Logarithmic Regret for Learning Linear Quadratic Regulators Efficiently

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Abstract

We consider the problem of learning in Linear Quadratic Control systems whose transition parameters are initially unknown. Recent results in this setting have demonstrated efficient learning algorithms with regret growing with the square root of the number of decision steps. We present new efficient algorithms that achieve, perhaps surprisingly, regret that scales only (poly)logarithmically with the number of steps in two scenarios: when only the state transition matrix $A$ is unknown, and when only the state-action transition matrix $B$ is unknown and the optimal policy satisfies a certain non-degeneracy condition. On the other hand, we give a lower bound that shows that when the latter condition is violated, square root regret is unavoidable.

1. Introduction

The linear-quadratic regulator model (LQR) is a classic model in optimal control theory. In this model, the dynamics of the environment are given as

$$x_{t+1} = A_x x_t + B_x u_t + w_t,$$

where $x_t$ and $u_t$ are the state and the action vectors at time $t$. $A_x$ and $B_x$ are transition matrices, and $w_t$ is a zero-mean i.i.d. Gaussian noise. The cost function is quadratic in both the state and the action. An interesting property of LQR systems is that a linear control policy minimizes the cost while keeping the system at a steady-state (stable) position.

In this work, we study the problem of designing an adaptive controller that regulates the system while learning its parameters. This problem has recently been approached through the lens of regret minimization, beginning in the work of Abbasi-Yadkori and Szepesvári (2011) that established an $O(\sqrt{T})$ regret bound for this setting albeit with a computationally inefficient algorithm. The problem of designing an efficient algorithm that enjoys $O(\sqrt{T})$ was later resolved by Cohen et al. (2019) and Mania et al. (2019). The former work relied on the “optimism in the face of uncertainty” principle and a reduction to an online semi-definite problem, and the latter work used a simpler greedy strategy.

Following this line of work, it has been believed that an $O(\sqrt{T})$ regret is tight for the problem. This appears natural as it is the typical rate for many imperfect information (bandit) optimization problems (e.g., Shamir, 2013). On the other hand, one could suspect that better, polylogarithmic regret bounds, are possible in the LQR setting thanks to the strongly convex structure of the cost functions. Often in optimization, this structure gives rise to faster convergence/regret rates, and indeed, in a recent work, Agarwal et al. (2019b) have demonstrated that such fast rates are attainable in the related, yet full-information online LQR problem endowed with any strongly convex loss functions.

In this paper, we show two interesting scenarios of learning unknown LQR systems in which an expected regret of $O(\log^2 T)$ is, in fact, achievable. In the first, we assume that the matrix $B_x$ is known and show that polylogarithmic regret can be attained by harnessing the intrinsic noise in the system dynamics for exploration. In the second, we assume that $A_x$ is known and show that polylogarithmic regret is, in fact, achievable. In the first, we assume that the optimal control policy $K_*$ is given by a full-rank matrix. Both results are attained using simple and efficient algorithms whose runtime per time step is polynomial in the natural parameters of the problem.

We complement our results with a lower bound showing that our assumptions are indeed necessary for obtaining improved regret guarantees. Specifically, we show that when $B_x$ is unknown and the optimal policy $K_*$ is near-degenerate (i.e., with very small singular values), any online algorithm, whether efficient or not, must suffer at least $\Omega(\sqrt{T})$ regret.

To the best of our knowledge, this is the first $\Omega(\sqrt{T})$ lower bound for learning linear quadratic regulators (that particularly holds even when the learner knows the entire set of system parameters but the matrix $B_x$, and even in a single-input single-output scenario). Concurrent to this work, Sim-

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We further assume that the learner has bounds on the transition matrices, as well as on the optimal cost; that is, there are known constants $\vartheta, \nu > 0$ such that

$$\|Q\|, \|R\| \leq \vartheta, \text{ and } J_\star \leq \nu.$$  

Finally, we assume that there is a known stable (not necessarily optimal) policy $K_0$ and $\nu_0 > 0$ such that $J(K_0) \leq \nu_0$.  

1.2. Main results

Our first result focuses on the case where the state-action transition matrix $B_\star$ is known (but the matrix $A_\star$ is unknown).

**Theorem 1.** There exists an efficient online algorithm (see Algorithm 1 in Section 3.1) that, given the matrix $B_\star$ as input, has expected regret

$$\mathbb{E}[R_T] = \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k) \, O(\log^2 T).$$  

Next, we consider the dual setup in which only the state-space matrix $A_\star$ is known. Here we require an additional non-degeneracy assumption for obtaining polylogarithmic regret.

**Theorem 2.** Suppose that the optimal policy of the system satisfies $K_\star K_\star^T \succeq \mu_\star I$ for some constant $\mu_\star > 0$ that is unknown to the learner. Then there exists an efficient online algorithm (see Algorithm 2 in Section 3.2) that, given the matrix $A_\star$ as input, has expected regret

$$\mathbb{E}[R_T] = \text{poly}(\mu_\star^{-1}, \alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k) \, O(\log^2 T).$$  

Finally, we show that our assumption regarding the non-degeneracy of the optimal policy $K_\star$ is necessary. Our next result shows that without it, the expected regret of any algorithm is unavoidably at least $\Omega(\sqrt{T})$, even in simple one-dimensional (single input, single output) systems.

**Theorem 3.** For any learning algorithm and any $\sigma > 0$, there exists an LQR system (in dimensions $d = k = 1$) which is stabilized by the policy $K_0 = 0$ and for which $\alpha_1 = \alpha_0 = 1$, $\vartheta = 1$ and $\nu = 2\sigma^2$, such that the expected regret of the algorithm is at least $\Omega(\sigma^2 \sqrt{T})$. This is true even if the algorithm receives the matrix $A_\star$ as input.

1.3. Discussion

Our results could be interpreted as a proof-of-concept that faster, polylogarithmic rates for learning in LQRs are possible under more limited uncertainty assumptions. This is perhaps surprising in light of the aforementioned work of Shamir (2013), that established $\Omega(\sqrt{T})$ regret lower bounds for online (bandit) optimization, even with quadratic and strongly convex objectives (as is the case in our LQR setup).
The questions of whether polylogarithmic regret guarantees are possible under more general uncertainty of the system parameters, as well as whether the squared dependence on $\log T$ is indeed tight, remain open. Our lower bound, however, shows that more assumptions are required for obtaining stronger positive results.

Our results focused on the expected regret compared to the infinite-horizon performance of the optimal policy $K_\star$. As far as we know, this is the first analysis that bounds the regret in expectation rather than in high-probability (and without additional assumptions, e.g., as in Ouyang et al., 1982) establish asymptotic convergence rates of system identification, while Polderman (1986a) show the necessity of said identification for optimal control. More generally, it is known that greedy control strategies only converge to a potentially large subset of the parameter space (see e.g., Kumar, 1983; 1985; Polderman, 1986b), and that in the context of our assumptions this subset is a singleton containing only the true system parameters. While this may allude to our positive findings, the asymptotic nature of existing results makes them insufficient for establishing finite-time (polylogarithmic) regret guarantees, which are the focus of this work.

The topic of learning in linear control has been attracting considerable attention in recent years. Since the early work of Abbasi-Yadkori and Szepesvári (2011), a long line of research has focused on obtaining improved regret bounds for learning in LQRs with a variety of algorithms (Ibrahimi et al., 2012; Faradonbeh et al., 2017; Abeille and Lazaric, 2018; Dean et al., 2018; Faradonbeh et al., 2018; Cohen et al., 2019; Abbasi-Yadkori et al., 2019a;b). To the best of our knowledge, our results are the first to exhibit logarithmic regret rates for LQRs albeit in a more restrictive setting.

A closely related line of work considered a non-stochastic variant of online control in which the cost functions can change arbitrarily from round to round (Cohen et al., 2018; Agarwal et al., 2019a). Other notable works have studied the sample complexity of estimating the unknown parameters of linear dynamical systems (Dean et al., 2017; Simchowitz et al., 2018; Sarkar and Rakhlin, 2019), improper prediction of linear systems (Hazan et al., 2017; 2018), as well as model-free learning of LQRs via policy gradient methods (Fazel et al., 2018; Malik et al., 2019).

2. Preliminaries

2.1. Linear Quadratic Control

We give a brief background on several basic properties and results in linear quadratic control that we require in the paper. For a given LQR system $(A, B)$ with cost matrices $Q, R > 0$, the optimal (infinite horizon) feedback controller is given by

$$K(A, B, Q, R) = -(R + B^T P B)^{-1} B^T P A,$$

where $P$ is the positive definite solution to the discrete Riccati equation

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A.$$  \hspace{1cm} (2)

In particular, for the system $(A_\star, B_\star)$ we have $K_\star = K(A_\star, B_\star, Q, R)$. For more background on linear control and derivation of the relations above, see Bertsekas (1995).

The following lemma, proved in Mania et al. (2019), relates the error in estimating a system’s parameters to the deviation of the corresponding estimated controller from the optimal one. This relation is given in terms of cost as well as in terms of distance in operator norm.
Lemma 4. There are explicit constants $C_0, \varepsilon_0 = \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k)$ such that, for any $0 \leq \varepsilon \leq \varepsilon_0$ and matrices $A, B$ such that $\|A - A_*\| \leq \varepsilon$ and $\|B - B_*\| \leq \varepsilon$, the policy $K = K(A, B, Q, R)$ satisfies
\[
\text{J}(K) - J_* \leq C_0 \varepsilon^2, \quad \text{and} \quad \|K - K_*\| \leq C_0 \varepsilon.
\]

Importantly, the lemma shows that the performance scales quadratically in the estimation error. This served Mania et al. (2019) as a key feature in showing that an $\varepsilon$-greedy algorithm obtains $O(\sqrt{T})$ regret. Here, we use this lemma to show that considerably improved regret bounds are achievable in certain scenarios.

Next, we recall the notion of strong stability (Cohen et al., 2018). This is essentially a quantitative version of classic stability notions in linear control.

Definition 5 (strong stability). A matrix $M$ is $(\kappa, \gamma)$–strongly stable for $\kappa \geq 1$ and $0 < \gamma < 1$ if there exists matrices $H > 0$ and $L$ such that $M = HLH^{-1} + (1 - \gamma) L$ and $\|H\|_F^2 \leq \kappa$. A controller $K$ for the system $(A, B)$ is $(\kappa, \gamma)$–strongly stable if $\|K\| \leq \kappa$ and the matrix $A + BK$ is $(\kappa, \gamma)$–strongly stable.

We remark that Cohen et al. (2018) also introduced the notion of sequential strong stability that is an analogous definition for an adaptive strategy that changes its linear policy over time. Here, we avoid this notion by ensuring that each linear policy is played in our algorithms for a sufficiently long duration.

2.2. Confidence bounds for least-squares estimation

Our algorithms use regularized least squares methods in order to estimate the system parameters. An analysis of this method for a general, possibly-correlated sample, was introduced in the context of linear bandit optimization (Abbasi-Yadkori et al., 2011), and was first used in the context of LQRs by Abbasi-Yadkori and Szepesvári (2011). We state the results in terms of a general sequence, since the estimation procedures differ between our two algorithms.

Let $\Theta_* \in \mathbb{R}^{d \times m}$, $\{y_{t+1}\}_{t=1}^\infty \in \mathbb{R}^d$, $\{z_i\}_{i=1}^\infty \in \mathbb{R}^m$, $\{w_i\}_{i=1}^\infty \in \mathbb{R}^d$ such that $y_{t+1} = \Theta_* z_t + w_t$, and $\{w_i\}_{i=1}^\infty$ are i.i.d. with distribution $\mathcal{N}(0, \sigma^2 I)$. Denote by
\[
\Theta_t \in \arg\min_{\Theta \in \mathbb{R}^{d \times m}} \left\{ \sum_{s=1}^{t-1} \|y_{s+1} - \Theta z_s\|^2 + \lambda \|\Theta\|^2_F \right\},
\]
the regularized least squares estimate of $\Theta_*$ with regularization parameter $\lambda$.

Lemma 6 (Abbasi-Yadkori and Szepesvári, 2011). Let $V_t = \lambda t + \sum_{s=1}^{t-1} z_s z_s^T$ and $\Delta_t = \Theta_* - \Theta_t$. With probability at least $1 - \delta$, we have for all $t \geq 1$
\[
\text{Tr}(\Delta_t^T V_t \Delta_t) \leq 4\sigma^2 d \log \left( \frac{d \det(V_t)}{\delta \det(V_1)} \right) + 2\lambda \|\Theta_*\|^2_F.
\]

3. Proofs and Algorithms

In this section we present our algorithms and illustrate the main ideas of our upper and lower bounds. The complete versions of the proofs are deferred to Appendices A to C.

3.1. Upper Bound for Unknown $A_*$

We start with the setting where $A_*$ is unknown, and show an efficient algorithm that achieves regret at most $O(\log^2 T)$. To that end, we propose Algorithm 1. The algorithm begins by playing the stable controller $K_0$ for a $\tau_0$-long warm-up period. It subsequently operates in phases whose length grows exponentially (quadrupling). Each phase begins by estimating the system parameters using Eq. (3) and computing the greedy controller with respect to said parameters using Eq. (1). It then proceeds to play greedily as long as a fail condition is not reached.

Algorithm 1

1: input: parameters $\tau_0, \kappa, \lambda, \nu, \kappa$.
2: initialize: $n_T = \frac{\log_2(T/\tau_0)}{\nu}$.
3: set: $\tau_i = \tau_0 \nu^i$.
4: for $t = 1, \ldots, n_T - 1$ do ▷ warm-up
5: play $u_t = K_0 x_t$.
6: for phase $i = 0, \ldots, n_T$ do ▷ main loop
7: $\hat{A}_i = \arg\min_{A} \sum_{s=1}^{\tau_i} \| (x_{s+1} - B_i u_s) - A x_s \|^2_F + \lambda \|A\|^2_F$.
8: $K_i = K(\hat{A}_i, B_i, Q, R)$.
9: for $t = \tau_i, \ldots, \tau_{i+1} - 1$ do
10: if $\|x_t\|^2 > \kappa$ then ▷ fail, abort
11: abort and play $K_0$ forever.
12: play $u_t = K_i x_t$.

We now give a quantified restatement of Theorem 1.

Theorem (Theorem 1 restated). Suppose Algorithm 1 is run with parameters
\[
\kappa_0 = \sqrt{\frac{\kappa_0}{\alpha_0 \sigma^2}}, \quad \kappa = \sqrt{\frac{\nu + \frac{2}{\nu} C_0}{\alpha_0 \sigma^2}}, \quad \tau_0 = \left[ \frac{80d \lambda (1 + \hat{\vartheta}^2)}{\sigma^2 \nu_0^2} \right],
\]
\[
\lambda = \kappa_0 = 135d \kappa^2 \sigma^2 \max\left\{ \kappa_0^6, 4 \kappa_0^6 \right\} \log(3T).
\]
Then for $T \geq \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k)$ we have $\mathbb{E}[R_T] \leq \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k) \log^2 T$.

We start by quantifying a high probability event on which the regret of the algorithm is small. The event holds when the error of the algorithm’s estimate of $A_*$ scales as $t^{-1/2}$, the states are bounded, and all control policies generated by the algorithm are strongly-stable. This is formally given by the following lemma.

Lemma 7. Let $\gamma = 1/2 \kappa^2$. With probability at least $1 - T^{-2}$,
(i) $K_\tau$ is $(\kappa, \gamma)$-strongly stable, for all $0 \leq i \leq n_T$;
(ii) $\|x_i\|^2 \leq \sigma_b$, for all $0 \leq i \leq T$;
(iii) $\|\Delta_{A_i}\| \leq \varepsilon_0 2^{-i}$, for all $0 \leq i \leq n_T$.

Here we give a sketch of the proof of Lemma 7, deferring technical details to the supplementary material.

**Proof (sketch).** Consider Lemma 6 with $z_i = x_i, y_{i+1} = x_{i+1} - B_s u_i, V_t = \lambda + \sum_{i=1}^{n_T} x_i x_i^T$ and $\Delta_A = A_t - A_s$, then we have with probability at least $1 - \frac{1}{4} T^2$

$$
\operatorname{Tr}(\Delta_A^T V_t \Delta_A) \leq 4 \sigma d \log \left( 3 d T^2 \frac{\det(V_j)}{\det(V_i)} \right) + 2 \lambda d \vartheta^2, \quad (4)
$$

for all $t \geq 1$. Transforming Eq. (4) into the desired bound requires that we bound $V_t$ from above and below. In what follows we show $\|V_t\| \leq \lambda$ on one hand, and $V_t \geq \frac{\sigma_0}{4} I$ on the other hand. Using the upper bound and choice of parameters, one can show that simplifying the right hand side of Eq. (4) yields $\operatorname{Tr}(\Delta_A^T V_t \Delta_A) \leq \sigma^2 \varepsilon_0^2 \lambda_0/40$. Complementing this with the lower bound gets us

$$
\|\Delta_A\|^2 \leq \operatorname{Tr}(\Delta_A^T V_t \Delta_A) \leq 40 \frac{\sigma^2}{\sigma_0^2} \lambda T \operatorname{Tr}(\Delta_A^T V_t \Delta_A) \leq \frac{\varepsilon_0^2 \lambda_0}{10},
$$

and taking the square root, we obtain the desired estimation error bound that indeed scales as $T^{-1/2}$ (up to logarithmic factors).

For a lower bound on $V_t$, notice that the system noise $w_t$ ensures that we have a sufficient exploration of the state space. Formally, we have

$$
\mathbb{E}[V_t] \geq \mathbb{E}[x_t x_t^T] \geq \lambda \sigma^2 I,
$$

where we used $\mathbb{E}[x_t x_t^T] \geq \mathbb{E}[w_t w_t^T] \geq \sigma^2 I$ and $\lambda \geq \sigma^2$. Applying a measure concentration argument yields the sought-after high-probability lower bound on $V_t$.

Now, for an upper bound on $V_t$, notice that

$$
\|V_t\| \leq \lambda + \sum_{i=1}^{T} \|x_i\|^2
$$

thus it suffices to show that $\|x_i\|^2 \leq \sigma_b = \lambda$. The proof of the lemma now follows inductively by the following argument. If the parameter estimation at time $\tau_i$ holds then $K_{\tau_i}$ is strongly-stable. This implies that the states throughout phase $i$ satisfy $\|x_i\|^2 \leq \sigma_b$ which in turn implies the upper bound on $V_{\tau_i+1}$. Thus we can bound the parameter estimation error at time $\tau_{i+1}$. We note that the initial parameter estimation, i.e., at time $\tau_0$, follows from the strong-stability of $K_0$ and by taking the warm-up duration $\tau_0$ to be sufficiently long.

**Proof of Theorem 1.** Let $E_A$ be the event where Lemma 7 hold, and notice that the algorithm does not abort on $E_A$. Defining $J_i = \sum_{t=\tau_i}^{\tau_{i+1} - 1} x_t^T (Q + K_t^T R K_t) x_t$, we have the following decomposition of the regret:

$$
\mathbb{E}[R_T] = R_1 + R_2 + R_3 - T \cdot J_*,
$$

where

$$
R_1 = \mathbb{E} \left[ \mathbb{I}\{E_A\} \sum_{i=0}^{n_T} J_i \right]; \quad R_2 = \mathbb{E} \left[ \mathbb{I}\{E_A^c\} \sum_{i=0}^{T} c_i \right]; \quad R_3 = \mathbb{E} \left[ \tau_0^{-1} \sum_{i=0}^{\tau_0 - 1} c_i \right],
$$

are the costs due to success, failure, and warm-up respectively. We now bound each of $R_1, R_2, R_3$ to conclude the proof.

Starting with $R_1$, the following lemma uses the strong-stability of $K_{\tau_i}$ (whenever $E_A$ holds) to show that $J_i$ is closely related to the steady-state cost of $K_{\tau_i}$.

**Lemma 8.** Fix some $i$ such that $0 \leq i \leq n_T$, and define the event $E_i = \{\|\Delta_{A_i}\| \leq \varepsilon_0 2^{-i}\}$. We have

$$
\mathbb{E} \left[ \mathbb{I}\{E_A\} J_i \right] \leq (\tau_{i+1} - \tau_i) \mathbb{E} \left[ \mathbb{I}\{E_i\} J(K_{\tau_i}) \right] + 4 \alpha_1 \kappa \varepsilon_b.
$$

We further relate the lemma’s bound to the cost of the optimal policy using Lemma 4. This gets us

$$
(\tau_{i+1} - \tau_i) \mathbb{E} \left[ \mathbb{I}\{E_i\} J(K_{\tau_i}) \right] \leq (\tau_{i+1} - \tau_i) J_* + C_0 \varepsilon_0^2 \lambda_0^2
$$

Next, summing over $i$, noticing that $\sum_{i=0}^{n_T} \tau_{i+1} - \tau_i \leq T$, and simplifying the arguments yields

$$
R_1 \leq T \cdot J_* + n_T \left( 6 C_0 \varepsilon_0^2 \lambda_0^2 + 8 \alpha_1 \kappa \varepsilon_b \right).
$$

Moving to $R_2$, let $T_{\text{abort}}$ be the time when the algorithm decides to abort, formally,

$$
T_{\text{abort}} = \min\{t \geq \tau_0 \ | \ \|x_t\|^2 > \sigma_b \text{ or } \|K_t\| > \kappa\},
$$

where we treat $\min \emptyset = T + 1$. Then we have the following bound on $R_2$.

$$
R_2 \leq \mathbb{E} \left[ \mathbb{I}\{E_A^c\} \sum_{i=0}^{T_{\text{abort}} - 1} c_i \right] + \mathbb{E} \left[ \sum_{i=T_{\text{abort}}}^{T} c_i \right].
$$

Now, the state and control policy before $T_{\text{abort}}$ are bounded by $x_b$ and $\kappa$ respectively hence $c_i \leq 2 \kappa \varepsilon_b$. Further recalling that $\mathbb{P}(E_A^c) \leq T^{-2}$ bounds the first term. After
The stable controller $K_0$ is played for the remaining period. This ensures that the state will not keep growing however some care is required as the state at $τ_{\text{abort}}$, $x_{\text{abort}}$, is not bounded. The above is made formal in the following lemma.

**Lemma 9.** $R_2 \leq J(K_0) + 2α_1κ^2x_b + o(1)$

Last, for $R_3$, the strongly stable controller $K_0$ is played throughout warm-up. Unlike $R_2$, here the initial state $x_0 = 0$ is clearly bounded and thus it is not difficult to show that $R_3$ scales linearly with the warm-up duration $τ_0$. Since the latter behaves as $O(\log T)$, the desired result is obtained. This is made formal in the following lemma.

**Lemma 10.** $R_3 \leq τ_0J(K_0)$.

The final bound now follows by combining the bounds of $R_1, R_2,$ and $R_3$ and from $n_T, x_0, τ_0$ being $O(\log T)$. ■

For a full proof of Lemmas 8 to 10, see the supplementary material.

### 3.2. Upper Bound for Unknown $B_*$

We move to a setting where $A_*$ is known, $B_*$ is unknown, but $K, K^T \succeq μ, I$ for some unknown constant $μ > 0$. We show an efficient algorithm that achieves regret at most $O(μ^2 \log^2 T)$. We propose Algorithm 2 to that end. The algorithm operates in a similar fashion to Algorithm 1 with warm-up with $K_0$ and then greedy with fail-safe, but with two main differences:

1. It adds artificial noise to the action during warm-up.
2. The warm-up length is not predetermined and implicitly depends on $μ$.

The first change ensures that the action space is explored uniformly during warm-up, and the second ensures that exploration continues at the same rate during the main loop where noise is not added. The specifics of these are made clear in what follows.

We now give a quantified restatement of Theorem 2.

**Theorem (Theorem 2 restated).** Suppose Algorithm 2 is run with parameters

$$κ_0 = \sqrt{\frac{b_0}{α_0σ^2}}, \quad κ = \sqrt{\frac{μ + 2^6C_0}{α_0σ^2}}, \quad τ_0 = \frac{80κ(1 + \vartheta^2)}{\vartheta(1 + \vartheta^2)}.$$

$x_b = 135dκ_s^2σ^2 \max\left\{(1 + \vartheta^2)κ_0^6, 4κ^6\right\} \log(4T), \quad λ = κ^2x_b, \quad μ_0 = 4κC_0σ^2.$

Then for $T \geq \text{poly}\left(κ_0^{-1}, α_1, \vartheta, ν, ν_0, d, k, μ^{-1}\right)$ we have $E[R_T] \leq \text{poly}\left(κ_0^{-1}, α_1, \vartheta, ν, ν_0, d, k, μ^{-1}\right) log^2 T$.

**Algorithm 2**

1. **input:** parameters $τ_0, x_0, κ, λ, μ_0$, a strongly stable controller $K_0$, and the state transition matrix $A_*$.
2. **initialize:** $n_T = \lceil \log(T/τ_0) \rceil, n_s = n_T + 1, τ_{n+1} = T + 1$.
3. **set:** $τ_i = τ_0^q, μ_i = μ_02^{-i}$ for all $0 \leq i \leq n_T$
4. **for** $t = 1, \ldots, τ_0 - 1$ do $\triangleright$ initial warm-up
5. **play:** $u_t \sim N(K_0x_t, σ^2I)$
6. **for** phase $i = 0, \ldots, n_T$ do $\triangleright$ adaptive warm-up
7. **play:** $u_t \sim N(K_0x_t, σ^2I)$
8. **for** phase $i = n_s, \ldots, n_T$ do $\triangleright$ main loop
9. **if** $M_i, K^2_t \geq (3μ/2)$ then $\triangleright$ fail, abort
10. **save** $n_s = i$ and break.
11. **for** $t = τ_i, \ldots, τ_{n+1} - 1$ do
12. **play:** $u_t \sim N(K_0x_t, σ^2I)$
13. **for** phase $i = n_s, \ldots, n_T$ do $\triangleright$ main loop
14. **play:** $u_t \sim N(K_0x_t, σ^2I)$
15. **for** $t = τ_i, \ldots, τ_{n+1} - 1$ do
16. **if** $||x_t||^2 > x_0$ or $||K^2_t|| > κ$ then $\triangleright$ fail, abort
17. **abort** and play $K_0$ forever.
18. **play:** $u_t = K_0x_t$.

We provide the main ideas required to prove Theorem 2. As in Algorithm 1, we first quantify the high probability event under which the regret of the algorithm is small. Let us first consider the parameter estimation error during warm-up, which is bounded by the following lemma.

**Lemma 11.** With probability at least $1 - T^{-2}$, it holds that $||Δ_{B_*}|| \leq ε_0 x_0^2$ for all $0 \leq i \leq n_T$.

Here we only give a sketch of the proof; for the full technical details, see the supplementary material.

**Proof (sketch).** Consider Lemma 6 with $z_i = u_i, y_{i+1} = x_{i+1} - A_*x_t, V_t = λI + ∑^{i-1}_{s=1} w_s^2$, and $Δ_{B_*} = B_* - B_*$, then with probability at least $1 - 1/4T^{-2}$

$$\text{Tr}(Δ^2_{B_*} V_i Δ_{B_*}) \leq 4σ^2 d \log\left(4d^2 T^2 \det(V_i)\right) + 2\lambda κ d^2,$$

for all $T \geq 1$. Hence, bounding $V_t$ from above and below as in Lemma 7 yields the desired parameter estimation error bound.

Now, during warm-up $u_t \sim N(K_0x_t, σ^2I)$ which is equivalent to having $u_t = K_0x_t + η_t$ where $η_t \sim N(0, σ^2I)$ are i.i.d. random variables. Note that just as $w_t$ provided exploration for $x_t$, here $η_t$ provides exploration for $u_t$. Indeed, for the lower bound, we have

$$E[V_t] \geq λ + ∑^{i-1}_{s=1} E[u_s u^T_s] \geq λ + ∑^{i-1}_{s=1} E[η_s η^T] \geq κ^2 x_0^2.$$
and thus a measure concentration argument yields the desired high probability lower bound. For the upper bound, notice that

$$
\|V_t\| \leq \lambda + \sum_{i=1}^{t-1} \|u_i\|^2 \leq \lambda + 2 \sum_{i=1}^{t-1} (\|K_0\|^2 \|x_i\|^2 + \|\eta_i\|^2),
$$

and so the strong-stability of $K_0$ together with a high probability bound on the system and artificial noises yields the desired upper bound on $V_t$. Combining both upper and lower bounds concludes the proof.

While the estimation rate during warm-up is desirable, adding constant magnitude noise to the action incurs regret that is linear in the warm-up length, even if $K_0 = K_*$, and as such we avoid this strategy during the main loop. Nonetheless, the following lemma shows that the estimation rate continues into the main loop albeit with slightly different constants.

**Lemma 12.** Let $\gamma = 1/2n^2$. With probability at least $1 - T^{-2}$,

(i) $K_\tau$ is $(\kappa, \gamma)$-strongly stable, $\forall n_s \leq i \leq n_T$;

(ii) $\|x_i\|^2 \leq \lambda_S$, $\forall 1 \leq t \leq T$;

(iii) $\|\Delta_{B_\tau}\| \leq \varepsilon_0 \min \{2^{-n_s}, 2 \mu_s^{-1/2} 2^{-i}\}$, $\forall n_s < i \leq n_T$.

We proceed with a proof sketch and defer details to the supplementary material.

**Proof (sketch).** The proof follows inductively by similar arguments to those of Lemma 7, yet with the caveat that the lower bound on $V_t$ may not hold when the controller is rank deficient.

To see this, recall that the algorithm plays $u_i = K_\tau x_i$ during the main loop as long as the abort state is not triggered, so we have

$$
E[u_i u_i^T | K_\tau] = K_\tau E[x_i x_i^T | K_\tau] K_\tau \succeq \sigma^2 K_\tau K_\tau^T.
$$

This means that transforming the exploration of states $x_i$, provided for by the system noise $w_i$, into exploration of actions $u_i$ depends on the controller $K_\tau$ being strictly non-degenerate. We show that with high probability, $K_\tau K_\tau^T \succeq (\mu_s/2)I$ thus ensuring the exploration and the parameter estimation rate.

First, suppose that the learner had knowledge of $\mu_s$ and recall that $\mu_0 = 4n^2 c_0 \varepsilon_0$. Taking $n_s \geq \max \{0, \log_2 (\mu_0/\mu_s)\}$, Lemma 11 implies that $\|\Delta_{B_\tau}\| \leq \varepsilon_0 \mu_s^{-1/2} \mu_s^{-1/2}$ and applying Lemma 4 we get that $\|K_\tau - K_\tau\| \leq \mu_s/4 \kappa$. Further assuming that $\|K_\tau\| \leq \kappa$, which is ensured by strong-stability, simple algebra yields that $K_\tau K_\tau^T \succeq (\mu_s/2)I$.

Now, when $\mu_s$ is unknown, we show that the break condition of the warm-up loop ensures that with high probability

$$
\max \left\{0, \log_2 \frac{\mu_0}{\mu_s} \right\} \leq n_s \leq 2 + \max \left\{0, \log_2 \frac{\mu_0}{\mu_s} \right\},
$$

a proof of which may be found in the supplementary material. The lower bound on $n_s$ ensures the desired non-degeneracy of $K_{\tau_0}$, and proceeding by induction, the same follows for subsequent controllers. We note that the purpose of the upper bound on $n_s$ is to ensure that the warm-up is not so long as to incur more than $O(\mu_s^2 \log^2 T)$ regret. ■

Proceeding from Lemma 12, we obtain a regret decomposition similar to that of Algorithm 1 with an added dependence on the random number of warm-up phases $n_s$. While this randomness introduces some additional technical challenges, the proof ideas remain largely the same. For the full proof of Theorem 2, see the supplementary material.

### 3.3. Lower Bound for Degenerate $K_*$

In this section we prove an $\Omega(\sqrt{T})$ lower bound for systems with a (nearly) degenerate optimal policy, stated in Theorem 3. By Yao’s principle, to establish the theorem it is enough to demonstrate a randomized construction of an LQR system such that the expected regret of any deterministic learning algorithm is large.

Fix $d = k = 1$ and consider the system

$$
\begin{align*}
&x_{t+1} = A x_t + B u_t + w_t, \\
&c_t = x_t^2 + u_t^2.
\end{align*}
$$

Here, $w_t \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. Gaussian random variables, $a = 1/\sqrt{5}$ and $b = \chi \sqrt{\epsilon}$ where $\chi$ is a Rademacher random variable (drawn initially) and $\epsilon > 0$ is a parameter whose value will be chosen later. For simplicity, we assume that $x_1 = 0$. Notice that for this system, we have the bounds $c_1 = 0 = 1$, $c_1 = 1$ and, as we will see below, the optimal cost of the system is bounded by $\nu = 2\sigma^2$. Further, note that the system is controllable and $k_0 = 0$ is a stabilizing policy. Our goal is to lower bound the regret, given by

$$
R_T = \sum_{i=1}^{T} (x_t^2 + u_t^2 - J(k_*)).
$$

**Theorem 3** follows directly from the following:

**Theorem 13.** Assume that $T \geq 12000$ and set $\epsilon = T^{-1/2}/4$. Then the expected regret of any deterministic learning algorithm on the system in Eq. (6) satisfies

$$
E[R_T] \geq \frac{1}{3100} \sigma^2 \sqrt{T} - 4\sigma^2.
$$

Here, the expectation is taken with respect to both the stochastic noise terms as well as the random variable $\chi$.

For the proof, we use the following notation. We use $k_*$ to denote the optimal policy for the system, which (recalling...
Eqs. (1) and (2)) is given by

\[ k_\star = \frac{abp_\star}{1 + b^2p_\star}, \]

where \( p_\star > 0 \) is a positive solution to the Riccati equation

\[ p_\star = 1 + a^2p_\star - \frac{a^2b^2p_\star^2}{1 + b^2p_\star} = 1 + \frac{a^2p_\star}{1 + b^2p_\star}. \]

Observe that for our choice of \( \epsilon \leq 1/400 \) we have that \(|b| \leq 1/20\), and so

\[ 1 \leq p_\star \leq 1/(1 - a^2) = 5/4, \quad 0.99\sqrt{e/5} \leq |k_\star| \leq \sqrt{e/3}. \]  

(7)

In particular, this means that the cost of the optimal policy is at most \( \sigma^2p_\star \leq 2\sigma^2 \). Further, the sign of \( k_\star \) is solely determined by the sign of \( \chi \).

Now, fix any deterministic learning algorithm. Let \( x^{(t)} = (x_t, \ldots, x_T) \) denote the trajectory generated by the learning algorithm up to and including time step \( t \). Denote by \( \mathbb{P}_+ \) and \( \mathbb{P}_- \) the probability laws with respect to the trajectory generated conditioned on \( \chi = 1 \) and \( \chi = -1 \) respectively.

First, we lower bound the expected regret in terms of the cumulative magnitude of the algorithm’s actions \( u_t \). The proof first relates the regret to the overall deviation of \( u_t \) from the actions of the optimal policy \( k_\star \) by using the fact that the action played by \( k_\star \) at any state minimizes the Q-function of the system. Since the actions of \( k_\star \) are small in expectation, the latter quantity can be in turn related to the total magnitude of the action.

**Lemma 14.** Suppose \( \epsilon \leq 1/400 \). The expected regret is lower bounded as

\[ \mathbb{E}[R_T] \geq 0.99 \mathbb{E}\left[ \sum_{t=1}^{T} (u_t - k_\star x_t)^2 \right] - 4\sigma^2, \]

and consequently,

\[ \mathbb{E}[R_T] \geq \frac{1}{2}\mathbb{E}\left[ \sum_{t=1}^{T} u_t^2 \right] - \frac{5}{6} \sigma^2 k_\star^2 T - 4\sigma^2. \]

Note that for the last bound to be meaningful, \( k_\star \) indeed has to be very small so that the additive term that scales with \( k_\star^2 T \) does not dominate the right hand side. The proofs of this as well as subsequent lemmas are deferred to the supplementary material.

Next, by standard information theoretic arguments, we obtain an upper bound on the statistical distance between the probability laws of \( x^{(T)} \) under \( \mathbb{P}_+ \) and \( \mathbb{P}_- \), that scales with the total magnitude of the actions \( u_t \).

**Lemma 15.** For the trajectory \( x^{(T)} \), it holds that

\[ TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \frac{\sqrt{\epsilon}}{\sigma} \sqrt{\mathbb{E}\left[ \sum_{t=1}^{T} u_t^2 \right]}. \]

Our final lemma shows that most of the states visited by the algorithm have a non-trivial (constant) magnitude. This is a straightforward consequence of the added Gaussian noise at each time step.

**Lemma 16.** Assume that \( T \geq 12000 \). With probability \( \geq \frac{7}{8} \), at least \( \frac{3}{4}T \) of the states \( x_1, \ldots, x_T \) satisfy \( |x_t| \geq 2\sigma \).

We are now ready to prove the main result of this section.

**Proof of Theorem 13.** Notice that if \( \mathbb{E}[\sum_{t=1}^{T} u_t^2] > \frac{1}{3}\sigma^2 \sqrt{T} \), then the desired lower bound is directly implied by the second inequality in Lemma 14, as \( k_\star^2 \leq \epsilon/3 = T^{-1/2}/12 \), so \( \mathbb{E}[R_T] \geq \frac{1}{100} \sigma^2 \sqrt{T} - 4\sigma^2 \). We henceforth assume that \( \mathbb{E}[\sum_{t=1}^{T} u_t^2] \leq \frac{1}{4}\sigma^2 \sqrt{T} \). Plugging this into the bound of Lemma 15 for the total variation distance between \( \mathbb{P}_+ \) and \( \mathbb{P}_- \), and using our choice \( \epsilon = T^{-1/4}/4 \), we obtain that

\[ TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \frac{\sqrt{\epsilon}}{\sigma} \cdot \frac{\sigma^2}{4} \sqrt{\epsilon} \sqrt{T} = \frac{1}{4}. \]

Now, let \( N_T \) denote the number of time steps in which \( u_t k_\star x_t \leq 0 \), i.e., the number of times in which the learner has guessed the sign of \( \chi \) incorrectly. We claim that \( \mathbb{P}[N_T \geq T/2] \geq 3/8 \). To see this, denote by \( N_t^\prime \) the number of time steps \( t \) in which \( u_t x_t \leq 0 \). Using the fact that \( N_t^\prime \) is a deterministic function of the trajectory \( x^{(T)} \) together with the bound on the total variation gives

\[ |\mathbb{P}_+[N_T^\prime \geq T/2] - \mathbb{P}_-[N_T^\prime \geq T/2]| \leq TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \frac{1}{4}. \]

Now, recall that the sign of \( k_\star \) is determined by that of \( \chi \). Thus, \( \mathbb{P}_-[N_T \geq T/2] = \mathbb{P}_-[N_T^\prime < T/2] \) and \( \mathbb{P}_+[N_T \geq T/2] = \mathbb{P}_+[N_T^\prime \geq T/2] \) from which

\[ \mathbb{P}_-[N_T \geq T/2] = \frac{1}{2} \mathbb{P}_+[N_T \geq T/2] + \frac{1}{2} \mathbb{P}_-[N_T \geq T/2] = \frac{1}{2}(1 + \mathbb{P}_+[N_T^\prime \geq T/2] - \mathbb{P}_-[N_T^\prime \geq T/2]) \geq \frac{3}{8}. \]

(8)

On the other hand, Lemma 16 implies that with probability at least 7/8, no less than 2T/3 of the states \( x_1, \ldots, x_T \) satisfy \( |x_t| \geq 2\sigma/5 \). Then by a union bound, with probability at least 1/4, at least 7T/6 instances of \( x_1, \ldots, x_T \) satisfy \( |x_t| \geq 2\sigma/5 \) and \( u_t k_\star x_t \leq 0 \). For these instances, we have

\[ (u_t - k_\star x_t)^2 \geq k_\star^2 x_t^2 \geq 0.99^2 \frac{4}{125} \sigma^2, \]
where we have bounded $k_\star$ as in Eq. (7). Hence, we can lower bound the regret using the first inequality in Lemma 14 as follows:

$$
E[R_T] \geq 0.99 \cdot E \left[ \sum_{t=1}^{T} (u_t - k_\star x_t)^2 \right] - 4\sigma^2 \\
\geq 0.99^3 \cdot \frac{1}{6} \cdot \frac{4}{125} \epsilon \sigma^2 - 4\sigma^2 \\
\geq \frac{1}{3100} \sigma^2 \sqrt{T} - 4\sigma^2,
$$

where the last transition used our choice of $\epsilon$. ■

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References


