mean of the entries. See (El Karoui et al., 2010) for an example of this calculation.

Similarly using the assumptions on X, we can bound the remaining diagonal matrix of eqn. (S6) as follows

$$\begin{aligned}\left\|\left(\frac{X^\top X}{n_0}\odot M-K_{\text{simp}}\right)\right\|&=(\eta'-\zeta)\left\|\operatorname{diag}(X^\top X/n_0)-I\right\|\\&=(\eta'-\zeta)\sup_a\left|\frac{1}{n_0}\sum_k X_{ka}^2-1\right|\\&=\mathcal{O}(n^{\varepsilon-1/2}).\end{aligned}\tag{S30}$$

Summing our bounds on Δ and eqns. (S27)-(S30) completes the proof of eqn. (S4).

S2. Gaussian equivalents

In this section we discuss the key arguments for existence of Gaussian equivalents and the linearizations of Sec. 4.2. As all the main elements of this argument have been established elsewhere, here we just provide the main intuitions and refer to prior work for the details.

Many of the statistics of random matrices are universal, that is, their limiting behavior as the matrix gets larger is insensitive to the detailed properties of their entries' distributions. Considerable work has gone into demonstrating universality for an increasingly large class of random matrices and a growing number of detailed statistics. In our case, the test loss is a global measurement of several random matrices. This perspective gives some intuition for why we are able to replace many of the intractable terms in the expressions we analyze with tractable terms, which only need to match quite superficial properties of the distributions to ensure the limiting test loss is the same.

In Secs. S3 and S4, we use this replacement strategy in two distinct situations. The first is for terms of the form

$$\operatorname{tr}(AB)=\sum_{ij}A_{ij}B_{ji},\tag{S31}$$

for deterministic A and random B. Under assumptions on A and B, standard concentration inequalities can be used to describe the limiting behavior of sums like eqn. (S31). In our setting, one finds that this behavior only depends on the the low-order moments of B. By matching these low-order moments with Gaussian random variables, we can replace B with a Gaussian random matrix with the same limiting behavior. Note, often A is not actually deterministic, we are simply conditioning on it and only considering the randomness in B. The approach is suitable for determining the average behavior of eqn. (S31) when we have control over the (weak) correlations in the entries of A and B. Linearizing the matrices A and B in this setting is just a convenient bookkeeping device for performing these computations.

When one of the matrices in eqn. (S31) is inverted, the situation is more complex, and indeed this is the case for the kernel matrix K in expressions for the training and test loss. To apply the linear pencil algorithm, we have to replace the NTK in all expressions with a linearized version (see eqn. (22)), which is a rational expression of the i.i.d. Gaussian matrices, X, W_1, *etc.* In Sec. S1, we bounded the difference between the first-layer kernel and its linearization, thus removing the Hadamard product structure. It remains to linearize the second-layer kernel, *i.e.* linearize F. This has been discussed in previous works, see (Mei & Montanari, 2019; Adlam et al., 2019; Péché et al., 2019; Benigni & Péché, 2019).

It should be expected that a linearized version of F will lead to the same asymptotic statistics due to some very general results on the limiting behavior of expressions of the form,

$$\operatorname{tr}\left(A\frac{1}{B-zI}\right), \tag{S32}$$

where A is symmetric and $z \in \mathbb{C}^+$. The resolvent matrix $(B-z)^{-1}$ is intimately related to the spectral properties of B. Recently, isotropic results for quite general A have been developed for matrices with correlated entries, which show that under certain assumptions the limiting behavior of eqn. (S32) depends only on the low-order moments of B. Specifically, the limiting behavior of eqn. (S32) is described by the matrix Dyson equation in many cases. For a summary of these results and related topics see e.g. (Erdos, 2019). While we do not explicitly show the correlation structure of K meets the conditions known to suffice for the matrix Dyson equation, the assumptions in Sec. 2 imply that the correlations between entries of K are weak, which is the essential ingredient.

Finding Gaussian equivalents for A and B in expressions like eqns. (S31) and (S32) is relatively simple in our case. We encounter terms for which the matrix B depends on some other random matrix C through a coordinate-wise nonlinear function $f(C)$. For such cases, Taylor expanding the function f is the key tool to finding these equivalents (see e.g. (Adlam et al., 2019) for more details on this type of approach).

S3. Exact asymptotics for the training loss

S3.1. Decomposition of terms

The model's predictions on the training set, $\hat{y}(X)$, take a simple form,

$$\hat{y}(X) = N_0(X) + (Y - N_0(X))K^{-1}K(X,X) \tag{S33}$$

$$= Y - \gamma(Y - N_0(X))K^{-1}\,. \tag{S34}$$

The expected training loss can be written as,

$$E_{\text{train}} = \frac{1}{m}\mathbb{E}\operatorname{tr}\left((Y-\hat{y}(X))(Y-\hat{y}(X))^\top\right) \tag{S35}$$

$$= \frac{\gamma^2}{m}\mathbb{E}\operatorname{tr}\left((Y-N_0(X))^\top(Y-N_0(X))K^{-2}\right) \tag{S36}$$

$$= T_1 + \nu T_2 \tag{S37}$$

where $\nu = 0$ with centering and $\nu = 1$ without it and,

$$T_1 = \frac{\gamma^2}{m}\mathbb{E}\operatorname{tr}(Y^\top Y K^{-2}) \tag{S38}$$

$$T_2 = \frac{\gamma^2}{m}\mathbb{E}\operatorname{tr}(N_0(X)^\top N_0(X) K^{-2})\,. \tag{S39}$$

Note we can suppress the terms linear in N_0 since they vanish in expectation owing to the linear dependence on the mean-zero random variable ω. Here $K = K(X,X) + \gamma I_m$ is the linearized NTK and is given by,

$$K = \sigma_{W_2}^2\left[(\eta' - \zeta)I_m + \frac{\zeta X^\top X}{n_0}\right] + \frac{F^\top F}{n_1} + \gamma I_m\,. \tag{S40}$$

This substitution can be justified using the result of Sec. S1:

$$\left|\frac{\gamma^2}{m}\mathbb{E}\operatorname{tr}\left(Y^\top Y K_{\text{simp}}^{-2}\right)) - \frac{\gamma^2}{m}\mathbb{E}\operatorname{tr}(Y^\top Y K^{-2})\right| = \left|\frac{\gamma^2}{m}\mathbb{E}Y\left(K_{\text{simp}}^{-2} - K^{-2}\right)Y^\top\right| \tag{S41}$$

$$\leq \frac{\gamma^2\zeta'}{mn_0}\mathbb{E}|Y\mathbf{1}^\top|_2 + \frac{\gamma^2}{m}\mathbb{E}\|Y\|_2^2\|R + K_{\text{simp}}^{-2} - K^{-2}\| = o(1). \tag{S42}$$

Eqn. (S39) is similar.

Note that taking the expectation over W_2 in eqn. (S39) and eqn. (S40) yields

$$\mathbb{E}_{W_2} N_0(X)^\top N_0(X) = \sigma_{W_2}^2 K - \sigma_{W_2}^2 \left[\sigma_{W_2}^2 \left(\eta' - \zeta\right) + \gamma\right] I_m - \sigma_{W_2}^4 \frac{\zeta X^\top X}{n_0}\,, \tag{S43}$$

since $\mathbb{E}_{W_2} N_0(X)^\top N_0(X) = \sigma_{W_2}^2/n_1 F^T F$.

Next we recall the substitution (14),

$$Y \to Y^{\mathrm{lin}} = \frac{1}{\sqrt{n_0 n_{\mathrm{T}}}} \omega \Omega X + \mathcal{E}\,, \tag{S44}$$

which can be used to calculate the expectation over ω and Ω to leading order (*i.e.* with remainder terms $o(1)$) using the approach of eqn. (S31). Concretely,

$$\frac{\gamma^2}{m} \mathbb{E}_{\omega,\Omega,\mathcal{E}} \operatorname{tr}(Y^\top Y K^{-2}) = \frac{\gamma^2}{m} \mathbb{E}_{\omega,\Omega,\mathcal{E}} \operatorname{tr}(\left(Y^{\mathrm{lin}}\right)^\top Y^{\mathrm{lin}} K^{-2}) + o(1) = \frac{\gamma^2}{m} \operatorname{tr}\left[\left(\frac{1}{n_0} X^\top X + \sigma_\varepsilon^2 I_m\right) K^{-2}\right] + o(1). \tag{S45}$$

Putting these pieces together, we can write for $\tau_1 = \tau_1(\gamma)$ and $\tau_2 = \tau_2(\gamma)$,

$$T_1 = -\gamma^2(\sigma_\varepsilon^2 \tau_1' + \tau_2') \tag{S46}$$

$$T_2 = \sigma_{W_2}^2 \gamma^2 \left(\tau_1 + (\sigma_{W_2}^2(\eta' - \zeta) + \gamma)\tau_1' + \sigma_{W_2}^2 \zeta \tau_2'\right)\,, \tag{S47}$$

where,

$$\tau_1 = \frac{1}{m} \mathbb{E} \operatorname{tr}(K^{-1})\,, \quad \text{and} \quad \tau_2 = \frac{1}{m} \mathbb{E} \operatorname{tr}(\frac{1}{n_0} X^\top X K^{-1})\,. \tag{S48}$$

Self-consistent equations for τ_1 and τ_2 can be computed using the resolvent method, as was done in (Adlam et al., 2019) for the case of $\sigma_{W_2} = 0$. In order to pave the way for the analysis of the test error, we instead demonstrate how to compute these traces using operator-valued free probability.

Remark S1. *In the remainder of this section, and in Sec. S4, we assume at times that σ is non-linear (so that $\eta' > \zeta$ and $\eta > \zeta$) and/or $\gamma > 0$ in order that certain denominator factors are non-zero. The linear and/or ridgeless cases can be obtained by limits of our general results, or through special cases of the pertinent intermediate formulas.*

S3.2. Linear pencils

To begin, we construct linear pencils for τ_1 and τ_2. Using the linearization eqn. (13), a straightforward block-matrix inversion confirms that

$$\tau_1 = \mathbb{E} \operatorname{tr}([Q_T^{-1}]_{1,1}) \quad \text{and} \quad \mathbb{E}\tau_2 = \operatorname{tr}([Q_T^{-1}]_{2,4})\,, \tag{S49}$$

where,

$$Q_T = \begin{pmatrix} I_m\left(\gamma + \sigma_{W_2}^2\left(\eta' - \zeta\right)\right) & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta - \zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta} X^\top}{\sqrt{n_0} n_1} \\ -X & I_{n_0} & 0 & 0 \\ -\sqrt{\eta - \zeta}\Theta_F & -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}} & I_{n_1} & 0 \\ 0 & 0 & \frac{\sqrt{\zeta}\psi W_1^\top}{\sqrt{n_0}\phi} & -\frac{\sqrt{\zeta}\psi I_{n_0}}{\sqrt{n_0}\phi} \end{pmatrix}. \tag{S50}$$

The matrix Q_T is not self-adjoint, but a self-adjoint representation can be obtained from it by doubling the dimensionality. In particular, letting

$$\bar{Q}_T = \begin{pmatrix} 0 & Q_T^\top \\ Q_T & 0 \end{pmatrix}\,, \tag{S51}$$

we have,

$$\tau_1 = \mathbb{E} \operatorname{tr}([\bar{Q}_T^{-1}]_{1,5})\,, \quad \text{and} \quad \mathbb{E} \operatorname{tr}([\bar{Q}_T^{-1}]_{2,8})\,. \tag{S52}$$

Observe that $\bar{Q}_T$ is a self-adjoint matrix whose blocks are either constants or proportional to one of $\{X, X^\top, W_1, W_1^\top, \Theta_F, \Theta_F^\top\}$; let us denote the constant terms as Z. As such, we can directly utilize the results of (Far et al., 2006; Mingo & Speicher, 2017) to compute the necessary traces.

S3.3. Operator-valued Stieltjes transform

The traces can be extracted from the operator-valued Stieltjes transform $G : M_d(\mathbb{C})^+ \to M_d(\mathbb{C})^+$, which is a solution of the equation,

$$ZG = I_d + \eta(G)G\,, \tag{S53}$$

where d is the number of blocks, $\eta : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ defined by

$$[\eta(D)]_{ij} = \sum_{kl} \sigma(i,k;l,j)\alpha_k D_{kl}\,, \tag{S54}$$

where α_k is dimensionality of the kth block and $\sigma(i,k;l,k)$ denotes the covariance between the entries of the blocks ij block of $\bar{Q}$ and entries of the kl block of $\bar{Q}$. Eqn. (S53) may admit many solutions, but there is a unique solution such that $\mathrm{Im}G \succ 0$ for $\mathrm{Im}Z \succ 0$.

The constants Z, the entries of σ, and therefore the equations (S54) are manifest by inspection of the block matrix representation for $\bar{Q}_T$. Although the matrix representation of the equations is too large to reproduce here, we can nevertheless extract the equations satisfied by each entry of G.

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_T$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}\,, \tag{S55}$$

where,

$$G_{12} = \begin{pmatrix} \tau_1 & 0 & 0 & 0 \\ 0 & g_3 & 0 & \tau_2 \\ 0 & 0 & g_4 & 0 \\ 0 & g_6 & 0 & g_5 \end{pmatrix} \tag{S56}$$

and the independent entry-wise component functions g_i, τ_1 and τ_2 satisfy the following system of polynomial equations,

$$0 = \sqrt{\zeta} g_6 \psi - \zeta g_3 g_4 \sqrt{n_0} \tag{S57}$$
$$0 = \sqrt{\zeta}\psi\big(\tau_2 - g_3\tau_1\big) \tag{S58}$$
$$0 = \sqrt{\zeta}\psi\big(g_5 - g_6\tau_1\big) + \sqrt{n_0}\phi \tag{S59}$$
$$0 = -\zeta g_4 g_5 - g_6\big(\zeta\tau_1\sigma_{W_2}^2 + \phi\big) \tag{S60}$$
$$0 = \sqrt{\zeta} g_5\psi + \sqrt{n_0}\big(\phi - \zeta g_4\tau_2\big) \tag{S61}$$
$$0 = \phi - g_4\big(\tau_1\psi(\eta - \zeta) + \zeta\tau_2\psi + \phi\big) \tag{S62}$$
$$0 = -\zeta g_4\tau_2 - g_3\big(\zeta\tau_1\sigma_{W_2}^2 + \phi\big) + \phi \tag{S63}$$
$$0 = -\sqrt{\zeta} g_5\tau_1\psi - \sqrt{n_0}\tau_2\big(\zeta\tau_1\sigma_{W_2}^2 + \phi\big) \tag{S64}$$
$$0 = \sqrt{n_0}\big(\phi - g_3\big(\zeta\tau_1\sigma_{W_2}^2 + \phi\big)\big) - \sqrt{\zeta} g_6\tau_1\psi \tag{S65}$$
$$0 = \sqrt{n_0}\big(1 - \tau_1\big(\gamma + g_4(\eta - \zeta) + \sigma_{W_2}^2\big(\eta' + \zeta\big(g_3 - 1\big)\big)\big)\big) - \sqrt{\zeta} g_6\tau_1\psi\,. \tag{S66}$$

It is straightforward algebra to eliminate g_3, g_4, g_5 and g_6 from the above equations. A simple set of equations for τ_1 and τ_2 follows,

$$0 = \phi\left(\zeta\tau_2\tau_1 + \phi(\tau_2 - \tau_1)\right) + \zeta\tau_1\tau_2\psi\left(\gamma\tau_1 - 1\right) + \zeta\tau_1\tau_2\sigma_{W_2}^2\left(\zeta\left(\tau_2 - \tau_1\right)\psi + \tau_1\psi\eta' + \phi\right) \tag{S67}$$
$$0 = \zeta\tau_1^2\tau_2\left(\eta' - \eta\right)\sigma_{W_2}^2 + \zeta\tau_1\tau_2\left(\gamma\tau_1 - 1\right) - \left(\tau_2 - \tau_1\right)\phi\left(\zeta\left(\tau_2 - \tau_1\right) + \eta\tau_1\right)\,. \tag{S68}$$

Although these equations admit multiple solutions, the general results of (Far et al., 2006; Mingo & Speicher, 2017) guarantee that the correct root is given by the unique solutions $\tau_1, \tau_2 : \mathbb{C}^+ \to \mathbb{C}^+$ which are analytic in the upper half-plane.

It will prove useful to obtain expressions for $\tau_1'(\gamma)$ and $\tau_2'(\gamma)$. By differentiating eqns. (S67) and (S68) with respect to γ, we find

$$\tau_1' = -\frac{\zeta^2\tau_2^2\big(\psi\tilde{\tau}_1^2 - \phi^2\big)}{\psi\tilde{\tau}_1^2\big(\zeta^2\big(\tilde{\tau}_2+1\big)^2 + \phi\big(\zeta\tilde{\tau}_2+\eta\big)\big(\zeta\tilde{\tau}_2\big(2\tilde{\tau}_2+3\big)+\eta\big)\big) + \zeta^2\phi^2\big(\tilde{\tau}_2+1\big)^2\big(\phi\tilde{\tau}_2^2 - 1\big)} \tag{S69}$$

$$\tau_2' = -\frac{\zeta\tau_2^2\big(\psi\tilde{\tau}_1^2(\zeta-\eta) - \zeta\phi^2\big(\tilde{\tau}_2+1\big)^2\big)}{\psi\tilde{\tau}_1^2\big(\zeta^2\big(\tilde{\tau}_2+1\big)^2 + \phi\big(\zeta\tilde{\tau}_2+\eta\big)\big(\zeta\tilde{\tau}_2\big(2\tilde{\tau}_2+3\big)+\eta\big)\big) + \zeta^2\phi^2\big(\tilde{\tau}_2+1\big)^2\big(\phi\tilde{\tau}_2^2-1\big)}\,, \tag{S70}$$

where we have introduced some auxiliary variables to ease the presentation,

$$\tilde{\tau}_1 = \sigma_{W_2}^2\zeta\tau_2 + \phi\tilde{\tau}_2 \quad \text{and} \quad \tilde{\tau}_2 = -1 + \tau_2/\tau_1\,. \tag{S71}$$

S4. Exact asymptotics for the test loss

S4.1. Decomposition of terms

As described in Sec. 4.3, the test loss can be written as,

$$E_{\text{test}} = \mathbb{E}(y - \hat{y}(\mathbf{x}))^2 = E_1 + E_2 + E_3 \tag{S72}$$

with

$$E_1 = \mathbb{E}\,\mathrm{tr}(y(\mathbf{x})y(\mathbf{x})^\top) + \mathbb{E}\,\mathrm{tr}(N_0(\mathbf{x})N_0(\mathbf{x})^\top) \tag{S73}$$

$$E_2 = -2\mathbb{E}\,\mathrm{tr}(K_{\mathbf{x}}^\top K^{-1}Y^\top y(\mathbf{x})) - 2\mathbb{E}\,\mathrm{tr}(K_{\mathbf{x}}^\top K^{-1}N_0(X)^\top N_0(\mathbf{x})) \tag{S74}$$

$$E_3 = \mathbb{E}\,\mathrm{tr}(K_{\mathbf{x}}^\top K^{-1}Y^\top Y K^{-1}K_{\mathbf{x}}) + \mathbb{E}\,\mathrm{tr}(K_{\mathbf{x}}^\top K^{-1}N_0(X)^\top N_0(X)K^{-1}K_{\mathbf{x}})\,. \tag{S75}$$

As in Sec. S3, we suppress the terms linear in ω as they vanish in expectation. The Neural Tangent Kernels $K = K(X,X) + \gamma I$ and $K_{\mathbf{x}} = K(X,\mathbf{x})$ are given by,

$$K = \sigma_{W_2}^2\big[(\eta'-\zeta)I_m + \frac{\zeta X^\top X}{n_0}\big] + \frac{F^\top F}{n_1} + \gamma I_m\,, \qquad \text{and} \qquad K_{\mathbf{x}} = \frac{\sigma_{W_2}^2\zeta}{n_0}X^\top\mathbf{x} + \frac{1}{n_1}F^\top f\,, \tag{S76}$$

where the substitution for the linearized NTK is justified as in Sec. S3 using the spectral norm bound of Sec. S1.

Using the cyclicity and linearity of the trace, the expectation over $\mathbf{x}$ requires the computation of

$$\mathbb{E}_{\mathbf{x}}K_{\mathbf{x}}K_{\mathbf{x}}^\top\,, \qquad \mathbb{E}_{\mathbf{x}}y(\mathbf{x})K_{\mathbf{x}}^\top\,, \qquad \mathbb{E}_{\mathbf{x}}y(\mathbf{x})y(\mathbf{x})^\top\,, \qquad \mathbb{E}_{\mathbf{x}}N_0(\mathbf{x})K_{\mathbf{x}}^\top\,, \qquad \text{and} \quad \mathbb{E}_{\mathbf{x}}N_0(\mathbf{x})N_0(\mathbf{x})^\top\,. \tag{S77}$$

As described in Sec. 4.2, without loss of generality we can consider the case of a linear teacher, so that $\eta_{\text{T}} = \zeta_{\text{T}} = 1$ and (16) and (15) become

$$y \to y^{\text{lin}} = \frac{\sqrt{\zeta_{\text{T}}}}{\sqrt{n_0 n_{\text{T}}}}\omega\Omega\mathbf{x} + \sqrt{\eta_{\text{T}}-\zeta_{\text{T}}}\frac{1}{\sqrt{n_{\text{T}}}}\omega\theta_y = \frac{1}{\sqrt{n_0 n_{\text{T}}}}\omega\Omega\mathbf{x} \qquad \text{and} \qquad f \to f^{\text{lin}} = \frac{\sqrt{\zeta}}{\sqrt{n_0}}W_1\mathbf{x} + \sqrt{\eta-\zeta}\theta_f\,. \tag{S78}$$

Using these substitutions, the expectations over $\mathbf{x}$ are now trivial and we readily find,

$$\mathbb{E}_{\mathbf{x}}K_{\mathbf{x}}K_{\mathbf{x}}^\top = \frac{\sigma_{W_2}^4\zeta^2}{n_0^2}X^\top X + \frac{\sigma_{W_2}^2\zeta^{3/2}}{n_0^{3/2}n_1}(X^\top W_1^T F + F^\top W_1 X) + \frac{1}{n_1^2}F^\top\big(\frac{\zeta}{n_0}W_1W_1^\top + (\eta-\zeta)I_{n_1}\big)F \tag{S79}$$

$$\mathbb{E}_{\mathbf{x}}y(\mathbf{x})K_{\mathbf{x}}^\top = \frac{\sigma_{W_2}^2\zeta}{n_0^{3/2}\sqrt{n_{\text{T}}}}\omega\Omega X + \frac{\sqrt{\zeta}}{n_0 n_1\sqrt{n_{\text{T}}}}\omega\Omega W_1^\top F \tag{S80}$$

$$\mathbb{E}_{\mathbf{x}}y(\mathbf{x})y(\mathbf{x})^\top = \frac{1}{n_0 n_{\text{T}}}\omega\Omega\Omega^\top\omega^\top \tag{S81}$$

$$\mathbb{E}_{\mathbf{x}}N_0(\mathbf{x})K_{\mathbf{x}}^\top = \frac{\sigma_{W_2}^2\zeta^{3/2}}{n_0^{3/2}\sqrt{n_1}}W_2W_1X + \frac{1}{n_1^{3/2}}W_2\big(\frac{\zeta}{n_0}W_1W_1^\top + (\eta-\zeta)I_{n_1}\big)F \tag{S82}$$

$$\mathbb{E}_{\mathbf{x}}\,\mathrm{tr}(N_0(\mathbf{x})N_0(\mathbf{x})^\top) = \sigma_{W_2}^2\eta\,. \tag{S83}$$

One may interpret the substitutions in eqn. (S78) as a tool to calculate the expectations above to leading order as it leads to terms like eqn. (S31). Next we recall the substitution (S44),

$$Y \to Y^{\text{lin}} = \frac{1}{\sqrt{n_0 n_{\text{T}}}}\omega\Omega X + \mathcal{E}\,. \tag{S84}$$

As above, we consider the leading order behavior with respect to the random variables ω, Ω, and W_2 using eqn. (S31) to find

$$\mathbb{E}_{\omega,\Omega,\mathcal{E}}\left[Y^\top Y\right] = \frac{1}{n_0}X^\top X + \sigma_\varepsilon^2 I_m \tag{S85}$$

$$\mathbb{E}_{\omega,\Omega,\mathcal{E},W_2}\left[Y^\top \mathbb{E}_{\mathbf{x}} y(\mathbf{x}) K_{\mathbf{x}}^\top\right] = \frac{\sigma_{W_2}^2\zeta}{n_0^2}X^\top X + \frac{\sqrt{\zeta}}{n_0^{3/2}n_1}X^\top W_1^\top F \tag{S86}$$

$$\mathbb{E}_{W_2}\left[N_0(X)^\top N_0(X)\right] = \frac{\sigma_{W_2}^2}{n_1}F^\top F \tag{S87}$$

$$\mathbb{E}_{W_2}\left[N_0(X)^\top \mathbb{E}_{\mathbf{x}} N_0(\mathbf{x}) K_{\mathbf{x}}^\top\right] = \frac{\sigma_{W_2}^4\zeta^{3/2}}{n_0^{3/2}n_1}F^\top W_1 X + \frac{\sigma_{W_2}^2}{n_1^2}F^\top\big(\frac{\zeta}{n_0}W_1 W_1^\top + (\eta-\zeta)I_{n_1}\big)F\,. \tag{S88}$$

Using (13),

$$F \to F^{\text{lin}} = \frac{\sqrt{\zeta}}{\sqrt{n_0}}W_1 X + \sqrt{\eta-\zeta}\,\Theta_F\,, \tag{S89}$$

we can write,

$$\frac{\sqrt{\zeta}}{\sqrt{n_0}}F^\top W_1 X + \frac{\sqrt{\zeta}}{\sqrt{n_0}}X^\top W_1^\top F = F^\top F + \frac{\zeta}{n_0}X^\top W_1^\top W_1 X - (\eta-\zeta)\Theta_F^\top\Theta_F\,. \tag{S90}$$

Putting these pieces together, we have

$$E_1 = 1 + \nu\sigma_{W_2}^2\eta \tag{S91}$$

$$E_2 = E_{21} + \nu E_{22} \tag{S92}$$

$$E_3 = E_{31} + E_{32} + \nu E_{33}\,, \tag{S93}$$

where $\nu = 0$ with centering and $\nu = 1$ without it,

$$E_{21} = -\mathbb{E}\,\mathrm{tr}\left(2\frac{\sigma_{W_2}^2\zeta}{n_0^2}XK^{-1}X^\top + \frac{1}{n_0 n_1}FK^{-1}F^\top + \frac{\zeta}{n_0^2 n_1}W_1 X K^{-1}X^\top W_1^\top - \frac{\eta-\zeta}{n_0 n_1}\Theta_F K^{-1}\Theta_F^\top\right) \tag{S94}$$

$$E_{22} = -\frac{2\sigma_{W_2}^2}{n_1}\mathbb{E}\,\mathrm{tr}\left(\frac{\sigma_{W_2}^2\zeta^{3/2}}{n_0^{3/2}}K^{-1}F^\top W_1 X + \frac{\zeta}{n_0 n_1}K^{-1}F^\top W_1 W_1^\top F + \frac{\eta-\zeta}{n_1}K^{-1}F^\top F\right) \tag{S95}$$

$$E_{31} = \sigma_\varepsilon^2\mathbb{E}\,\mathrm{tr}\left(K^{-1}\Sigma_3 K^{-1}\right) \tag{S96}$$

$$E_{32} = \frac{1}{n_0}\mathbb{E}\,\mathrm{tr}\left(XK^{-1}\Sigma_3 K^{-1}X^\top\right) \tag{S97}$$

$$E_{33} = \frac{\sigma_{W_2}^2}{n_1}\mathbb{E}\,\mathrm{tr}\left(FK^{-1}\Sigma_3 K^{-1}F^\top\right)\,, \tag{S98}$$

and,

$$\Sigma_3 = \frac{\sigma_{W_2}^4\zeta^2}{n_0^2}X^\top X + \big(\frac{\sigma_{W_2}^2\zeta}{n_0 n_1} + \frac{\eta-\zeta}{n_1^2}\big)F^\top F + \frac{\zeta}{n_0 n_1^2}F^\top W_1 W_1^\top F + \frac{\sigma_{W_2}^2\zeta^2}{n_0^2 n_1}X^\top W_1^\top W_1 X - \frac{\sigma_{W_2}^2\zeta(\eta-\zeta)}{n_0 n_1}\Theta_F^\top\Theta_F\,. \tag{S99}$$

S4.2. Linear pencils

Repeated application of the Schur complement formula for block matrix inversion establishes the following representations for $E_{21}, E_{22}, E_{31}, E_{32}, E_{33}$.

S4.2.1. E_{21}

A linear pencil for E_{21} follows from the representation,

$$E_{21} = \mathbb{E}\,\mathrm{tr}(U_{21}^T Q_{21}^{-1} V_{21})\,, \tag{S100}$$

where,

$$U_{21}^T = \begin{pmatrix} 0 & -\frac{2\zeta I_{n_0}\sigma_{W_2}^2}{n_0} & 0 & 0 & 0 & \frac{(\eta-\zeta)I_{n_1}}{n_0} & 0 & 0 & 0 & 0 & 0 & -\frac{I_{n_1}}{n_0} & 0 & 0 \end{pmatrix} \tag{S101}$$

$$V_{21}^T = \begin{pmatrix} 0 & 0 & 0 & -\frac{\sqrt{n_0} n_1 I_{n_0}}{\sqrt{\zeta}} & 0 & 0 & 0 & 0 & 0 & I_{n_1} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{S102}$$

and,

$$Q_{21} = \begin{pmatrix} Q_{21}^{11} & 0 & 0 \\ 0 & Q_{21}^{22} & Q_{21}^{23} \\ 0 & 0 & Q_{21}^{33} \end{pmatrix} \tag{S103}$$

with,

$$Q_{21}^{11} = \begin{pmatrix} I_m \left(\gamma + \sigma_{W_2}^2 \left(\eta' - \zeta\right)\right) & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta} X^\top}{\sqrt{n_0} n_1} \\ -X & I_{n_0} & 0 & 0 \\ -\sqrt{\eta-\zeta}\Theta_F & -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}} & I_{n_1} & 0 \\ 0 & 0 & -W_1^\top & I_{n_0} \end{pmatrix} \tag{S104}$$

$$Q_{21}^{22} = \begin{pmatrix} I_m \left(\gamma + \sigma_{W_2}^2 \left(\eta' - \zeta\right)\right) & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta} X^\top}{\sqrt{n_0} n_1} \\ -\Theta_F & I_{n_1} & -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}\sqrt{\eta-\zeta}} & 0 & 0 \\ -X & 0 & I_{n_0} & 0 & 0 \\ -\sqrt{\eta-\zeta}\Theta_F & 0 & -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}} & I_{n_1} & \\ 0 & 0 & 0 & -W_1^\top & I_{n_0} \end{pmatrix} \tag{S105}$$

$$Q_{21}^{23} = \begin{pmatrix} -\Theta_F^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\zeta} W_1}{\sqrt{n_0}(\eta-\zeta)} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ I_{n_1} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{S106}$$

$$Q_{21}^{33} = \begin{pmatrix} -\sqrt{\eta-\zeta}\Theta_F^\top & I_m \left(\gamma + \sigma_{W_2}^2 \left(\eta' - \zeta\right)\right) & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\zeta} X^\top}{\sqrt{n_0} n_1} \\ 0 & -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}} & 0 \\ 0 & -X & 0 & I_{n_0} & 0 \\ n_1 W_1^\top & 0 & -W_1^\top & 0 & I_{n_0} \end{pmatrix}. \tag{S107}$$

S4.2.2. E_{22}

A linear pencil for E_{22} follows from the representation,

$$E_{22} = \mathbb{E}\,\mathrm{tr}(U_{22}^T Q_{22}^{-1} V_{22})\,, \tag{S108}$$

where,

$$U_{22}^T = \begin{pmatrix} 0 & -\frac{2\sqrt{\zeta} I_{n_1} \sigma_{W_2}^2 \left(n_0(\eta-\zeta)+\zeta n_1 \sigma_{W_2}^2\right)}{n_0^{3/2} n_1} & 0 & \frac{2(\zeta-\eta) I_{n_1} \sigma_{W_2}^2}{n_1} & 0 & 0 & 0 \end{pmatrix} \tag{S109}$$

$$V_{22}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -n_1 I_{n_1} & 0 \end{pmatrix} \tag{S110}$$

and,

$$Q_{22} = \begin{pmatrix} I_{n_0} & 0 & -X & 0 & 0 & 0 & 0 \\ -W_1 & I_{n_1} & 0 & 0 & -\frac{\sqrt{n_0} W_1}{\sqrt{\zeta} n_1 \sigma_{W_2}^2} & 0 & 0 \\ \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & 0 & I_m\left(\gamma+\sigma_{W_2}^2\left(\eta'-\zeta\right)\right) & 0 & 0 & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta} X^\top}{\sqrt{n_0} n_1} \\ 0 & 0 & -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & \frac{W_1}{n_1 \sigma_{W_2}^2} & 0 & 0 \\ 0 & -\frac{\sqrt{\zeta} W_1^\top}{\sqrt{n_0}} & 0 & -W_1^\top & I_{n_0} & 0 & 0 \\ -\frac{\sqrt{\zeta} W_1}{\sqrt{n_0}} & 0 & -\sqrt{\eta-\zeta}\Theta_F & 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -W_1^\top & I_{n_0} \end{pmatrix}. \tag{S111}$$

S4.2.3. E_{31}

A linear pencil for E_{31} follows from the representation,

$$E_{31} = \mathbb{E}\,\mathrm{tr}(U_{31}^T Q_{31}^{-1} V_{31})\,, \tag{S112}$$

where,

$$U_{31}^T = \begin{pmatrix} m\sigma_\varepsilon^2 I_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad V_{31}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \end{pmatrix} \tag{S113}$$

and, for $\beta = \big(n_0(\zeta-\eta) - \zeta n_1 \sigma_{W_2}^2\big)$,

$$Q_{31} = \begin{pmatrix}
I_m\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} & -\frac{\zeta^2 X^\top \sigma_{W_2}^4}{n_0^2} & 0 & \frac{\sqrt{\eta-\zeta}\Theta_F^\top \beta}{n_0 n_1^2} & \frac{\sqrt{\zeta}X^\top \beta}{n_0^{3/2} n_1^2} \\
-X & I_{n_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{\eta-\zeta}\Theta_F & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & I_{n_1} & 0 & 0 & -\frac{\zeta\sqrt{\eta-\zeta}\Theta_F \sigma_{W_2}^2}{n_0} & 0 & \frac{\zeta W_1}{n_0 n_1} \\
0 & 0 & -W_1^\top & I_{n_0} & 0 & 0 & \frac{\zeta W_1^\top \sigma_{W_2}^2}{n_0} & 0 \\
0 & 0 & 0 & 0 & I_{n_0} & -X & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & I_m\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} \\
0 & 0 & 0 & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -W_1^\top & I_{n_0}
\end{pmatrix}. \tag{S114}$$

S4.2.4. E_{32}

A linear pencil for E_{32} follows from the representation,

$$E_{32} = \mathbb{E}\,\mathrm{tr}(U_{32}^T Q_{32}^{-1} V_{32})\,, \tag{S115}$$

where,

$$U_{32}^T = \begin{pmatrix} 0 & I_{n_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad V_{32}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{n_0}n_1 I_{n_0}}{\sqrt{\zeta}} \end{pmatrix} \tag{S116}$$

and, for $\beta = \big(n_0(\zeta-\eta) - \zeta n_1 \sigma_{W_2}^2\big)$

$$Q_{32} = \begin{pmatrix}
I_m\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} & -\frac{\zeta^2 X^\top \sigma_{W_2}^4}{n_0^2} & 0 & \frac{\sqrt{\eta-\zeta}\Theta_F^\top \beta}{n_0 n_1^2} & 0 \\
-X & I_{n_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-X & 0 & I_{n_0} & 0 & 0 & 0 & 0 & \frac{\sqrt{\zeta}W_1^\top}{\sqrt{n_0}n_1} & 0 \\
-\sqrt{\eta-\zeta}\Theta_F & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & I_{n_1} & 0 & 0 & -\frac{\zeta\sqrt{\eta-\zeta}\Theta_F \sigma_{W_2}^2}{n_0} & 0 & 0 \\
0 & 0 & 0 & -W_1^\top & I_{n_0} & 0 & 0 & W_1^\top\left(\frac{\eta-\zeta}{n_1}+\frac{\zeta\sigma_{W_2}^2}{n_0}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & I_{n_0} & -X & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & I_m\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} \\
0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_1^\top & I_{n_0}
\end{pmatrix}. \tag{S117}$$

S4.2.5. E_{33}

A linear pencil for E_{33} follows from the representation,

$$E_{33} = \mathbb{E}\,\mathrm{tr}(U_{33}^T Q_{33}^{-1} V_{33})\,, \tag{S118}$$

where,

$$U_{33}^T = \begin{pmatrix} 0 & I_{n_1}\sigma_{W_2}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{S119}$$

$$V_{33}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -n_1 I_{n_1} & 0 \end{pmatrix} \tag{S120}$$

and, for $\beta = \left(n_0(\zeta - \eta) - \zeta n_1 \sigma_{W_2}^2\right)$,

$$Q_{33} = \begin{pmatrix} I_m\left(\gamma+\sigma_{W_2}^2\left(\eta'-\zeta\right)\right) & 0 & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} & -\frac{\zeta^2 X^\top \sigma_{W_2}^4}{n_0^2} & 0 & \frac{\sqrt{\eta-\zeta}\Theta_F^\top \beta}{n_0 n_1^2} & 0 & 0 \\ -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -X & 0 & I_{n_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -X & 0 & 0 & I_{n_0} & 0 & 0 & 0 & 0 & \frac{\sqrt{\zeta}W_1^\top}{\sqrt{n_0}n_1} & 0 & 0 \\ -\sqrt{\eta-\zeta}\Theta_F & 0 & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & I_{n_1} & 0 & 0 & -\frac{\zeta\sqrt{\eta-\zeta}\Theta_F \sigma_{W_2}^2}{n_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -W_1^\top & I_{n_0} & 0 & 0 & W_1^\top\left(\frac{\eta-\zeta}{n_1}+\frac{\zeta\sigma_{W_2}^2}{n_0}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{n_0} & -X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\zeta X^\top \sigma_{W_2}^2}{n_0} & I_m\left(\gamma+\sigma_{W_2}^2\left(\eta'-\zeta\right)\right) & 0 & \frac{\sqrt{\eta-\zeta}\Theta_F^\top}{n_1} & \frac{\sqrt{\zeta}X^\top}{\sqrt{n_0}n_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & -\sqrt{\eta-\zeta}\Theta_F & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{\zeta}W_1}{\sqrt{n_0}} & -\sqrt{\eta-\zeta}\Theta_F & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_1^\top & I_{n_0} \end{pmatrix}. \tag{S121}$$

S4.3. Operator-valued Stieltjes transform

Even though the individual error terms $E_{21}, E_{22}, E_{31}, E_{32}, E_{33}$ can be written as the trace of self-adjoint matrices, the individual Q matrices are not themselves self-adjoint. However, by enlarging the dimensionality by a factor of two, equivalent self-adjoint representations can easily be constructed. To do so, we simply utilize the identity,

$$U^T Q V = \bar{U}^\top \bar{Q} \bar{V} \equiv \left(\tfrac{1}{2}U^\top \quad V^\top\right) \begin{pmatrix} 0 & Q^\top \\ Q & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}U \\ V \end{pmatrix}. \tag{S122}$$

Observe that $\bar{Q}_{21}, \bar{Q}_{22}, \bar{Q}_{31}, \bar{Q}_{32}$ and $\bar{Q}_{33}$ are all self-adjoint block matrices whose blocks are either constants or proportional to one of $\{X, X^\top, W_1, W_1^\top, \Theta_F, \Theta_F^\top\}$; let us denote the constant terms as Z. As such, we can directly utilize the results of (Far et al., 2006; Mingo & Speicher, 2017) to compute the error terms in question.

For each linear pencil, the corresponding error term can be extracted from the operator-valued Stieltjes transform $G : M_d(\mathbb{C})^+ \to M_d(\mathbb{C})^+$, which is a solution of the equation,

$$ZG = I_d + \eta(G)G\,, \tag{S123}$$

where d is the number of blocks, $\eta : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ defined by

$$[\eta(D)]_{ij} = \sum_{kl} \sigma(i,k;l,j)\alpha_k D_{kl}\,, \tag{S124}$$

where α_k is dimensionality of the kth block and $\sigma(i,k;l,k)$ denotes the covariance between the entries of the ij block of $\bar{Q}$ and entries of the kl block of $\bar{Q}$. Eqn. (S123) may admit many solutions, but there is a unique solution such that Im$G \succ 0$ for Im$Z \succ 0$.

The constants Z, the entries of σ, and therefore the equations (S124) are manifest by inspection of the block matrix representations for Q. Although the matrix representations are too large to reproduce here, we can nevertheless extract the equations satisfied by each entry of G, which we present in the subsequent sections.

S4.3.1. E_{21}

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_{21}$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}, \tag{S125}$$

where,

$$G_{12} = \begin{pmatrix} g_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_9 & 0 & g_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{12} & 0 & g_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_1 & 0 & g_5 & 0 & g_4 & 0 & g_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_9 & 0 & g_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{11} & 0 & g_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_{12} & 0 & g_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_2 & 0 & g_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_9 & g_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{12} & g_{10} \end{pmatrix}, \tag{S126}$$

and the independent entry-wise component functions g_i combine to produce the error E_{21} through the relation,

$$E_{21} = \frac{g_4(\eta-\zeta)}{n_0} + \frac{2\sqrt{\zeta}g_6\sqrt{n_0}\sigma_{W_2}^2}{\psi} - \frac{g_2}{n_0}\,, \tag{S127}$$

and themselves satisfy the following system of polynomial equations,

$$0 = 1 - g_1 \tag{S128a}$$
$$0 = \sqrt{\zeta}g_9 g_{11}\sqrt{n_0} - g_{12}\psi \tag{S128b}$$
$$0 = \sqrt{\zeta}g_6 g_{11}\sqrt{n_0} - g_{10}\psi + \psi \tag{S128c}$$
$$0 = g_7(\eta-\zeta) + \sqrt{\zeta}g_6 g_{11}\sqrt{n_0} \tag{S128d}$$
$$0 = g_8 g_{11} n_0\sqrt{\eta-\zeta} - g_3\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128e}$$
$$0 = -\sqrt{\zeta}g_8 g_9\psi - g_6\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128f}$$
$$0 = -\sqrt{\zeta}g_8 g_{12}\psi - (g_{10}-1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128g}$$
$$0 = g_6\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) + g_8\big(\sqrt{\zeta}g_{10}\psi + \zeta g_6\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S128h}$$
$$0 = g_8 g_{11}\psi(\eta-\zeta) - \phi\big(g_5\sqrt{\eta-\zeta} - \sqrt{\zeta}g_6 g_{11}\sqrt{n_0}\big)\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big) \tag{S128i}$$
$$0 = (g_9-1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) + g_8\big(\sqrt{\zeta}g_{12}\psi + \zeta g_9\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S128j}$$
$$0 = g_1 g_8 n_0\sqrt{\eta-\zeta} + g_3\big(g_8\psi(\zeta-\eta) + \phi\big(\sqrt{\zeta}g_6\sqrt{n_0}-1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big) \tag{S128k}$$
$$0 = \sqrt{\zeta}g_{10}g_{11}\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big) + g_{12}\psi\big(\gamma\phi + \sigma_{W_2}^2\big(-\zeta\phi + \phi\eta' + \zeta g_8\big)\big) \tag{S128l}$$
$$0 = g_{11}\big(g_8\psi(\zeta-\eta) + \phi\big(\sqrt{\zeta}g_6\sqrt{n_0}-1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big) + \phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128m}$$
$$0 = g_{11}n_0\big(g_8\psi(\eta-\zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big)\big) - g_2\psi\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128n}$$
$$0 = g_9\psi\big(\gamma\phi + \sigma_{W_2}^2\big(\phi(\eta'-\zeta) + \zeta g_8\big)\big) - \phi\big(\sqrt{\zeta}g_6 g_{11}\sqrt{n_0} + \psi\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128o}$$
$$0 = g_8\big(-\sqrt{\zeta}g_{12}\psi - \sqrt{n_0}\big(\gamma + g_{11}(\eta-\zeta) + \sigma_{W_2}^2\big(\eta' + \zeta(g_9-1)\big)\big)\big) + \sqrt{n_0}\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S128p}$$
$$0 = \sqrt{\zeta}g_1 g_6\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big) - g_7(\zeta-\eta)\big(g_8\psi(\zeta-\eta) + \phi\big(\sqrt{\zeta}g_6\sqrt{n_0}-1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big) \tag{S128q}$$
$$\begin{aligned}0 &= g_1 n_0\big(g_8\psi(\eta-\zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big)\big) + g_2\psi\big(g_8\psi(\zeta-\eta) \\ &\quad + \phi\big(\sqrt{\zeta}g_6\sqrt{n_0}-1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big)\end{aligned} \tag{S128r}$$
$$\begin{aligned}0 &= g_1\big(g_8\psi(\eta-\zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big)\big) + g_5\sqrt{\eta-\zeta}\big(g_8\psi(\eta-\zeta) \\ &\quad - \phi\big(\sqrt{\zeta}g_6\sqrt{n_0}-1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big)\end{aligned} \tag{S128s}$$
$$\begin{aligned}0 &= n_0\big(-\zeta g_5 g_8\psi\sqrt{\eta-\zeta} + \eta g_5 g_8\psi\sqrt{\eta-\zeta} + g_8\psi(\zeta-\eta)\big(g_7(\zeta-\eta) - g_1\big) \\ &\quad + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(g_7(\zeta-\eta) + g_1\big)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big) + g_4\psi\phi(\zeta-\eta)\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\end{aligned} \tag{S128t}$$
$$\begin{aligned}0 &= \sqrt{n_0}\sqrt{\eta-\zeta}\big(g_1 g_8\sqrt{n_0}\psi(\eta-\zeta) + \sqrt{\zeta}g_1 g_6 n_0\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big) - \sqrt{\zeta}g_2 g_6\psi\phi\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\big) \\ &\quad + g_3\psi(\zeta-\eta)\big(g_8\psi(\eta-\zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta') - \gamma\big)\big) + g_4\psi(-\phi)(\eta-\zeta)^{3/2}\big(\gamma + \sigma_{W_2}^2(\eta'-\zeta)\big)\,.\end{aligned} \tag{S128u}$$

After some straightforward algebra, one can eliminate all g_i except for g_6 and g_8, which satisfy coupled polynomial equations. Those equations can be shown to be identical to eqn. (S48) by invoking the change of variables,

$$g_6 = -\frac{\sqrt{\zeta}\psi}{\sqrt{n_0}\phi}\tau_2\,, \quad \text{and} \quad g_8 = \big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big)\tau_1\,. \tag{S129}$$

In terms of these variables, the error E_{21} is given by,

$$E_{21} = 2(\tau_2/\tau_1 - 1)\,. \tag{S130}$$

S4.3.2. E_{22}

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_{22}$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}\,, \tag{S131}$$

where,

$$G_{12} = \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 & 0 & g_7 \\ 0 & g_5 & 0 & g_2 & 0 & g_9 & 0 \\ 0 & 0 & g_{10} & 0 & 0 & 0 & 0 \\ 0 & g_3 & 0 & g_4 & 0 & g_8 & 0 \\ g_{14} & 0 & 0 & 0 & g_1 & 0 & g_6 \\ 0 & 0 & 0 & 0 & 0 & g_{13} & 0 \\ g_{14} & 0 & 0 & 0 & 0 & 0 & g_{12} \end{pmatrix}\,, \tag{S132}$$

and the independent entry-wise component functions g_i combine to produce the error E_{22} through the relation,

$$E_{22} = \frac{2\sqrt{\zeta}g_9\sigma_{W_2}^2\big(\psi(\eta-\zeta) + \zeta\sigma_{W_2}^2\big)}{\sqrt{n_0}\psi} + 2g_8(\eta-\zeta)\sigma_{W_2}^2\,, \tag{S133}$$

and themselves satisfy the following system of polynomial equations,

$$0 = \sqrt{\zeta}g_{11}g_{13}\sqrt{n_0} - g_{14}\psi \tag{S134a}$$
$$0 = \sqrt{\zeta}g_7g_{13}\sqrt{n_0} - g_{12}\psi + \psi \tag{S134b}$$
$$0 = g_1\psi\big(g_3\sqrt{n_0} - \sqrt{\zeta}g_4\big) - g_3\sqrt{n_0}\sigma_{W_2}^2 \tag{S134c}$$
$$0 = -g_1\psi\big(\sqrt{\zeta}g_5 + g_3\sqrt{n_0}\big) - g_3\sqrt{n_0}\sigma_{W_2}^2 \tag{S134d}$$
$$0 = g_1\psi\big(g_5\sqrt{n_0} - \sqrt{\zeta}g_2\big) - \sqrt{\zeta}g_2\sigma_{W_2}^2 \tag{S134e}$$
$$0 = g_1\psi\big(\sqrt{\zeta}g_2 + g_4\sqrt{n_0}\big) - \sqrt{\zeta}g_2\sigma_{W_2}^2 \tag{S134f}$$
$$0 = g_1\psi\big(g_5\sqrt{n_0} - \sqrt{\zeta}g_2\big) - \big(g_5 - 1\big)\sqrt{n_0}\sigma_{W_2}^2 \tag{S134g}$$
$$0 = -g_1\psi\big(\sqrt{\zeta}g_2 + g_4\sqrt{n_0}\big) - \big(g_4 - 1\big)\sqrt{n_0}\sigma_{W_2}^2 \tag{S134h}$$
$$0 = g_1\psi\big(g_3\sqrt{n_0} - \sqrt{\zeta}g_4\big) - \sqrt{\zeta}\big(g_4 - 1\big)\sigma_{W_2}^2 \tag{S134i}$$
$$0 = g_1\psi\big(\sqrt{\zeta}g_5 + g_3\sqrt{n_0}\big) - \sqrt{\zeta}\big(g_5 - 1\big)\sigma_{W_2}^2 \tag{S134j}$$
$$0 = -\sqrt{\zeta}g_{10}g_{11}\psi - g_7\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big) \tag{S134k}$$
$$0 = -\sqrt{\zeta}g_{10}g_{14}\psi - g_6\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big) \tag{S134l}$$
$$0 = -\sqrt{\zeta}g_{10}g_{14}\psi - \big(g_{12} - 1\big)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big) \tag{S134m}$$
$$0 = g_1\big(-\zeta g_2 + \sqrt{\zeta}\big(g_5 - g_4\big)\sqrt{n_0} + g_3 n_0\big) - \sqrt{\zeta}\big(g_1 - 1\big)\sqrt{n_0}\sigma_{W_2}^2 \tag{S134n}$$
$$0 = g_1\psi\big(\sqrt{\zeta}g_9 + g_8\sqrt{n_0}\big) + \sqrt{\zeta}\big(g_7g_{13}n_0 - g_9\big)\sigma_{W_2}^2 + g_6g_{13}\sqrt{n_0}\psi \tag{S134o}$$
$$0 = g_7\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big) + g_{10}\big(\sqrt{\zeta}g_{12}\psi + \zeta g_7\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S134p}$$
$$0 = \big(g_{11} - 1\big)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2\big(\eta' - \zeta\big)\big) + g_{10}\big(\sqrt{\zeta}g_{14}\psi + \zeta g_{11}\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S134q}$$
$$0 = \sqrt{\zeta}g_{12}g_{13}\sqrt{n_0}\phi\big(\sigma_{W_2}^2\big(\zeta - \eta'\big) - \gamma\big) + g_{14}\psi\big(\gamma\phi + \sigma_{W_2}^2\big(-\zeta\phi + \phi\eta' + \zeta g_{10}\big)\big) \tag{S134r}$$

$$0 = g_{13}\big(g_{10}\psi(\zeta-\eta) + \phi(\sqrt{\zeta}g_7\sqrt{n_0}-1)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big) + \phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S134s}$$

$$0 = g_6\psi\big(-\zeta g_2 + \sqrt{\zeta}(g_5-g_4)\sqrt{n_0}+g_3n_0\big) + \sqrt{\zeta}\sqrt{n_0}\sigma_{W_2}^2\big(g_7\big(\zeta g_9+\sqrt{\zeta}(g_5+g_8)\sqrt{n_0}+g_3n_0\big)-g_6\psi\big) \tag{S134t}$$

$$0 = g_{11}\psi\big(\gamma\phi+\sigma_{W_2}^2\big(\phi(\eta'-\zeta)+\zeta g_{10}\big)\big) - \phi\big(\sqrt{\zeta}g_7g_{13}\sqrt{n_0}+\psi\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S134u}$$

$$0 = g_{10}\big(-\sqrt{\zeta}g_{14}\psi - \sqrt{n_0}\big(\gamma+g_{13}(\eta-\zeta)+\sigma_{W_2}^2\big(\eta'+\zeta(g_{11}-1)\big)\big)\big) + \sqrt{n_0}\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S134v}$$

$$\begin{aligned}0 &= g_{14}\psi\big(-\zeta g_2+\sqrt{\zeta}(g_5-g_4)\sqrt{n_0}+g_3n_0\big) + \sqrt{\zeta}\sqrt{n_0}\sigma_{W_2}^2\big(g_{11}\big(\zeta g_9\\ &\quad + \sqrt{\zeta}(g_5+g_8)\sqrt{n_0}+g_3n_0\big) - g_{14}\psi\big)\end{aligned} \tag{S134w}$$

$$\begin{aligned}0 &= \sqrt{\zeta}g_6g_{13}\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta-\eta')-\gamma\big) - g_1\phi\big(\zeta g_9+\sqrt{\zeta}(g_5+g_8)\sqrt{n_0}+g_3n_0\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\\ &\quad + g_{14}\psi\big(\gamma\phi+\sigma_{W_2}^2\big(-\zeta\phi+\phi\eta'+\zeta g_{10}\big)\big)\end{aligned} \tag{S134x}$$

$$\begin{aligned}0 &= g_1\psi\phi\big(\sqrt{\zeta}g_9+g_8\sqrt{n_0}\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) + \sqrt{n_0}\big(\sigma_{W_2}^2\big(g_{10}g_{13}\psi(\eta-\zeta)+g_8\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big)\\ &\quad + g_6g_{13}\psi\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big)\end{aligned} \tag{S134y}$$

$$\begin{aligned}0 &= \sqrt{\zeta}g_8\sigma_{W_2}^2\big(g_{10}\psi(\eta-\zeta)-\phi\big(\sqrt{\zeta}g_7\sqrt{n_0}-1\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big) - g_3\sqrt{n_0}\phi\big(\sqrt{\zeta}g_7\sqrt{n_0}\sigma_{W_2}^2+g_6\psi\big)\\ &\quad \big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) + \sqrt{\zeta}g_4\psi\big(g_6\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)+g_{10}(\eta-\zeta)\sigma_{W_2}^2\big)\end{aligned} \tag{S134z}$$

$$\begin{aligned}0 &= \sqrt{\zeta}g_9\sigma_{W_2}^2\big(g_{10}\psi(\eta-\zeta)-\phi\big(\sqrt{\zeta}g_7\sqrt{n_0}-1\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big) - g_5\sqrt{n_0}\phi\big(\sqrt{\zeta}g_7\sqrt{n_0}\sigma_{W_2}^2+g_6\psi\big)\\ &\quad \big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) + \sqrt{\zeta}g_2\psi\big(g_6\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)+g_{10}(\eta-\zeta)\sigma_{W_2}^2\big)\end{aligned} \tag{S134aa}$$

After some straightforward algebra, one can eliminate all g_i except for g_7 and g_{10}, which satisfy coupled polynomial equations. Those equations can be shown to be identical to eqn. (S48) by invoking the change of variables,

$$g_7 = -\frac{\sqrt{\zeta}\psi}{\sqrt{n_0}\phi}\tau_2\,, \quad \text{and} \quad g_{10} = \big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\tau_1\,. \tag{S135}$$

The error E_{22} is then given by,

$$E_{22} = 2\zeta\big(\frac{\tau_2}{\tau_1}-1\big) + \frac{2\psi\big(\zeta(\tau_2-\tau_1)+\eta\tau_1\big)^2\big((\tau_2-\tau_1)\phi+\zeta\tau_1\tau_2\sigma_{W_2}^2\big)}{\zeta\tau_1^2\tau_2\phi}\,. \tag{S136}$$

S4.3.3. E_{31}

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_{31}$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}\,, \tag{S137}$$

where,

$$G_{12} = \begin{pmatrix}
g_5 & 0 & 0 & 0 & 0 & g_2 & 0 & 0\\
0 & g_6 & 0 & g_1 & g_3 & 0 & 0 & g_4\\
0 & 0 & g_8 & 0 & 0 & 0 & g_{12} & 0\\
0 & g_{11} & 0 & g_7 & g_{10} & 0 & 0 & g_9\\
0 & 0 & 0 & 0 & g_6 & 0 & 0 & g_1\\
0 & 0 & 0 & 0 & 0 & g_5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & g_8 & 0\\
0 & 0 & 0 & 0 & g_{11} & 0 & 0 & g_7
\end{pmatrix}\,, \tag{S138}$$

and the independent entry-wise component functions g_i give the error E_{31} through the relation,

$$E_{31} = \frac{g_2 n_0 \sigma_\varepsilon^2}{\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)}\,, \tag{S139}$$

and themselves satisfy the following system of polynomial equations,

$$0 = \sqrt{\zeta}g_6g_8\sqrt{n_0} - g_{11}\psi \tag{S140a}$$

$$0 = \sqrt{\zeta}g_1g_8\sqrt{n_0} - g_7\psi + \psi \tag{S140b}$$

$$0 = -\sqrt{\zeta} g_5 g_6 \psi - g_1 \sqrt{n_0} \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \tag{S140c}$$

$$0 = -\sqrt{\zeta} g_5 g_{11} \psi - (g_7 - 1)\sqrt{n_0} \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \tag{S140d}$$

$$0 = -\zeta g_7 g_8 \psi + \sqrt{\zeta}\sqrt{n_0}\big((g_4 g_8 + g_1 g_{12}) n_0 - \zeta g_1 g_8 \sigma_{W_2}^2\big) - g_9 n_0 \psi \tag{S140e}$$

$$0 = g_1 \sqrt{n_0} \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + g_5\big(\sqrt{\zeta} g_7 \psi + \zeta g_1 \sqrt{n_0} \sigma_{W_2}^2\big) \tag{S140f}$$

$$0 = \sqrt{\zeta} g_6 g_{12} n_0^{3/2} - g_8\big(\zeta g_{11} \psi + \sqrt{\zeta}\sqrt{n_0}\big(\zeta g_6 \sigma_{W_2}^2 - g_3 n_0\big)\big) - g_{10} n_0 \psi \tag{S140g}$$

$$0 = (g_6 - 1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + g_5\big(\sqrt{\zeta} g_{11} \psi + \zeta g_6 \sqrt{n_0} \sigma_{W_2}^2\big) \tag{S140h}$$

$$0 = \sqrt{\zeta} g_7 g_8 \sqrt{n_0} \phi\big(\sigma_{W_2}^2 (\zeta - \eta') - \gamma\big) + g_{11}\psi\big(\gamma\phi + \sigma_{W_2}^2\big(- \zeta\phi + \phi\eta' + \zeta g_5\big)\big) \tag{S140i}$$

$$0 = g_8\big(g_5 \psi(\zeta - \eta) + \phi\big(\sqrt{\zeta} g_1 \sqrt{n_0} - 1\big)\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big)\big) + \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \tag{S140j}$$

$$0 = g_6 \psi\big(\gamma\phi + \sigma_{W_2}^2\big(- \zeta\phi + \phi\eta' + \zeta g_5\big)\big) - \phi\big(\sqrt{\zeta} g_1 g_8 \sqrt{n_0} + \psi\big)\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \tag{S140k}$$

$$0 = g_5\big(\sqrt{\zeta} g_{11} \psi + \sqrt{n_0}\big(\gamma + g_8 (\eta - \zeta) + \sigma_{W_2}^2\big(\eta' + \zeta(g_6 - 1)\big)\big)\big) - \sqrt{n_0}\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \tag{S140l}$$

$$\begin{aligned}0 = {}& \sqrt{\zeta} g_5 \psi\big(g_6\big(\psi(\eta - \zeta) + \zeta\sigma_{W_2}^2\big) - g_3 n_0\big) - \sqrt{n_0}\big(\sqrt{\zeta} g_2 g_6 \sqrt{n_0} \psi + g_4 n_0 \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \\ &+ \zeta g_1 g_8 \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big)\big)\end{aligned} \tag{S140m}$$

$$\begin{aligned}0 = {}& \sqrt{\zeta} g_5 \psi\big(g_{11}\big(\psi(\eta - \zeta) + \zeta\sigma_{W_2}^2\big) - g_{10} n_0\big) - \sqrt{n_0}\big(\sqrt{\zeta} g_2 g_{11} \sqrt{n_0} \psi + g_9 n_0 \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \\ &+ \zeta g_7 g_8 \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big)\big)\end{aligned} \tag{S140n}$$

$$\begin{aligned}0 = {}& g_5\big(- \sqrt{\zeta} g_9 n_0 \psi + \zeta\sqrt{n_0}\sigma_{W_2}^2\big(\zeta g_1 \sigma_{W_2}^2 - g_4 n_0\big) + \sqrt{\zeta} g_7 \psi\big(\psi(\eta - \zeta) + \zeta\sigma_{W_2}^2\big)\big) \\ &- n_0\big(g_4 \sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + g_2\big(\sqrt{\zeta} g_7 \psi + \zeta g_1 \sqrt{n_0}\sigma_{W_2}^2\big)\big)\end{aligned} \tag{S140o}$$

$$\begin{aligned}0 = {}& g_5\big(- \sqrt{\zeta} g_{10} n_0 \psi + \zeta\sqrt{n_0}\sigma_{W_2}^2\big(\zeta g_6 \sigma_{W_2}^2 - g_3 n_0\big) + \sqrt{\zeta} g_{11} \psi\big(\psi(\eta - \zeta) + \zeta\sigma_{W_2}^2\big)\big) \\ &- n_0\big(g_3 \sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + g_2\big(\sqrt{\zeta} g_{11} \psi + \zeta g_6 \sqrt{n_0}\sigma_{W_2}^2\big)\big)\end{aligned} \tag{S140p}$$

$$\begin{aligned}0 = {}& g_2 g_8 n_0 \psi(\eta - \zeta) - g_5 \psi(\zeta - \eta)\big(g_8 \psi(\zeta - \eta) + g_{12} n_0\big) - \sqrt{n_0}\phi\big(g_{12}\big(\sqrt{\zeta} g_1 n_0 - \sqrt{n_0}\big) \\ &+ \sqrt{\zeta} g_8\big(g_4 n_0 - \zeta g_1 \sigma_{W_2}^2\big)\big)\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + \zeta g_7 g_8 \psi\phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big)\end{aligned} \tag{S140q}$$

$$\begin{aligned}0 = {}& g_2 n_0\big(- \sqrt{\zeta} g_{11} \psi - \sqrt{n_0}\big(\gamma + g_8 (\eta - \zeta) + \sigma_{W_2}^2\big(\eta' + \zeta(g_6 - 1)\big)\big)\big) + g_5\big(\sqrt{n_0}\big(g_8 \psi(\zeta - \eta)^2 \\ &+ g_{12} n_0 (\zeta - \eta) - \sqrt{\zeta} g_{10} \sqrt{n_0} \psi - \zeta g_3 n_0 \sigma_{W_2}^2 + \zeta^2 g_6 \sigma_{W_2}^4\big) + \sqrt{\zeta} g_{11} \psi\big(\psi(\eta - \zeta) + \zeta\sigma_{W_2}^2\big)\big)\end{aligned} \tag{S140r}$$

$$\begin{aligned}0 = {}& g_3 n_0 \psi\big(\gamma\phi + \sigma_{W_2}^2\big(\phi(\eta' - \zeta) + \zeta g_5\big)\big) - \sqrt{\zeta}\big(g_4 g_8 n_0^{3/2} \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \\ &+ g_1 \sqrt{n_0}\phi\big(g_{12} n_0 - \zeta g_8 \sigma_{W_2}^2\big)\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + \sqrt{\zeta} g_6 \psi \sigma_{W_2}^2\big(\zeta g_5 \sigma_{W_2}^2 - g_2 n_0\big)\big)\end{aligned} \tag{S140s}$$

$$\begin{aligned}0 = {}& g_{10} n_0 \psi\big(\gamma\phi + \sigma_{W_2}^2\big(\phi(\eta' - \zeta) + \zeta g_5\big)\big) - \sqrt{\zeta}\big(g_7 g_{12} n_0^{3/2} \phi\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) \\ &+ g_8 \sqrt{n_0}\phi\big(g_9 n_0 - \zeta g_7 \sigma_{W_2}^2\big)\big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big) + \sqrt{\zeta} g_{11} \psi \sigma_{W_2}^2\big(\zeta g_5 \sigma_{W_2}^2 - g_2 n_0\big)\big)\end{aligned} \tag{S140t}$$

After some straightforward algebra, one can eliminate all g_i except for g_1 and g_5, which satisfy coupled polynomial equations. Those equations can be shown to be identical to eqn. (S48) by invoking the change of variables,

$$g_1 = -\frac{\sqrt{\zeta}\psi}{\sqrt{n_0}\phi}\tau_2\,, \quad \text{and} \quad g_5 = \big(\gamma + \sigma_{W_2}^2 (\eta' - \zeta)\big)\tau_1\,. \tag{S141}$$

The error E_{31} can then be written in terms of τ_1 and its derivative τ_1' (S69),

$$E_{31} = \sigma_\varepsilon^2\big(- \tau_1'/\tau_1^2 - 1\big)\,. \tag{S142}$$

S4.3.4. E_{32}

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_{32}$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}\,, \tag{S143}$$

where,

$$G_{12} = \begin{pmatrix} g_9 & 0 & 0 & 0 & 0 & 0 & g_6 & 0 & 0 \\ 0 & g_1 & g_3 & 0 & g_4 & g_7 & 0 & 0 & g_2 \\ 0 & 0 & g_{10} & 0 & g_4 & g_{13} & 0 & 0 & g_5 \\ 0 & 0 & 0 & g_{12} & 0 & 0 & 0 & g_{16} & 0 \\ 0 & 0 & g_{15} & 0 & g_{11} & g_{14} & 0 & 0 & g_8 \\ 0 & 0 & 0 & 0 & 0 & g_{10} & 0 & 0 & g_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{15} & 0 & 0 & g_{11} \end{pmatrix}, \tag{S144}$$

and the independent entry-wise component functions g_i give the error E_{32} through the relation,

$$E_{32} = -g_2 n_0^{3/2}/(\sqrt{\zeta}\psi)\,, \tag{S145}$$

and themselves satisfy the following system of polynomial equations,

$$0 = \sqrt{\zeta} g_{10} g_{12} \sqrt{n_0} - g_{15}\psi \tag{S146a}$$
$$0 = \sqrt{\zeta} g_4 g_{12} \sqrt{n_0} - g_{11}\psi + \psi \tag{S146b}$$
$$0 = -\sqrt{\zeta} g_9 g_{10}\psi - g_4\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146c}$$
$$0 = -\sqrt{\zeta} g_9 g_{15}\psi - (g_{11}-1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146d}$$
$$0 = -\sqrt{\zeta} g_9\psi - \sqrt{\zeta} g_3 g_9\psi - g_4\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146e}$$
$$0 = -\sqrt{\zeta} g_6 g_{10}\psi - \sqrt{\zeta} g_9 g_{13}\psi - g_5\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146f}$$
$$0 = -\sqrt{\zeta} g_9 g_{14}\psi - \sqrt{\zeta} g_6 g_{15}\psi - g_8\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146g}$$
$$0 = \sqrt{\zeta} g_5 g_{12} n_0 + \sqrt{\zeta} g_4\big(g_{16} n_0 + g_{12}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2)\big) + g_8\sqrt{n_0}(-\psi) \tag{S146h}$$
$$0 = g_4\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_9\big(\sqrt{\zeta} g_{11}\psi + \zeta g_4\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S146i}$$
$$0 = g_3\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_9\big(\sqrt{\zeta} g_{15}\psi + \zeta g_{10}\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S146j}$$
$$0 = \sqrt{\zeta} g_{12} g_{13} n_0 + \sqrt{\zeta} g_{10}\big(g_{16} n_0 + g_{12}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2)\big) + g_{14}\sqrt{n_0}(-\psi) \tag{S146k}$$
$$0 = (g_{10}-1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_9\big(\sqrt{\zeta} g_{15}\psi + \zeta g_{10}\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S146l}$$
$$0 = -\sqrt{\zeta}\big((g_1+g_3)g_6 + g_7 g_9\big)\psi - \gamma g_2\sqrt{n_0}\phi + \zeta g_2\sqrt{n_0}\phi\sigma_{W_2}^2 + g_2\sqrt{n_0}(-\phi)\eta'\sigma_{W_2}^2 \tag{S146m}$$
$$0 = \sqrt{\zeta} g_{11} g_{12}\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta - \eta') - \gamma\big) + g_{15}\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_9)\big) \tag{S146n}$$
$$0 = g_{12}\Big(g_9\psi(\zeta-\eta) + \phi\big(\sqrt{\zeta} g_4\sqrt{n_0} - 1\big)\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\Big) + \phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146o}$$
$$0 = g_{10}\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_9)\big) - \phi\big(\sqrt{\zeta} g_4 g_{12}\sqrt{n_0} + \psi\big)\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146p}$$
$$0 = g_9\Big(\sqrt{\zeta} g_{15}\psi + \sqrt{n_0}\big(\gamma + g_{12}(\eta-\zeta) + \sigma_{W_2}^2(\eta' + \zeta(g_{10}-1))\big)\Big) - \sqrt{n_0}\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S146q}$$
$$0 = -\sqrt{\zeta} g_4 g_{12}\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_3\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_9)\big) + \zeta g_9\psi\sigma_{W_2}^2 \tag{S146r}$$
$$0 = g_7 n_0\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_6\big(\sqrt{\zeta} g_{15}\sqrt{n_0}\psi + \zeta g_{10} n_0\sigma_{W_2}^2\big) + g_9\big(\sqrt{\zeta} g_{14}\sqrt{n_0}\psi + \zeta\sigma_{W_2}^2(g_{13} n_0 - \zeta g_{10}\sigma_{W_2}^2)\big) \tag{S146s}$$
$$\begin{aligned} 0 = {}& \gamma g_2 n_0\phi + \sqrt{\zeta} g_8 g_9\sqrt{n_0}\psi + g_6\big(\sqrt{\zeta} g_{11}\sqrt{n_0}\psi + \zeta g_4 n_0\sigma_{W_2}^2\big) - \zeta g_2 n_0\phi\sigma_{W_2}^2 + \zeta g_5 g_9 n_0\sigma_{W_2}^2 \\ & + g_2 n_0\phi\eta'\sigma_{W_2}^2 - \zeta^2 g_4 g_9\sigma_{W_2}^4 \end{aligned} \tag{S146t}$$
$$\begin{aligned} 0 = {}& g_6\Big(-\sqrt{\zeta} g_{15}\sqrt{n_0}\psi - n_0\big(\gamma + g_{12}(\eta-\zeta) + \sigma_{W_2}^2(\eta' + \zeta(g_{10}-1))\big)\Big) + g_9\big(g_{12}\psi(\zeta-\eta)^2 \\ & + g_{16} n_0(\zeta-\eta) - \sqrt{\zeta} g_{14}\sqrt{n_0}\psi - \zeta g_{13} n_0\sigma_{W_2}^2 + \zeta^2 g_{10}\sigma_{W_2}^4\big) \end{aligned} \tag{S146u}$$
$$\begin{aligned} 0 = {}& \gamma g_5 n_0\phi + \sqrt{\zeta} g_8 g_9\sqrt{n_0}\psi + \sqrt{\zeta} g_6 g_{11}\sqrt{n_0}\psi + \zeta g_4\big(g_6 n_0\sigma_{W_2}^2 + g_{12}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) - \zeta g_9\sigma_{W_2}^4\big) \\ & - \zeta g_5 n_0\phi\sigma_{W_2}^2 + \zeta g_5 g_9 n_0\sigma_{W_2}^2 + g_5 n_0\phi\eta'\sigma_{W_2}^2 \end{aligned} \tag{S146v}$$
$$\begin{aligned} 0 = {}& \gamma g_{13} n_0\phi + \sqrt{\zeta} g_6 g_{15}\sqrt{n_0}\psi + \zeta g_{10}\big(g_6 n_0\sigma_{W_2}^2 + g_{12}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) - \zeta g_9\sigma_{W_2}^4\big) \\ & + g_9\big(\sqrt{\zeta} g_{14}\sqrt{n_0}\psi + \zeta g_{13} n_0\sigma_{W_2}^2\big) - \zeta g_{13} n_0\phi\sigma_{W_2}^2 + g_{13} n_0\phi\eta'\sigma_{W_2}^2 \end{aligned} \tag{S146w}$$
$$\begin{aligned} 0 = {}& -\sqrt{\zeta} g_{12}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\big(\sqrt{n_0}\big(g_8 n_0 + g_{11}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2)\big) - \sqrt{\zeta} g_{15}\psi\big) \\ & + g_{14} n_0\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_9)\big) - \sqrt{\zeta} g_{11} g_{16} n_0^{3/2}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + \zeta g_{15}\psi\sigma_{W_2}^2\big(g_6 n_0 - \zeta g_9\sigma_{W_2}^2\big) \end{aligned} \tag{S146x}$$

$$0 = g_9\psi(-(\zeta-\eta))\big(g_{12}\psi(\zeta-\eta)+g_{16}n_0\big) - \sqrt{\zeta}g_4\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big(g_{16}n_0+g_{12}(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2)\big) + n_0\big(g_6g_{12}\psi(\eta-\zeta)+\phi\big(g_{16}-\sqrt{\zeta}g_5g_{12}\sqrt{n_0}\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big)+\zeta g_{10}g_{12}\psi\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S146y}$$

$$0 = g_{13}n_0\psi\big(\gamma\phi+\sigma_{W_2}^2\big(-\zeta\phi+\phi\eta'+\zeta g_9\big)\big) - \sqrt{\zeta}g_4\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)+\sqrt{\zeta}g_5g_{12}n_0^{3/2}\phi\big(\sigma_{W_2}^2(\zeta-\eta')-\gamma\big) \big(g_{16}n_0+g_{12}(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2)\big)+\zeta g_{10}\psi\big(g_6n_0\sigma_{W_2}^2+g_{12}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)-\zeta g_9\sigma_{W_2}^4\big) \tag{S146z}$$

$$0 = -\gamma\sqrt{\zeta}g_2g_{12}n_0^{3/2}\phi+\gamma g_7n_0\psi\phi-\sqrt{\zeta}g_4\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\big(g_{16}n_0+g_{12}(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2)\big) + \zeta g_3\psi\big(g_6n_0\sigma_{W_2}^2+g_{12}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)-\zeta g_9\sigma_{W_2}^4\big)+\zeta^{3/2}g_2g_{12}n_0^{3/2}\phi\sigma_{W_2}^2-\zeta^2g_9\psi\sigma_{W_2}^4 + n_0\phi\eta'\sigma_{W_2}^2\big(g_7\psi-\sqrt{\zeta}g_2g_{12}\sqrt{n_0}\big)+\zeta g_6n_0\psi\sigma_{W_2}^2+\zeta g_7g_9n_0\psi\sigma_{W_2}^2-\zeta g_7n_0\psi\phi\sigma_{W_2}^2 \tag{S146aa}$$

After some straightforward algebra, one can eliminate all g_i except for g_4 and g_9, which satisfy coupled polynomial equations. Those equations can be shown to be identical to eqn. (S48) by invoking the change of variables,

$$g_4 = -\frac{\sqrt{\zeta}\psi}{\sqrt{n_0}\phi}\tau_2\,,\quad \text{and}\quad g_9 = \big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big)\tau_1\,. \tag{S147}$$

In terms of τ_1, τ_2, and τ_2' (S70), the error E_{32} is given by,

$$E_{32} = 1-2\tau_2/\tau_1-\tau_2'/\tau_1^2\,. \tag{S148}$$

S4.3.5. E_{33}

The equations satisfied by the operator-valued Stieltjes transform G of $\bar{Q}_{32}$ induce the following structure on G,

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{12}^\top & 0 \end{pmatrix}\,, \tag{S149}$$

where,

$$G_{12} = \begin{pmatrix}
g_{13} & 0 & 0 & 0 & 0 & 0 & 0 & g_8 & 0 & 0 & 0\\
0 & g_1 & 0 & 0 & g_5 & 0 & 0 & 0 & g_{11} & g_3 & 0\\
0 & 0 & g_1 & g_4 & 0 & g_6 & g_9 & 0 & 0 & 0 & g_2\\
0 & 0 & 0 & g_{14} & 0 & g_6 & g_{17} & 0 & 0 & 0 & g_7\\
0 & 0 & 0 & 0 & g_{16} & 0 & 0 & 0 & g_{20} & g_{12} & 0\\
0 & 0 & 0 & g_{19} & 0 & g_{15} & g_{18} & 0 & 0 & 0 & g_{10}\\
0 & 0 & 0 & 0 & 0 & 0 & g_{14} & 0 & 0 & 0 & g_6\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{13} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{16} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & g_{19} & 0 & 0 & 0 & g_{15}
\end{pmatrix}, \tag{S150}$$

and the independent entry-wise component functions g_i give the error E_{32} through the relation,

$$E_{33} = -g_3n_0\sigma_{W_2}^2/\psi\,, \tag{S151}$$

and themselves satisfy the following system of polynomial equations,

$$0 = \sqrt{\zeta}g_{14}g_{16}\sqrt{n_0}-g_{19}\psi \tag{S152a}$$

$$0 = \sqrt{\zeta}g_6g_{16}\sqrt{n_0}-g_{15}\psi+\psi \tag{S152b}$$

$$0 = -\sqrt{\zeta}g_{13}g_{14}\psi-g_6\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152c}$$

$$0 = -\sqrt{\zeta}g_{13}g_{19}\psi-\big(g_{15}-1\big)\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152d}$$

$$0 = -\sqrt{\zeta}g_{13}\psi-\sqrt{\zeta}g_4g_{13}\psi-g_6\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152e}$$

$$0 = -\sqrt{\zeta}g_8g_{14}\psi-\sqrt{\zeta}g_{13}g_{17}\psi-g_7\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152f}$$

$$0 = -\sqrt{\zeta}g_{13}g_{18}\psi-\sqrt{\zeta}g_8g_{19}\psi-g_{10}\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152g}$$

$$0 = g_{13}g_{16}\psi(\zeta-\eta)-\phi\big(g_5-\sqrt{\zeta}g_6g_{16}\sqrt{n_0}\big)\big(\gamma+\sigma_{W_2}^2(\eta'-\zeta)\big) \tag{S152h}$$

$$0 = g_6\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_{13}\big(\sqrt{\zeta}g_{15}\psi + \zeta g_6\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S152i}$$

$$0 = g_4\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_{13}\big(\sqrt{\zeta}g_{19}\psi + \zeta g_{14}\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S152j}$$

$$0 = (g_{14} - 1)\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_{13}\big(\sqrt{\zeta}g_{19}\psi + \zeta g_{14}\sqrt{n_0}\sigma_{W_2}^2\big) \tag{S152k}$$

$$0 = -\sqrt{\zeta}\big((g_4 + 1)g_8 + g_9 g_{13}\big)\psi - \gamma g_2\sqrt{n_0}\phi + \zeta g_2\sqrt{n_0}\phi\sigma_{W_2}^2 + g_2\sqrt{n_0}(-\phi)\eta'\sigma_{W_2}^2 \tag{S152l}$$

$$0 = \sqrt{\zeta}g_{15}g_{16}\sqrt{n_0}\phi\big(\sigma_{W_2}^2(\zeta - \eta') - \gamma\big) + g_{19}\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_{13})\big) \tag{S152m}$$

$$0 = g_{16}\big(g_{13}\psi(\zeta - \eta) + \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) + \phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S152n}$$

$$0 = g_{13}\big(\sqrt{\zeta}g_{19}\psi + \sqrt{n_0}(\gamma + g_{16}(\eta - \zeta) + \sigma_{W_2}^2(\eta' + \zeta(g_{14} - 1)))\big) - \sqrt{n_0}\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S152o}$$

$$0 = g_{14}\psi\big(\gamma\phi + \sigma_{W_2}^2(\phi(\eta' - \zeta) + \zeta g_{13})\big) - \phi\big(\sqrt{\zeta}g_6 g_{16}\sqrt{n_0} + \psi\big)\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S152p}$$

$$0 = -\sqrt{\zeta}g_6 g_{16}\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_4\psi\big(\gamma\phi + \sigma_{W_2}^2(-\zeta\phi + \phi\eta' + \zeta g_{13})\big) + \zeta g_{13}\psi\sigma_{W_2}^2 \tag{S152q}$$

$$0 = \sqrt{\zeta}\big(g_7 g_{16} + g_6(g_{12} + g_{20})\big)n_0 + g_{10}\sqrt{n_0}(-\psi) + \sqrt{\zeta}g_6\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big) + \sqrt{\zeta}g_5 g_6\big(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2\big) \tag{S152r}$$

$$0 = \sqrt{\zeta}\big(g_{16}g_{17} + g_{14}(g_{12} + g_{20})\big)n_0 + g_{18}\sqrt{n_0}(-\psi) + \sqrt{\zeta}g_{14}\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big) + \sqrt{\zeta}g_5 g_{14}\big(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2\big) \tag{S152s}$$

$$0 = g_{13}\psi(\zeta - \eta) + g_5\big(g_{13}\psi(\zeta - \eta) + \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) + \sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \tag{S152t}$$

$$0 = g_9 n_0\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + g_8\big(\sqrt{\zeta}g_{19}\sqrt{n_0}\psi + \zeta g_{14}n_0\sigma_{W_2}^2\big) + g_{13}\big(\sqrt{\zeta}g_{18}\sqrt{n_0}\psi + \zeta\sigma_{W_2}^2(g_{17}n_0 - \zeta g_{14}\sigma_{W_2}^2)\big) \tag{S152u}$$

$$\begin{aligned}0 = {} & \gamma g_2 n_0\phi + \sqrt{\zeta}g_{10}g_{13}\sqrt{n_0}\psi + g_8\big(\sqrt{\zeta}g_{15}\sqrt{n_0}\psi + \zeta g_6 n_0\sigma_{W_2}^2\big) - \zeta g_2 n_0\phi\sigma_{W_2}^2 \\ & + \zeta g_7 g_{13}n_0\sigma_{W_2}^2 + g_2 n_0\phi\eta'\sigma_{W_2}^2 - \zeta^2 g_6 g_{13}\sigma_{W_2}^4\end{aligned} \tag{S152v}$$

$$\begin{aligned}0 = {} & g_{13}g_{16}\psi(-(\zeta - \eta))\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big) \\ & - \phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\big(-\zeta g_{14}g_{16}\psi + \sqrt{\zeta}g_6 g_{16}\sqrt{n_0}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2) - g_{20}n_0\big)\end{aligned} \tag{S152w}$$

$$\begin{aligned}0 = {} & -\sqrt{\zeta}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\big(g_6\sqrt{n_0}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2) - \sqrt{\zeta}g_{14}\psi\big) + g_{20}n_0\big(g_{13}\psi(\eta - \zeta) \\ & - \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) + g_{13}\psi(-(\zeta - \eta))\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big)\end{aligned} \tag{S152x}$$

$$\begin{aligned}0 = {} & \big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big)\big(g_{13}\psi(\eta - \zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi(\sigma_{W_2}^2(\zeta - \eta') - \gamma)\big) + n_0\big(g_{13}g_{20}\psi(\eta - \zeta) \\ & + \phi(g_{11} - \sqrt{\zeta}g_6 g_{20}\sqrt{n_0})(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) + \zeta g_4\psi\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\end{aligned} \tag{S152y}$$

$$\begin{aligned}0 = {} & \gamma g_7 n_0\phi + \sqrt{\zeta}g_{10}g_{13}\sqrt{n_0}\psi + \sqrt{\zeta}g_8 g_{15}\sqrt{n_0}\psi - \zeta g_7 n_0\phi\sigma_{W_2}^2 + \zeta g_6 g_8 n_0\sigma_{W_2}^2 + \zeta g_7 g_{13}n_0\sigma_{W_2}^2 + g_7 n_0\phi\eta'\sigma_{W_2}^2 \\ & + \zeta g_6\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + \zeta g_5 g_6\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) - \zeta^2 g_6 g_{13}\sigma_{W_2}^4\end{aligned} \tag{S152z}$$

$$\begin{aligned}0 = {} & \gamma g_{17}n_0\phi + \sqrt{\zeta}g_{13}g_{18}\sqrt{n_0}\psi + \sqrt{\zeta}g_8 g_{19}\sqrt{n_0}\psi - \zeta g_{17}n_0\phi\sigma_{W_2}^2 + \zeta g_8 g_{14}n_0\sigma_{W_2}^2 + \zeta g_{13}g_{17}n_0\sigma_{W_2}^2 + g_{17}n_0\phi\eta'\sigma_{W_2}^2 \\ & + \zeta g_{14}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) + \zeta g_5 g_{14}\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) - \zeta^2 g_{13}g_{14}\sigma_{W_2}^4\end{aligned} \tag{S152aa}$$

$$\begin{aligned}0 = {} & g_5\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big)\big(g_{13}\psi(\eta - \zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi(\sigma_{W_2}^2(\zeta - \eta') - \gamma)\big) + n_0\big(\phi(g_3 - \sqrt{\zeta}(g_6 g_{12} + g_2 g_{16})\sqrt{n_0}) \\ & (\gamma + \sigma_{W_2}^2(\eta' - \zeta)) - (g_{12}g_{13} + g_8 g_{16})\psi(\zeta - \eta)\big) + \zeta g_4 g_5\psi\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\end{aligned} \tag{S152ab}$$

$$\begin{aligned}0 = {} & \big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big)\big(g_{13}\psi(\eta - \zeta) + \sqrt{\zeta}g_6\sqrt{n_0}\phi(\sigma_{W_2}^2(\zeta - \eta') - \gamma)\big) + g_5\big(g_{13}\psi(-(\zeta - \eta))(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2) \\ & - \sqrt{\zeta}\phi(\gamma + \sigma_{W_2}^2(\eta' - \zeta))(g_6\sqrt{n_0}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2) - \sqrt{\zeta}g_{14}\psi)\big) + g_{11}n_0\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big) \\ & + \zeta g_4\psi\phi\big(\gamma + \sigma_{W_2}^2(\eta' - \zeta)\big)\end{aligned} \tag{S152ac}$$

$$\begin{aligned}0 = {} & g_{12}n_0\big(g_{13}\psi(\eta - \zeta) - \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) - g_{16}\big(g_8 n_0\psi(\zeta - \eta) + \sqrt{\zeta}g_7 n_0^{3/2}\phi(\gamma + \sigma_{W_2}^2(\eta' - \zeta)) \\ & + \zeta g_{13}\psi(\zeta - \eta)\sigma_{W_2}^2\big) + g_5\big(g_{13}\psi(-(\zeta - \eta))(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2) \\ & - \sqrt{\zeta}\phi(\gamma + \sigma_{W_2}^2(\eta' - \zeta))(g_6\sqrt{n_0}(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2) - \sqrt{\zeta}g_{14}\psi)\big)\end{aligned} \tag{S152ad}$$

$$\begin{aligned}0 = {} & \gamma\sqrt{\zeta}g_7 g_{16}n_0^{3/2}\phi + \gamma\sqrt{\zeta}g_6 g_{20}n_0^{3/2}\phi + g_8 g_{16}n_0\psi(\zeta - \eta) + \zeta g_{13}g_{20}n_0\psi - \eta g_{13}g_{20}n_0\psi \\ & + g_{12}n_0\big(g_{13}\psi(\zeta - \eta) + \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) - \zeta^{3/2}g_7 g_{16}n_0^{3/2}\phi\sigma_{W_2}^2 - \zeta^{3/2}g_6 g_{20}n_0^{3/2}\phi\sigma_{W_2}^2 \\ & + \sqrt{\zeta}(g_7 g_{16} + g_6 g_{20})n_0^{3/2}\phi\eta'\sigma_{W_2}^2 + \zeta^2 g_{13}g_{16}\psi\sigma_{W_2}^2 - \zeta\eta g_{13}g_{16}\psi\sigma_{W_2}^2\end{aligned} \tag{S152ae}$$

$$\begin{aligned}0 = {} & -\gamma g_8 n_0 - \sqrt{\zeta}g_{13}g_{18}\sqrt{n_0}\psi - \sqrt{\zeta}g_8 g_{19}\sqrt{n_0}\psi + \zeta g_{12}g_{13}n_0 + \zeta g_8 g_{16}n_0 + \zeta g_{13}g_{20}n_0 - \eta g_{12}g_{13}n_0 \\ & - \eta g_8 g_{16}n_0 - \eta g_{13}g_{20}n_0 + \zeta g_8 n_0\sigma_{W_2}^2 - \zeta g_8 g_{14}n_0\sigma_{W_2}^2 - \zeta g_{13}g_{17}n_0\sigma_{W_2}^2 - g_8 n_0\eta'\sigma_{W_2}^2 + \zeta^2 g_{13}g_{14}\sigma_{W_2}^4 \\ & + \zeta^2 g_{13}g_{16}\sigma_{W_2}^2 + g_{13}(\zeta - \eta)\big(\psi(\zeta - \eta) - \zeta\sigma_{W_2}^2\big) + g_5 g_{13}(\zeta - \eta)\big(\zeta\psi - \eta\psi - \zeta\sigma_{W_2}^2\big) - \zeta\eta g_{13}g_{16}\sigma_{W_2}^2\end{aligned} \tag{S152af}$$

$$\begin{aligned}0 = {} & \gamma\sqrt{\zeta}g_5 g_7 n_0^{3/2}\phi + \gamma\sqrt{\zeta}g_6 g_{11}n_0^{3/2}\phi + \zeta g_5 g_8 n_0\psi + \zeta g_{11}g_{13}n_0\psi - \eta g_5 g_8 n_0\psi - \eta g_{11}g_{13}n_0\psi \\ & + g_3 n_0\big(g_{13}\psi(\zeta - \eta) + \phi(\sqrt{\zeta}g_6\sqrt{n_0} - 1)(\gamma + \sigma_{W_2}^2(\eta' - \zeta))\big) + n_0\big(g_8\psi(\zeta - \eta)\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\zeta}g_2\sqrt{n_0}\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\big)-\zeta^{3/2}g_5g_7n_0^{3/2}\phi\sigma_{W_2}^2-\zeta^{3/2}g_6g_{11}n_0^{3/2}\phi\sigma_{W_2}^2+\sqrt{\zeta}g_5g_7n_0^{3/2}\phi\eta'\sigma_{W_2}^2\\
&+\sqrt{\zeta}g_6g_{11}n_0^{3/2}\phi\eta'\sigma_{W_2}^2+\zeta^2g_5g_{13}\psi\sigma_{W_2}^2-\zeta\eta g_5g_{13}\psi\sigma_{W_2}^2 \qquad \text{(S152ag)}
\end{aligned}
$$

$$
\begin{aligned}
0=&-\sqrt{\zeta}g_6\sqrt{n_0}\phi\big(\psi(\zeta-\eta)-\zeta\sigma_{W_2}^2\big)\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)-\sqrt{\zeta}g_5g_6\sqrt{n_0}\phi\big(\psi(\zeta-\eta)-\zeta\sigma_{W_2}^2\big)\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\\
&+g_9n_0\psi\big(\gamma\phi+\sigma_{W_2}^2\big(\phi\big(\eta'-\zeta\big)+\zeta g_{13}\big)\big)+\zeta g_4\psi\big(g_8n_0\sigma_{W_2}^2+g_5\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)-\zeta g_{13}\sigma_{W_2}^4\big)\\
&-\sqrt{\zeta}\big(g_2g_{16}+g_6\big(g_{12}+g_{20}\big)\big)n_0^{3/2}\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)+\zeta\psi\sigma_{W_2}^2\big(g_8n_0-\zeta g_{13}\sigma_{W_2}^2\big)\\
&+\zeta g_4\psi\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big) \qquad \text{(S152ah)}
\end{aligned}
$$

$$
\begin{aligned}
0=&-\gamma\sqrt{\zeta}g_6g_{12}n_0^{3/2}\phi-\gamma\sqrt{\zeta}g_7g_{16}n_0^{3/2}\phi-\gamma\sqrt{\zeta}g_6g_{20}n_0^{3/2}\phi+\gamma g_{17}n_0\psi\phi-\sqrt{\zeta}\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\\
&\big(g_6\sqrt{n_0}\big(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2\big)-\sqrt{\zeta}g_{14}\psi\big)-\sqrt{\zeta}g_5\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\big(g_6\sqrt{n_0}\big(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2\big)-\sqrt{\zeta}g_{14}\psi\big)\\
&+\zeta^{3/2}g_6g_{12}n_0^{3/2}\phi\sigma_{W_2}^2+\zeta^{3/2}g_7g_{16}n_0^{3/2}\phi\sigma_{W_2}^2+\zeta^{3/2}g_6g_{20}n_0^{3/2}\phi\sigma_{W_2}^2-n_0\phi\eta'\sigma_{W_2}^2\big(\sqrt{\zeta}\big(g_7g_{16}+g_6\big(g_{12}+g_{20}\big)\big)\sqrt{n_0}\\
&-g_{17}\psi\big)+\zeta g_8g_{14}n_0\psi\sigma_{W_2}^2+\zeta g_{13}g_{17}n_0\psi\sigma_{W_2}^2-\zeta g_{17}n_0\psi\phi\sigma_{W_2}^2-\zeta^2g_{13}g_{14}\psi\sigma_{W_2}^4 \qquad \text{(S152ai)}
\end{aligned}
$$

$$
\begin{aligned}
0=&-\gamma\sqrt{\zeta}g_{12}g_{15}n_0^{3/2}\phi-\gamma\sqrt{\zeta}g_{10}g_{16}n_0^{3/2}\phi-\gamma\sqrt{\zeta}g_{15}g_{20}n_0^{3/2}+\gamma g_{18}n_0\psi\phi-\sqrt{\zeta}\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\\
&\big(g_{15}\sqrt{n_0}\big(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2\big)-\sqrt{\zeta}g_{19}\psi\big)-\sqrt{\zeta}g_5\phi\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\big(g_{15}\sqrt{n_0}\big(\zeta\psi-\eta\psi-\zeta\sigma_{W_2}^2\big)-\zeta g_{18}n_0\psi\phi\sigma_{W_2}^2\\
&-\sqrt{\zeta}g_{19}\psi\big)+\zeta^{3/2}g_{12}g_{15}n_0^{3/2}\phi\sigma_{W_2}^2+\zeta^{3/2}g_{10}g_{16}n_0^{3/2}\phi\sigma_{W_2}^2+\zeta^{3/2}g_{15}g_{20}n_0^{3/2}\phi\sigma_{W_2}^2-\zeta^2g_{13}g_{19}\psi\sigma_{W_2}^4\\
&-n_0\phi\eta'\sigma_{W_2}^2\big(\sqrt{\zeta}\big(g_{10}g_{16}+g_{15}\big(g_{12}+g_{20}\big)\big)\sqrt{n_0}-g_{18}\psi\big)+\zeta g_{13}g_{18}n_0\psi\sigma_{W_2}^2+\zeta g_8g_{19}n_0\psi\sigma_{W_2}^2 \qquad \text{(S152aj)}
\end{aligned}
$$

After some straightforward algebra, one can eliminate all g_i except for g_6 and g_{13}, which satisfy coupled polynomial equations. Those equations can be shown to be identical to eqn. (S48) by invoking the change of variables,

$$g_6=-\frac{\sqrt{\zeta}\psi}{\sqrt{n_0}\phi}\tau_2\,,\quad \text{and}\quad g_{13}=\big(\gamma+\sigma_{W_2}^2\big(\eta'-\zeta\big)\big)\tau_1\,. \qquad \text{(S153)}$$

In terms of τ_1, τ_2, and their derivatives τ_1' (S69), τ_2' (S70), the error E_{33} is given by,

$$E_{33}=\sigma_{W_2}^2\left[\left(\tau_1+(\sigma_{W_2}^2(\eta'-\zeta)+\gamma)\tau_1'+\sigma_{W_2}^2\zeta\tau_2'\right)/\tau_1^2-\eta\right]-E_{22}\,. \qquad \text{(S154)}$$

S4.4. Total test error

Recall from eqns. (S72, S91-S93) that the total test error can be written as

$$E_{\text{test}}=1+E_{21}+E_{31}+E_{32}+\nu\big(\eta\sigma_{W_2}^2+E_{22}+E_{33}\big)\,, \qquad \text{(S155)}$$

where $\nu=0$ with centering and $\nu=1$ without it. Combining the results from the previous subsections, we find

$$
\begin{aligned}
E_{\text{test}}&=1+2(\tau_2/\tau_1-1)+\sigma_\varepsilon^2\big(\tau_1'/\tau_1^2-1\big)+1-2\tau_2/\tau_1+\tau_2'/\tau_1^2 && \text{(S156)}\\
&\quad+\nu\sigma_{W_2}^2\left[\left(\tau_1+(\sigma_{W_2}^2(\eta'-\zeta)+\gamma)\tau_1'+\sigma_{W_2}^2\zeta\tau_2'\right)/\tau_1^2\right] && \text{(S157)}\\
&=\tau_2'/\tau_1^2+\sigma_\varepsilon^2\tau_1'/\tau_1^2+\nu\sigma_{W_2}^2\left[\left(\tau_1+(\sigma_{W_2}^2(\eta'-\zeta)+\gamma)\tau_1'+\sigma_{W_2}^2\zeta\tau_2'\right)/\tau_1^2\right]-\sigma_\varepsilon^2 && \text{(S158)}\\
&=(\gamma\tau_1)^{-2}E_{\text{train}}-\sigma_\varepsilon^2\,, && \text{(S159)}
\end{aligned}
$$

thereby establishing the result of the main theorem (27).