Goodness-of-Fit Tests for Inhomogeneous Random Graphs: Supplementary Materials

This supplementary material is organized as follows. In Section 1 we give the proof of Theorem 2.1. The proofs of Theorem 2.2 and Theorem 2.3 are given in Section 2 and Section 3, respectively. In Section 4 we prove Theorem 3.1. Additional simulations are given in Section 5.

1 Proof of Theorem 2.1

Denote by $\mathscr{G}_n$ the set of all graphs on n vertices, and by $\mathrm{IER}(P^{(n)})^{\otimes m}$ the product measure $\mathscr{G}_n^{\otimes m}$, which is the measure induced by the collection $G_1, G_2, \ldots, G_m$ i.i.d. samples from $\mathrm{IER}(P^{(n)})$. By the sub-additivity of the Hellinger's distance

$$\begin{aligned}
\mathrm{H}^2(\mathrm{IER}(P^{(n)})^{\otimes m}, \mathrm{IER}(Q^{(n)})^{\otimes m}) &\leq m\mathrm{H}^2(\mathrm{IER}(P^{(n)}), \mathrm{IER}(Q^{(n)})) \\
&\leq m \sum_{1 \leq i < j \leq n} \mathrm{H}^2(\mathrm{Ber}(p_{ij}), \mathrm{Ber}(q_{ij})) \\
&= m \sum_{1 \leq i < j \leq n} F(p_{ij}, q_{ij}), \qquad (1.1)
\end{aligned}$$

where $F(p_{ij}, q_{ij}) = 1 - \sqrt{p_{ij}q_{ij}} - \sqrt{(1-p_{ij})(1-q_{ij})}$.

Denote by $N_0(Q^{(n)})$ and $N_1(Q^{(n)})$ the number of elements in $Q^{(n)}$ which is 0 or 1, respectively. Hereafter, we will fix $0 < \delta < \frac{1}{2}$, and consider the following cases:

- If $q_{ij} = 0$, then set $p_{ij} = \delta$. Then
$$\sum_{\substack{1 \leq i < j \leq n \\ q_{ij}=0}} F(0, p_{ij}) = N_0(Q^{(n)})(1 - \sqrt{1-\delta}) \lesssim n^2\delta, \qquad (1.2)$$
using $N_0(Q^{(n)}) \leq n^2$ and $1 - \sqrt{1-\delta} \lesssim \delta$, for $\delta \in (0, 1)$.
- If $q_{ij} = 1$, then set $p_{ij} = 1 - \delta$. Then
$$\sum_{\substack{1 \leq i < j \leq n \\ q_{ij}=1}} F(1, p_{ij}) = N_1(Q^{(n)})(1 - \sqrt{1-\delta}) \lesssim n^2\delta, \qquad (1.3)$$
using $N_1(Q^{(n)}) \leq n^2$.
- If $0 < q_{ij} < 1/2$, then set $p_{ij} = q_{ij} + \delta q_{ij}$.
$$\begin{aligned}
\sum_{\substack{1 \leq i < j \leq n \\ 0 < q_{ij} < \frac{1}{2}}} F(q_{ij}, p_{ij}) &= \sum_{\substack{1 \leq i < j \leq n \\ 0 < q_{ij} < \frac{1}{2}}} \left(1 - q_{ij}\sqrt{1+\delta} - (1-q_{ij})\sqrt{1 - \frac{q_{ij}\delta}{1-q_{ij}}}\right) \\
&\lesssim \delta^2 \sum_{\substack{1 \leq i < j \leq n \\ 0 < q_{ij} < \frac{1}{2}}} \frac{q_{ij}}{1-q_{ij}} \lesssim \delta^2 n^2, \qquad (1.4)
\end{aligned}$$
where the second last step uses $\sqrt{1 \pm x} - 1 \mp \frac{x}{2} \geq -\frac{x^2}{4}$, for $x \in (0, 1)$, and the last step uses $1 - q_{ij} \geq \frac{1}{2}$.

- If $\frac{1}{2} \leq q_{ij} < 1$, then set $1 - p_{ij} = 1 - q_{ij} + \delta(1 - q_{ij})$. Then, as in the previous case, it can be shown that

$$\sum_{\substack{1 \leq i < j \leq n \\ \frac{1}{2} \leq q_{ij} \leq 1}} F(q_{ij}, p_{ij}) \lesssim \delta^2 n^2. \tag{1.5}$$

Note that for $Q^{(n)}$ and any $\delta > 0$, the construction of $P^{(n)}$ ensures $||P^{(n)} - Q^{(n)}||_0 = n(n-1)$. Therefore, combining (1.2), (1.3), (1.4), and (1.5) with (1.1), gives

$$\mathrm{TV}^2(\mathrm{IER}(P^{(n)})^{\otimes m}, \mathrm{IER}(Q^{(n)})^{\otimes m}) \leq \mathrm{H}^2(\mathrm{IER}(P^{(n)})^{\otimes m}, \mathrm{IER}(Q^{(n)})^{\otimes m}) \lesssim mn^2\delta \ll 1,$$

by choosing any $0 < \delta \ll \frac{1}{mn^2}$. Since the construction of $P^{(n)}$ above, attains the maximum possible value of the zero-norm, this shows that all tests are asymptotically powerless, for any $Q^{(n)}$ and $\varepsilon > 0$.

2 Proof of Theorem 2.2

In the section we prove Theorem 2.2, which derives the optimal sample complexity for goodness-of-fit testing in the Frobenius norm. The proof of the upper bound is given below in Section 2.1, and the lower bound is proved in Section 2.2.

2.1 Upper Bound: Proof of Theorem 2.2 (a)

Recall, from (2.1), the definition of the test statistic $T_{m,n}$:

$$T_{m,n} = \sum_{1 \leq i < j \leq n} \left(\sum_{s \leq \frac{m}{2}} (a_{ij}(G_s) - q_{ij}) \right) \left(\sum_{s > \frac{m}{2}} (a_{ij}(G_s) - q_{ij}) \right).$$

To begin with, note that $\mathbb{E}_{P^{(n)}}(T_{m,n}) = \frac{m^2}{8} ||P^{(n)} - Q^{(n)}||_F^2$. We now compute the variance of $T_{m,n}$.

Lemma 2.1. *Given $G_1, G_2, \ldots, G_m$ i.i.d. samples from* $\mathrm{IER}(P^{(n)})$,

$$\mathrm{Var}_{P^{(n)}}(T_{m,n}) \asymp m^2 \sum_{1 \leq i \neq j \leq n} \left(p_{ij}^2 (1 - p_{ij})^2 + 2 p_{ij}(1 - p_{ij}) \Delta_{ij}^2 \right) + m^3 \sum_{1 \leq i \neq j \leq n} p_{ij}(1 - p_{ij}) \Delta_{ij}^2,$$

where $\Delta_{ij} = p_{ij} - q_{ij}$.

Proof. By the independence of $G_1, G_2, \ldots, G_m$,

$$\begin{aligned}
\mathrm{Var}_{P^{(n)}}(T_{m,n}) &= \sum_{1 \leq i < j \leq n} \mathrm{Var}_{P^{(n)}} \left[\left(\sum_{s \leq \frac{m}{2}} (a_{ij}(G_s) - q_{ij}) \right) \left(\sum_{s > \frac{m}{2}} (a_{ij}(G_s) - q_{ij}) \right) \right] \\
&= \sum_{1 \leq i < j \leq n} \mathrm{Var}_{P^{(n)}} \left[\sum_{s \leq \frac{m}{2}} \sum_{s' > \frac{m}{2}} (a_{ij}(G_s) - q_{ij})(a_{ij}(G_{s'}) - q_{ij}) \right] \\
&= \sum_{1 \leq i < j \leq n} \left\{ T_{ij}^{(1)} + T_{ij}^{(2)} \right\},
\end{aligned} \tag{2.1}$$

where

$$T_{ij}^{(1)} := \sum_{s \leq \frac{m}{2}} \sum_{s' > \frac{m}{2}} \mathrm{Var}_{P^{(n)}}((a_{ij}(G_s) - q_{ij})(a_{ij}(G_{s'}) - q_{ij})),$$

and

$$T_{ij}^{(2)} := 2 \sum_{s \leq \frac{m}{2}} \sum_{\frac{m}{2} < s' \neq s'' \leq m} \mathrm{Cov}_{P^{(n)}}((a_{ij}(G_s) - q_{ij})(a_{ij}(G_{s'}) - q_{ij}), (a_{ij}(G_s) - q_{ij})(a_{ij}(G_{s''}) - q_{ij})).$$

Note that $\mathbb{E}_{P^{(n)}}(a_{ij}(G_s) - q_{ij}) = \Delta_{ij}$ and $\mathbb{E}_{P^{(n)}}(a_{ij}(G_s) - q_{ij})^2 = p_{ij}(1 - p_{ij}) + \Delta_{ij}^2$. This implies, for $s \neq s'$,

$$\mathrm{Var}_{P^{(n)}}((a_{ij}(G_s) - q_{ij})(a_{ij}(G_{s'}) - q_{ij})) = \mathbb{E}_{P^{(n)}}\left((a_{ij}(G_s) - q_{ij})^2 (a_{ij}(G_{s'}) - q_{ij})^2\right) - \Delta_{ij}^4$$

$$
\begin{aligned}
&= \mathbb{E}_{P^{(n)}}(a_{ij}(G_s)-q_{ij})^2\mathbb{E}_{P^{(n)}}(a_{ij}(G_{s'})-q_{ij})^2-\Delta_{ij}^4\\
&= \left(p_{ij}(1-p_{ij})+\Delta_{ij}^2\right)^2-(p_{ij}-q_{ij})^4\\
&= p_{ij}^2(1-p_{ij})^2+2p_{ij}(1-p_{ij})\Delta_{ij}^2.
\end{aligned}
$$

This implies,

$$
\sum_{1\le i<j\le n} T_{ij}^{(1)} \asymp m^2 \sum_{1\le i\ne j\le n}\Big(p_{ij}^2(1-p_{ij})^2+2p_{ij}(1-p_{ij})\Delta_{ij}^2\Big). \tag{2.2}
$$

Next, observe that

$$
\begin{aligned}
&\mathrm{Cov}_{P^{(n)}}((a_{ij}(G_s)-q_{ij})(a_{ij}(G_{s'})-q_{ij}),(a_{ij}(G_s)-q_{ij})(a_{ij}(G_{s''})-q_{ij}))\\
&\qquad= \mathbb{E}_{P^{(n)}}((a_{ij}(G_s)-q_{ij}))^2\mathbb{E}_{P^{(n)}}(a_{ij}(G_{s'})-q_{ij})\mathbb{E}_{P^{(n)}}(a_{ij}(G_{s''})-q_{ij})-\Delta_{ij}^4\\
&\qquad= (p_{ij}(1-p_{ij})+\Delta_{ij}^2)\Delta_{ij}^2-\Delta_{ij}^4\\
&\qquad= p_{ij}(1-p_{ij})\Delta_{ij}^2.
\end{aligned}
$$

This implies,

$$
\sum_{1\le i<j\le n} T_{ij}^{(2)} \asymp m^3 \sum_{1\le i\ne j\le n} p_{ij}(1-p_{ij})\Delta_{ij}^2. \tag{2.3}
$$

Combining (2.2) and (2.3) with (2.1), the lemma follows. □

This lemma can now be used to bound the worst-case risk (1.2) of $\phi_{m,n}$ as follows: Under the null H_0, $\Delta_{ij}=0$, for all $1\le i,j\le n$. Then the probability of Type I error is

$$
\mathbb{P}_{H_0}\left(T_{m,n}>\tfrac{m^2\varepsilon^2}{16}\right)\lesssim \frac{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}{m^4\varepsilon^4}\lesssim \frac{\sum_{1\le i\ne j\le n} q_{ij}^2(1-q_{ij})^2}{m^2\varepsilon^4}\le \frac{n^2}{m^2\varepsilon^4}\ll 1, \tag{2.4}
$$

whenever $m\gg n/\varepsilon^2$.

To compute the worst case Type II error, consider $P^{(n)}$ such that $||P^{(n)}-Q^{(n)}||_F>\varepsilon$. Then

$$
\begin{aligned}
\mathbb{P}_{P^{(n)}}\left(T_{m,n}\le \tfrac{m^2\varepsilon^2}{16}\right) &\le \mathbb{P}_{P^{(n)}}\left(T_{m,n}-\mathbb{E}_{P^{(n)}}(T_{m,n})\le \tfrac{m^2\varepsilon^2}{16}-\tfrac{m^2||P^{(n)}-Q^{(n)}||_F^2}{8}\right)\\
&\le \mathbb{P}_{P^{(n)}}\left(T_{m,n}-\mathbb{E}_{P^{(n)}}(T_{m,n})\le -\tfrac{m^2||P^{(n)}-Q^{(n)}||_F^2}{16}\right)\\
&\lesssim \frac{\mathrm{Var}_{P^{(n)}}(T_{m,n})}{m^4||P^{(n)}-Q^{(n)}||_F^4}\\
&\lesssim \frac{m^2||P^{(n)}||_F^2+m^3\sum_{1\le i\ne j\le n}p_{ij}(1-p_{ij})\Delta_{ij}^2}{m^4||P^{(n)}-Q^{(n)}||_F^4} \qquad \text{(by Lemma 2.1)}\\
&\lesssim \frac{m^2||P^{(n)}-Q^{(n)}||_F^2+m^2||Q^{(n)}||_F^2+m^3\sum_{1\le i\ne j\le n}p_{ij}\Delta_{ij}^2}{m^4||P^{(n)}-Q^{(n)}||_F^4},
\end{aligned} \tag{2.5}
$$

Now, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\left(\sum_{1\le i\ne j\le n} p_{ij}\Delta_{ij}^2\right)^2 &\le \sum_{1\le i\ne j\le n} p_{ij}^2 \sum_{1\le i\ne j\le n}\Delta_{ij}^4\\
&\lesssim (||P^{(n)}-Q^{(n)}||_F^2+||Q^{(n)}||_F^2)||P^{(n)}-Q^{(n)}||_F^4.
\end{aligned} \tag{2.6}
$$

Then using the bound above,

$$
\begin{aligned}
\frac{\left(m^3\sum_{1\le i\ne j\le n}p_{ij}\Delta_{ij}^2\right)^2}{m^8||P^{(n)}-Q^{(n)}||_F^8} &\le \frac{1}{m^2||P^{(n)}-Q^{(n)}||_F^2}+\frac{||Q^{(n)}||_F^2}{m^2||P^{(n)}-Q^{(n)}||_F^4}\\
&\le \frac{1}{m^2\varepsilon^2}+\frac{||Q^{(n)}||_F^2}{m^2\varepsilon^4}\ll 1,
\end{aligned}
$$

whenever $m \gg \max\{\frac{1}{\varepsilon}, \frac{||Q^{(n)}||_F}{\varepsilon^2}\}$. Moreover, the first and second terms in (2.6) becomes

$$\frac{m^2||Q^{(n)}||_F^2}{m^4||P^{(n)} - Q^{(n)}||_F^4} \leq \frac{||Q^{(n)}||_F^2}{m^2\varepsilon^4} \ll 1 \quad \text{and} \quad \frac{m^2||P^{(n)} - Q^{(n)}||_F^2}{m^4||P^{(n)} - Q^{(n)}||_F^4} \leq \frac{1}{m^2\varepsilon^2} \ll 1, \tag{2.7}$$

whenever $m \gg \max\{\frac{1}{\varepsilon}, \frac{||Q^{(n)}||_F}{\varepsilon^2}\}$.

This implies, using (2.5), $\mathbb{P}_{P^{(n)}}(T_{m,n} > \frac{m^2\varepsilon^2}{16}) \ll 1$, whenever $m \gg n/\varepsilon^2 \geq \max\{\frac{1}{\varepsilon}, \frac{||Q^{(n)}||_F}{\varepsilon^2}\}$ (since $\varepsilon \leq n$).

2.2 Lower Bound: Proof of Theorem 2.2 (b)

Given a sequence of prior probability distribution $\pi^{(n)}$ on the alternative H_1 (which corresponds to the set of all symmetric matrices $P^{(n)}$ such that $||P^{(n)} - Q^{(n)}||_F > \varepsilon$), the Bayes risk of a test function $\phi_{m,n}$ is defined as

$$\mathcal{R}_m(Q^{(n)}, \phi_{m,n}, ||\cdot||_F, \pi^{(n)}) = \mathbb{P}_{Q^{(n)}}(\phi_{m,n} = 1) + \mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(\phi_{m,n} = 0)\right]. \tag{2.8}$$

Note that for any prior $\pi^{(n)}$ and any test function $\phi_{m,n}$, the worst case risk ((1.2))

$$\mathcal{R}_m(Q^{(n)}, \phi_{m,n}, ||\cdot||_F) \geq \mathcal{R}_m(Q^{(n)}, \phi_{m,n}, ||\cdot||_F, \pi^{(n)}). \tag{2.9}$$

Moreover, denoting $\mathscr{G}_{m,n}$ the set of possible collections of m graphs on n vertices,

$$\begin{aligned}
\mathcal{R}_m(Q^{(n)}, \phi_{m,n}, ||\cdot||_F, \pi^{(n)}) &\geq \inf_{\phi_{m,n}} \left\{\mathbb{P}_{Q^{(n)}}(\phi_{m,n} = 1) + \mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(\phi_{m,n} = 0)\right]\right\} \\
&\geq 1 - \sup_{\phi_{m,n}} \left|\mathbb{P}_{Q^{(n)}}(\phi_{m,n} = 1) - \mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(\phi_{m,n} = 1)\right]\right| \\
&\geq 1 - \sup_{A \in \mathscr{G}_{m,n}} \left|\mathbb{P}_{Q^{(n)}}(A) - \mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(A)\right]\right| \\
&\geq 1 - \frac{1}{2}\sum_{\omega \in \mathscr{G}_{m,n}} \left|\frac{\mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(\omega)\right]}{\mathbb{P}_{Q^{(n)}}(\omega)} - 1\right| \mathbb{P}_{Q^{(n)}}(\omega) \\
&= 1 - \tfrac{1}{2}\mathbb{E}_{H_0}|L_{\pi^{(n)}} - 1| \\
&\geq 1 - \tfrac{1}{2}\sqrt{\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) - 1},
\end{aligned} \tag{2.10}$$

where $L_{\pi^{(n)}} = \frac{\mathbb{E}_{P^{(n)} \sim \pi^{(n)}}\left[\mathbb{P}_{P^{(n)}}(\omega)\right]}{\mathbb{P}_{Q^{(n)}}(\omega)}$ is the $\pi^{(n)}$-integrated likelihood ratio, and the last step uses the Cauchy-Schwarz inequality.

Therefore, by (2.9) and (2.10), to show a test function $\phi_{n,m}$ is asymptotically powerless for (1.1), it suffices to prove $\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) \leq 1 + o(1)$. To show this we follow arguments similar those in [2], where the analogous problem for two-sample testing was studied.

We will choose $Q^{(n)}$ to be the matrix corresponding to $\text{ER}(n, \frac{1}{2})$, that is, $q_{ij} = \frac{1}{2}$, for all $1 \leq i < j \leq n$, and define $\delta_{ij} = \frac{\varepsilon q_{ij}}{\|Q\|_F}$, and consider the prior $\pi^{(n)}$ as follows: Let

$$q'_{ij} = q_{ij} + \delta_{ij}\gamma_{ij},$$

where $\{\gamma_{ij}\}$ are i.i.d. ± 1 with probability $\frac{1}{2}$. (Note that $q'_{ij} \in [0, 1]$, for all $1 \leq i < j \leq n$.) Let $S = \sum_{s=1}^m A(G_s)$, which is the sum of the adjacency matrices of the graphs $G_1, G_2, \ldots, G_m$. Then

$$L_{\pi^{(n)}} = \mathbb{E}_{\boldsymbol{\gamma}}\left[\prod_{1 \leq i < j \leq n} \frac{(q_{ij} + \gamma_{ij}\delta_{ij})^{S_{ij}}(1 - q_{ij} - \gamma_{ij}\delta_{ij})^{m - S_{ij}}}{q_{ij}^{S_{ij}}(1 - q_{ij})^{m - S_{ij}}}\right],$$

where the expectation above is over the randomness of $\boldsymbol{\gamma} = ((\gamma_{ij}))$. This implies,

$L^2_{\pi^{(n)}}$

$$= \mathbb{E}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}\left[\prod_{1\le i<j\le n}\frac{\left((q_{ij}+\gamma_{ij}\delta_{ij})(q_{ij}+\gamma'_{ij}\delta_{ij})\right)^{S_{ij}}\left((1-q_{ij}-\gamma_{ij}\delta_{ij})(1-q_{ij}-\gamma'_{ij}\delta_{ij})\right)^{m-S_{ij}}}{q_{ij}^{2S_{ij}}(1-q_{ij})^{2m-2S_{ij}}}\right]$$

$$= \mathbb{E}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}\left[\prod_{1\le i<j\le n}\left(\frac{(1-q_{ij})^2(q_{ij}+\gamma_{ij}\delta_{ij})(q_{ij}+\gamma'_{ij}\delta_{ij})}{q_{ij}^2(1-q_{ij}-\gamma_{ij}\delta_{ij})(1-q_{ij}-\gamma'_{ij}\delta_{ij})}\right)^{S_{ij}}\kappa_{ij}^m\right]. \tag{2.11}$$

where $\kappa_{ij} = \frac{(1-q_{ij}-\gamma_{ij}\delta_{ij})(1-q_{ij}-\gamma'_{ij}\delta_{ij})}{(1-q_{ij})^2}$. Note that, under the null H_0, $S_{ij}\sim \mathrm{Bin}(m,q_{ij})$, and $\mathbb{E}_{H_0}\lambda^{S_{ij}} = (1-q_{ij}+\lambda q_{ij})^m$, for any $\lambda\in\mathbb{R}$. Using this and taking expectation over H_0, the RHS of (2.11) simplifies to

$$\begin{aligned}\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) &= \mathbb{E}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}\left[\prod_{1\le i<j\le n}\left(1+\frac{\gamma_{ij}\gamma'_{ij}\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right)^m\right] \qquad (2.12)\\ &= \prod_{1\le i<j\le n}\left[\frac{1}{2}\left(1+\frac{\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right)^m+\frac{1}{2}\left(1-\frac{\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right)^m\right]\\ &\le \prod_{1\le i<j\le n}\left[\frac{1}{2}\exp\left\{\frac{m\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right\}+\frac{1}{2}\exp\left\{-\frac{m\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right\}\right]\\ &= \prod_{1\le i<j\le n}\cosh\left(\frac{m\delta_{ij}^2}{q_{ij}(1-q_{ij})}\right)\le \exp\left\{\sum_{1\le i<j\le n}\frac{m^2\delta_{ij}^4}{2q_{ij}^2(1-q_{ij})^2}\right\}.\end{aligned}$$

Now, recalling $q_{ij}=\frac{1}{2}$, for all $1\le i<j\le n$, the RHS above simplifies to

$$\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) \le e^{\Theta\left(\frac{m^2\varepsilon^4}{n^2}\right)} \le 1+o(1),$$

whenever $m \ll n/\varepsilon^2$.

3 Proof of Theorem 2.3

The upper bound follows by using the same test as in the Frobenius case, that is, reject H_0 when $T_{m,n} > \frac{1}{16}m^2\varepsilon^2$, where $T_{m,n}$ is as defined in (2.1). To see this, note that the probability of Type I error is controlled as in (2.4). For the probability of Type II error, consider a $P^{(n)}$ such that $||P^{(n)}-Q^{(n)}||_{\mathrm{op}}\ge\varepsilon$. This implies, $||P^{(n)}-Q^{(n)}||_F \ge ||P^{(n)}-Q^{(n)}||_{\mathrm{op}}\ge\varepsilon$, and by combining (2.5), (2.6), and (2.7) it follows that $\mathbb{P}_{P^{(n)}}\left(T_{m,n}>\frac{m^2\varepsilon^2}{16}\right)\ll 1$, whenever $m\gg n/\varepsilon^2$.

For the lower bound, as in the case of the Frobenius norm, we choose $Q^{(n)}=((q_{ij}))$ to be the edge-probability matrix of $\mathrm{ER}(n,\frac{1}{2})$, that is, $q_{ij}=\frac{1}{2}$, for all $1\le i<j\le n$. Now, consider the prior $\pi^{(n)}$ as follows: Let $P^{(n)}=((p_{ij}))$ where

$$p_{ij} = q_{ij}+\delta\gamma_i\gamma_j,$$

where $\gamma_1,\gamma_2,\ldots,\gamma_n$ are i.i.d. ±1 with probability $\frac{1}{2}$ and $\delta=\frac{\varepsilon}{n-1}$. Note that$\|P^{(n)}-Q^{(n)}\|_{\mathrm{op}} = \delta\|\boldsymbol{\gamma}\boldsymbol{\gamma}^T-\mathrm{I}\|_{op}=\varepsilon$.

Now, by arguments identical to the proof to (2.12),

$$\begin{aligned}\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) &= \mathbb{E}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}\left[\prod_{1\le i<j\le n}\left(1+4\gamma_i\gamma_j\gamma'_i\gamma'_j\delta^2\right)^m\right]\\ &\le \mathbb{E}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}\left[\exp\left\{4m\delta^2\sum_{1\le i<j\le n}\gamma_i\gamma_j\gamma'_i\gamma'_j\right\}\right]\end{aligned}$$

$$\leq \mathbb{E}_{\bar{\boldsymbol{\gamma}}}\left[\exp\left\{4m\delta^2 \sum_{1\leq i<j\leq n} \bar{\gamma}_i\bar{\gamma}_j\right\}\right], \tag{3.1}$$

where $\bar{\gamma}_i = \gamma_i\gamma_i'$ and $\bar{\boldsymbol{\gamma}} = (\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_n)$ are i.i.d. ± 1 with probability $\frac{1}{2}$. Denoting by $S_n = \sum_{i=1}^n \bar{\gamma}_i$, the RHS of (3.1), for n large enough, simplifies to,

$$\mathbb{E}_{H_0}(L^2_{\pi^{(n)}}) = \mathbb{E}_{\bar{\boldsymbol{\gamma}}}\left[e^{2m\delta^2(S_n^2-n)}\right] \leq e^{2m\delta^2 n} \leq 1 + o(1),$$

whenever $m \ll n/\varepsilon^2$, where the second inequality uses [2, Claim 3].

4 Proof of Theorem 3.1

In this section we derive the asymptotic distribution of

$$Z_{m,n} = \frac{T_{m,n}}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}} = \frac{\sum_{1\leq i<j\leq n} T_{m,n}^{(i,j)}}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}},$$

where

$$T_{m,n}^{(i,j)} = \sum_{s\leq \frac{m}{2}}(a_{ij}(G_s) - q_{ij}) \sum_{s>\frac{m}{2}}(a_{ij}(G_s) - q_{ij}).$$

Recall that $\mathbb{E}_{Q^{(n)}}(T_{m,n}) = 0$ and, from the proof of Lemma 2.1, $\mathrm{Var}_{Q^{(n)}}(T_{m,n}) = \frac{m^2}{8}\sum_{1\leq i\neq j\leq n} q_{ij}^2(1-q_{ij})^2$. To prove the asymptotic normality of $Z_{m,n}$ we invoke the Berry-Essen theorem [1], which states that

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}_{Q^{(n)}}(Z_{m,n}\leq x) - \Phi(x)\right| \lesssim \frac{1}{\mathrm{Var}_{Q^{(n)}}(T_{m,n})^{\frac{3}{2}}} \sum_{1\leq i<j\leq n} \mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^3), \tag{4.1}$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal $N(0,1)$.

Note that

$$\begin{aligned}\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^4) &= \mathbb{E}_{Q^{(n)}}\left(\sum_{s\leq\frac{m}{2}}(a_{ij}(G_s)-q_{ij})\right)^4 \mathbb{E}_{Q^{(n)}}\left(\sum_{s>\frac{m}{2}}(a_{ij}(G_s)-q_{ij})\right)^4 \\ &\lesssim \left(m(q_{ij}(1-q_{ij})^4 + q_{ij}^4(1-q_{ij})) + m^2q_{ij}^2(1-q_{ij})^2\right)^2.\end{aligned}$$

Then, using the Cauchy-Schwarz inequality and $\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^2) = \frac{m^2}{4}q_{ij}^2(1-q_{ij})^2$, gives

$$\begin{aligned}\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^3) &\leq \sqrt{\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^2)\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^4)} \\ &\lesssim mq_{ij}(1-q_{ij})\left\{m(q_{ij}(1-q_{ij})^4 + q_{ij}^4(1-q_{ij})) + m^2q_{ij}^2(1-q_{ij})^2\right\} \\ &\lesssim m^3q_{ij}^2(1-q_{ij})^2\end{aligned}$$

This implies,

$$\frac{1}{\mathrm{Var}_{Q^{(n)}}(T_{m,n})^{\frac{3}{2}}}\sum_{1\leq i<j\leq n}\mathbb{E}_{Q^{(n)}}(|T_{m,n}^{(i,j)}|^3) \lesssim \frac{1}{\sqrt{\sum_{1\leq i<j\leq n} q_{ij}^2(1-q_{ij})^2}} \lesssim \frac{1}{||Q^{(n)}||_F} \ll 1,$$

since, the assumption $\max_{1\leq i<j\leq n} q_{ij} \gtrsim 1$, implies $\sum_{1\leq i<j\leq n} q_{ij}^2(1-q_{ij})^2 \gtrsim ||Q^{(n)}||_F^2$. This shows, under the conditions of Theorem 3.1, the RHS of (4.1) goes to zero, that is, $Z_{m,n}$ converges in distribution to $N(0,1)$ under H_0. Hence, the probability of Type-I error is $\mathbb{P}_{Q^{(n)}}(|Z_{m,n}| > z_{\alpha/2}) = 2(1-\Phi(z_{\alpha/2})) + o(1) = \alpha + o(1)$, as required in (3.1).

To show consistency, note that

$$\frac{m^2||P^{(n)} - Q^{(n)}||_F^2}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}} \asymp \frac{m||P^{(n)} - Q^{(n)}||_F^2}{\sqrt{\sum_{1\leq i\neq j\leq n} q_{ij}^2(1-q_{ij})^2}} \geq \frac{m||P^{(n)} - Q^{(n)}||_F^2}{||Q^{(n)}||_F} \gg 1,$$

by the assumption in Theorem 3.1. Therefore, $z_{\alpha/2} \leq \frac{\frac{m^2}{16}||P^{(n)}-Q^{(n)}||_F^2}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}}$, and the probability of Type-II error becomes,

$$\begin{aligned}
\mathbb{P}_{Q^{(n)}}(|Z_{m,n}| \leq z_{\alpha/2}) &\leq \mathbb{P}_{Q^{(n)}}\left(\frac{T_{m,n} - \mathbb{E}_{P^{(n)}}(T_{m,n})}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}} \leq z_{\alpha/2} - \frac{\frac{m^2}{8}||P^{(n)} - Q^{(n)}||_F^2}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}}\right) \\
&\leq \mathbb{P}_{Q^{(n)}}\left(\frac{T_{m,n} - \mathbb{E}_{P^{(n)}}(T_{m,n})}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}} \leq -\frac{\frac{m^2}{16}||P^{(n)} - Q^{(n)}||_F^2}{\sqrt{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}}\right) \\
&\lesssim \frac{\mathrm{Var}_{Q^{(n)}}(T_{m,n})}{m^4||P^{(n)} - Q^{(n)}||_F^2} \\
&\lesssim \frac{m^2||P^{(n)}||_F^2 + m^3\sum_{1\leq i\neq j\leq n} p_{ij}\Delta_{ij}^2}{m^4||P^{(n)} - Q^{(n)}||_F^4} \\
&\leq \frac{m^2||P^{(n)}||_F^2 + m^3||P^{(n)} - Q^{(n)}||_F^2}{m^4||P^{(n)} - Q^{(n)}||_F^4} \\
&= \frac{||P^{(n)}||_F^2}{m^2||P^{(n)} - Q^{(n)}||_F^4} + \frac{1}{m||P^{(n)} - Q^{(n)}||_F^2},
\end{aligned} \tag{4.2}$$

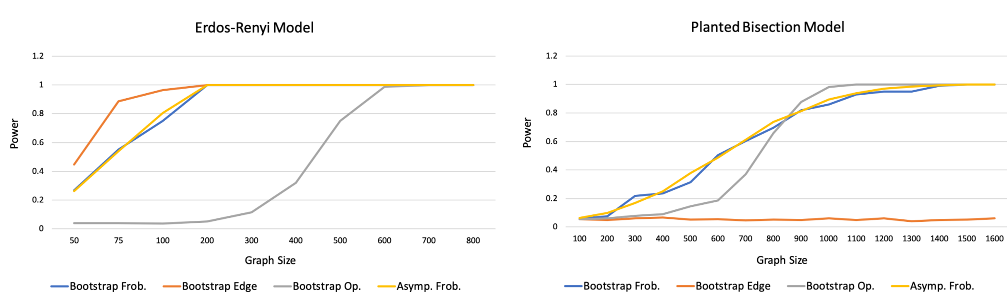
by (2.5) and (2.6). For the first term above,

$$\begin{aligned}
\frac{||P^{(n)}||_F^2}{m^2||P^{(n)} - Q^{(n)}||_F^4} &\lesssim \frac{||Q^{(n)}||_F^2}{m^2||P^{(n)} - Q^{(n)}||_F^4} + \frac{1}{m^2||P^{(n)} - Q^{(n)}||_F^2} \\
&\ll \frac{||Q^{(n)}||_F^2}{m^2||P^{(n)} - Q^{(n)}||_F^4} + \frac{1}{m||Q^{(n)}||_F} \\
&\qquad\qquad (\text{since } m||P^{(n)} - Q^{(n)}||_F^2 \gg ||Q^{(n)}||_F) \\
&\ll 1,
\end{aligned}$$

since $m||P^{(n)} - Q^{(n)}||_F \gg ||Q^{(n)}||_F$ and $m||Q^{(n)}||_F \geq ||Q^{(n)}||_F \gg 1$ by assumption. The second term in (2.6) goes to zero similarly. Therefore, the RHS of (4.2) goes to zero, completing the proof of (3.2).

5 Additional Simulations

We conclude with some additional simulations, comparing the performance of the different tests by varying the number of vertices n of the graph (keeping the separation and sample size m fixed) in Section 5.1, and by varying the sample size (keeping the separation and the size n of the graph fixed) in Section 5.2.



(a) (b)

Figure 1: Empirical power of the different tests as a function of increasing graph size in (a) the Erdős-Rényi model with $m = 2$, (b) the planted bisection model with $m = 4$.

5.1 Dependence on the Size of the Graph

Here, we fix the sample size m, and a reference edge-probability matrix $Q^{(n)}$ (which corresponds to the null) and another edge-probability matrix $P^{(n)}$ (which is at a fixed perturbation from $Q^{(n)}$), and consider samples $G_1, G_2, \ldots, G_m$ i.i.d from $\mathrm{IER}(P^{(n)})$, for increasing values of n. The figures below show the empirical power of the tests over 1000 iterations (calibrated either using the asymptotic distribution or the parametric bootstrap at $\alpha = 0.05$) as the size of the graph increases. We consider the following three scenarios:

- In Figure 1(a) the reference matrix $Q^{(n)}$ corresponds to $\mathrm{ER}(n, 0.5)$ and the matrix $P^{(n)}$ corresponds to $\mathrm{ER}(n, 0.59)$, and the power curve is shown as a function of increasing n. As in the case of increasing separation in Figure 4(a), the `Bootstrapped Edge Test` has the highest power, while the `Asymptotic Frobenius Test` and the `Bootstrapped Frobenius Test` have slightly less but comparable power, while the `Bootstrapped Operator-Norm Test` has the least power.
- In Figure 1(b) the reference matrix $Q^{(n)}$ corresponds to $\mathrm{PB}(n, 0.6, 0.4)$ and the matrix $P^{(n)}$ corresponds to $\mathrm{PB}(n, 0.59, 0.41)$, and the power curve is shown as a function of increasing n. Here, as expected, the `Bootstrapped Edge Test` is powerless. However, the `Asymptotic Frobenius Test`, `Bootstrapped Frobenius Test`, and `Bootstrapped Operator-Norm Test` all have power converging to 1 with increasing n. However, the convergence to 1 is slower than in the Erdős-Rényi case.

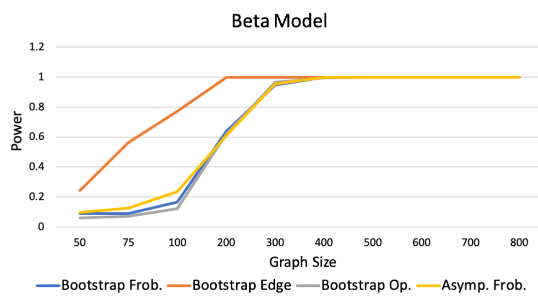


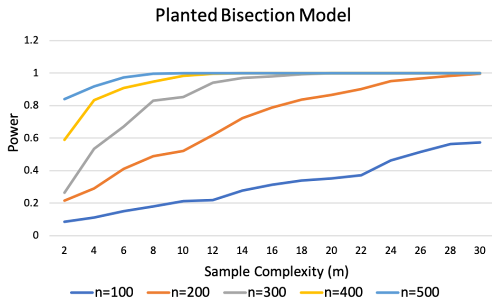
Figure 2: Empirical power of the different tests as a function of increasing graph size for a sample of $m = 8$ graphs in the β-model.

- In Figure 2 the reference matrix $Q^{(n)}$ corresponds to the β-model $\mathscr{B}(n, \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ is chosen uniformly from the surface of ball in $\mathbb{R}^n$ with radius 20, and $P^{(n)}$ corresponds to the β-model $\mathscr{B}(n, \boldsymbol{\beta} + 0.02)$, and the power curve is shown as a function of increasing n. Here, the `Bootstrapped Edge Test` has the highest power. The `Asymptotic Frobenius Test`, `Bootstrapped Frobenius Test`, `Bootstrapped Operator-Norm Test` all have similar performances with power converging to 1 around $n = 300$.

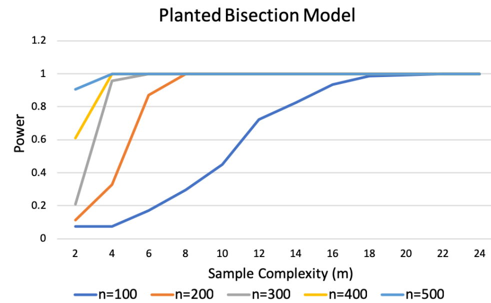
5.2 Dependence on Sample Size

To understand the dependence on sample size, we will fix the number of vertices n, a reference edge-probability matrix $Q^{(n)}$ (which corresponds to the null) and another edge-probability matrix $P^{(n)}$ (which is at a fixed perturbation from $Q^{(n)}$), and consider the power of the different tests as a function of the sample size m.

We expect the power of the tests to increase with sample size and the size of the graph. We illustrate this for the planted bisection model in Figure 3, where the reference matrix $Q^{(n)}$ corresponds to $\mathrm{PB}(n, 0.6, 0.4)$ and the matrix $P^{(n)}$ corresponds to $\mathrm{PB}(n, 0.58, 0.42)$. The plot in Figure 3 (a) shows the power of the `Bootstrapped Frobenius Test` and the plot in Figure 3 (b) shows the power of the `Bootstrapped Operator-Norm Test`, when the sample size m increases, for various values of n. In both cases, as the size of graph n increases the power converges to power 1 faster. However, the rate of increase is quicker for the `Bootstrapped Operator-Norm Test`.



(a) Bootstrapped Frobenius norm test



(b) Bootstrapped Operator norm test

Figure 3: Empirical power of the two tests as a function of increasing separation sample size m for various sizes n of the graph.

References

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[2] D. Ghoshdastidar, M. Gutzeit, A. Carpentier, and U. von Luxburg, Two-sample hypothesis testing for inhomogeneous random graphs, *Annals of Statistics*, to appear, 2020.