A. Closure of SDDM Matrices under Schur complement

Lemma A.1. *If* $\mathbf{M}$ *is an SDDM matrix and* $T = V \setminus \{x\}$ *is a subset of its columns, then* $\mathbf{S} := \mathbf{SC}(\mathbf{M}, T)$ *is also an SDDM matrix.*

Proof. Recall that

$$\mathbf{SC}(\mathbf{M}, T) = \mathbf{M}_{T,T} - \frac{\mathbf{M}_{T,x}\mathbf{M}_{T,x}^{\top}}{\mathbf{D}'_{x,x}}$$

and observe that $\mathbf{D}'_{x,x} = \mathbf{M}_{x,x}$. By definition of SDDM matrices, we need to show that $\mathbf{S}$ is (i) symmetric, (ii) its off-diagonal entries are non-positive and (iii) for all $i \in [n-1]$ we have $\mathbf{S}_{ii} \geq -\sum_{j \neq i} \mathbf{S}_{ij}$. An easy inspection shows that $\mathbf{S}$ satisfies (i) and (ii). We next show that (iii) holds.

To this end, by definition of $\mathbf{S}$, we have that

$$\begin{aligned} -\sum_{j \neq i} \mathbf{S}_{ij} &= \sum_{j \neq i} \left(-\mathbf{M}_{ij} + \frac{\mathbf{M}_{ix}\mathbf{M}_{xj}}{\mathbf{M}_{xx}} \right) \\ &= -\sum_{j \neq i} \mathbf{M}_{ij} + \frac{\mathbf{M}_{ix}}{\mathbf{M}_{xx}} \left(\sum_{j \neq i} \mathbf{M}_{xj} \right) \end{aligned} \tag{9}$$

As $\mathbf{M}$ is an SDDM matrix, the following inequality holds for the x-th row of $\mathbf{M}$

$$-\sum_{j \neq i} \mathbf{M}_{xj} \leq \mathbf{M}_{ix},$$

or equivalently

$$\mathbf{M}_{ix} \left(\sum_{j \neq i} \mathbf{M}_{xj} \right) \leq -\mathbf{M}_{ix}^2. \tag{10}$$

Plugging Eq. (10) in Eq. (9) and using the fact that $-\sum_{j \neq i} \mathbf{M}_{ij} \leq \mathbf{M}_{ii}$, we get that

$$-\sum_{j \neq i} \mathbf{S}_{ij} \leq \mathbf{M}_{ii} - \frac{\mathbf{M}_{ix}^2}{\mathbf{M}_{xx}} = \mathbf{S}_{ii},$$

which completes the proof of the lemma. □