Supplementary material

We now provide a detailed analysis on two fundamental aspects of our games: convergence and identifiability. That is, we characterize conditions under which players converge to an equilibrium, and when the game parameters can be provably recovered from observed outcomes.

A. Identifiability of our games

We begin with the results on provably recovering the structure of our one-shot games from data. Specifically, we characterize the conditions under which our games with one step of dynamics become identifiable, and provide an algorithm to recover the structure of the game, i.e., the neighbors for each player $i \in [n]$ with the signs (positive or negative) of their respective influences.

Our recovery procedure adapts the primal-dual witness method (Wainwright, 2009) for structure estimation in games. The method has previously been applied in several non-strategic settings such as Lasso (Wainwright, 2009) and Ising models (Ravikumar et al., 2010). Recently, (Ghoshal and Honorio, 2017) employed this method to recover a set of pure strategy Nash equilibria (PSNE) from data consisting of a subset of PSNE, and a small fraction of non-equilibrium outcomes assumed to be sampled under their noise models in the setting of linear influence games. However, the problem of structure recovery is significantly harder: it is known (Honorio and Ortiz, 2015; Ghoshal and Honorio, 2017) that the problem becomes non-identifiable in the setting of PSNE, since multiple game structures may pertain to the same of PSNE. We leverage dynamics to fill this gap by characterizing conditions under which our one-shot games become identifiable.

Our approach follows the general proof structure of primal-dual witness method in the context of model selection for Ising models (Ravikumar et al., 2010). However, our setting is significantly different from the setting in (Ravikumar et al., 2010) where context and dynamics play no part, and all the observed data is assumed to be sampled from a common (global) distribution expressible in a closed form. In contrast, each observed outcome in our setting is sampled from a separate joint strategy profile following one-step of dynamics initiated under a different context.

Specifically, in the one-shot setting, consider a dataset $D = \{(x^{(m)}, a^{(m)}) \in \mathcal{X} \times \mathcal{Y}, m \in [M]\}$ where $a^{(m)}$ is the action profile (i.e. observed outcome) sampled from the joint player strategies after one round of communication. Assume that the type parameters $\theta = (\theta_1, \ldots, \theta_n)$ are known. Then, since types for any context are determined by the parameters θ, we have access to the player types $z^{(m)}(x^{(m)}) = (z_1^{(m)}, \ldots, z_n^{(m)})$, which in turn determine the initial strategies for all the examples $m \in [M]$. We focus on binary actions here since they let us simplify the exposition while conveying the essential ideas. Specifically, each player $i \in [n]$ initially plays action 1 with probability

$$\phi_i^{(m)} = \xi(z_i^{(m)}) \triangleq \frac{1}{1 + \exp(-z_i^{(m)})} ,$$

and the action 0 with probability $1 - \phi_i^{(m)}$. We define $\phi^{(m)} = (\phi_1^{(m)}, \ldots, \phi_n^{(m)})$, and $\Phi_{-i}^{(m)} = (\phi_j^{(m)})_{j \neq i}$. We focus on the gradient update setting where after one round of communication, player i responds to its neighbors with its updated strategy $(\sigma_i^{*(m)}, 1 - \sigma_i^{*(m)})$, where

$$\sigma_i^{*(m)} \triangleq \xi \left(\phi_i^{(m)} + \alpha (\sum_{j \neq i} w_{ij}^* \phi_j^{(m)} - z_i^{(m)}) \right) ,$$

such that $\alpha > 0$, and $w_{ij}^* \in \mathbb{R}$ is the true influence (i.e. interaction weight) of player $j \in [n] \setminus \{i\}$ on i. Recall that we call player j a neighbor of i if $|w_{ij}^*| > 0$. Finally, action $a_i^{(m)}$ is sampled from the updated strategy, and we obtain the joint profile $a^{(m)} = \{a_i^{(m)}, i \in [n]\}$ as the observed outcome. Our goal is to estimate, from D and α, the *support* S_i, or the set of neighbors j for i, i.e., the players that have influence $w_{ij}^* \neq 0$. We can thus separate the influence of neighbors of i from the non-neighbors by defining the set of non-zero weights $w_{i,S}^* = \{w_{ij}^* | j \in S_i\}$. We denote the complement of a set A by A^c. Thus, $w_{ij}^* = 0$ for $j \in S_i^c$. We equivalently write $w_{i,S^c}^* = \mathbf{0}$. We are interested in recovering not only the support of each player i, but also the correct sign of influence (i.e. positive or negative) of each neighbor j on i.

We consider the average cross-entropy loss between the strategy under w_i and the observed outcome.

$$\ell_i(w_i; D) \;=\; \frac{1}{M} \sum_{m=1}^{M} - \left(a_i^{(m)} \log(\sigma_i^{(m)}) + (1 - a_i^{(m)}) \log(1 - \sigma_i^{(m)}) \right) \; . \tag{14}$$

We compute the gradient and the Hessian of the sample loss:

$$\nabla \ell_i(w_i; D) \;=\; \frac{\alpha}{M} \sum_{m=1}^{M} (\sigma_i^{(m)} - a_i^{(m)}) \, \Phi_{-i}^{(m)} \; , \tag{15}$$

$$H_i^M \;\triangleq\; \nabla^2 \ell_i(w_i; D) \;=\; \frac{\alpha^2}{M} \sum_{m=1}^{M} \sigma_i^{(m)} (1 - \sigma_i^{(m)}) \, \Phi_{-i}^{(m)} {\Phi_{-i}^{(m)}}^\top \; . \tag{16}$$

We will often use the variance function $\eta_i(w_i; m) \triangleq \alpha^2 \sigma_i^{(m)} (1 - \sigma_i^{(m)})$ as a shorthand, and write

$$H_i^M = \frac{1}{M} \sum_{m=1}^{M} \eta_i(w_i; m) \, \Phi_{-i}^{(m)} {\Phi_{-i}^{(m)}}^\top \; . \tag{17}$$

We denote by $H_{i,SS}^{*M}$ the submatrix obtained by restricting the Hessian H_i^{*M}, pertaining to true weights, to rows and columns corresponding to neighbors, i.e., players in S_i. Likewise, H_{i,SS^c}^{*M} denotes the submatrix restricted to rows pertaining to S_i (neighbors) and columns to S_i^c (non-neighbors).

We will provide detailed analysis under sample Fisher matrix assumptions. We will omit the analysis for the population setting that can be derived by imposing analogous assumptions directly on the population matrices, and making concentration arguments that show these assumptions hold in the sampled setting with high probability. Recall from the main text that we make the following assumptions that are reminiscent of those for support recovery under Lasso (Wainwright, 2009), and model selection in Ising models (Ravikumar et al., 2010). We first recall our assumptions from the main text.

Assumptions.

$$\Lambda_{\min} \left(H_{i,SS}^{*M} \right) \;\geq\; \alpha^2 C_{min} \; . \tag{18}$$

$$\Lambda_{\max} \left(\frac{1}{M} \sum_{m=1}^{M} \Phi_{-i}^{(m)} {\Phi_{-i}^{(m)}}^\top \right) \;\leq\; C_{max} \; . \tag{19}$$

$$|||H_{i,S^cS}^{*M} (H_{i,SS}^{*M})^{-1}|||_\infty \;\leq\; 1 - \gamma \; , \tag{20}$$

such that $C_{\min} > 0$, $C_{\max} < \infty$, and $\gamma \in (0, 1]$. In our notation, $|||A|||_\infty$ denotes the maximum ℓ_1 norm across rows of matrix A, and $|||A|||_2$ denotes the spectral norm (i.e. maximum singular value) of A. $\Lambda_{min}(A)$ and $\Lambda_{max}(A)$ refer respectively to the minimum and the maximum eigenvalue of a square matrix A.

Analysis. We propose to solve the following regularized problem for each player $i \in [n]$ separately.

$$\arg \min_{w_i \in \mathbb{R}^{n-1}} \; \ell_i(w_i; D) \;+\; \lambda_{M,n,d} ||w_i||_1 \; , \tag{21}$$

where $\lambda_{M,n,d} > 0$ is a regularization parameter that depends on the sample size M, the number of players n, and the maximum *degree* (i.e. number of neighbors) d of any player. For brevity, we will omit the dependence of this parameter on n and d, and simply write λ_M. This problem is convex but not differentiable everywhere because of the L_1 penalty. Note that since the problem is not strictly convex, it might have multiple minimizing solutions. For any such optimal solution $\hat{w}_i$, we must have by KKT conditions,

$$\nabla \ell_i(\hat{w}_i; D) \;+\; \lambda_M \hat{\kappa}_i \;=\; \mathbf{0} \; , \tag{22}$$

where the subgradient $\hat{\kappa}_i \in \mathbb{R}^{n-1}$ is such that

$$\hat{\kappa}_{ij} = \text{sign}(\hat{w}_{ij}) \in \{\pm 1\} \text{ if } \hat{w}_{ij} \neq 0, \text{ and } |\hat{\kappa}_{ij}| \leq 1 \text{ otherwise.} \tag{23}$$

We would like to ensure the following conditions in order to recover the signed neighborhood for i.

$$\text{sign}(\hat{\kappa}_{ij}) = \text{sign}(w^*_{ij}),\ \forall j \in S_i \tag{24}$$
$$\hat{w}_{ij} = 0,\ \forall j \in S_i^c\ . \tag{25}$$

Our analysis is built on the *primal-dual witness* (PDW) method (Wainwright, 2009). This method has the following steps. First, only for the sake of analysis, we presuppose that some Oracle provides the true neighbors S_i. Therefore, we solve the following problem to recover the signs of true neighbors.

$$\hat{w}_{i,S} = \arg\min_{(w_{i,S},\mathbf{0})\in\mathbb{R}^{n-1}} \ell_i(w_i; D)\ +\ \lambda_M ||w_{i,S}||_1\ , \tag{26}$$

We then set the components of the dual vector κ_i that pertain to neighbors of i to the sign of corresponding components in $\hat{w}_{i,S}$. That is, $\hat{\kappa}_{i,j} = \text{sign}(\hat{w}_{i,j}),\ \forall j \in S_i$. We next set $\hat{w}_{i,S^c} = \mathbf{0}$, and thus (25) is satisfied. We then solve for $\hat{\kappa}_{i,S^c}$ by plugging $\hat{w}_{i,S}$, $\hat{\kappa}_{i,S}$, and $\hat{w}_{i,S^c}$ in (22). Thus, we are left to show that (23) and (24) are satisfied. We impose conditions on M, n, and d under which these conditions are satisfied with high probability. In fact, we prove a stronger result for (23), namely, strict dual feasibility for non-neighbors, i.e., $|\hat{\kappa}_{i,j}| < 1$ for all $j \in S_i^c$.

We argue that our construction yields a unique optimal primal solution $\hat{w}_i$. Specifically, we invoke Lemma 1 from (Ravikumar et al., 2010) that states that so long as $||\hat{\kappa}_{i,S^c}||_\infty < 1$, any optimal primal solution $\tilde{w}_i$ satisfies $\tilde{w}_{i,S^c} = \mathbf{0}$. This is established by our construction above. Moreover, Lemma 1 asserts that $\hat{w}_i$ is the unique solution to (21) if $\Lambda_{\min}(\hat{H}^M_{i,SS}) > 0$, i.e., if the sample Hessian under $\hat{w}_i$ is positive definite when restricted to the rows and columns in the true support S_i. We show that assumption (18) implies $\Lambda_{\min}\left(\hat{H}^M_{i,SS}\right) \geq \alpha^2 \dfrac{C_{min}}{2} > 0$, and this guarantees that we correctly recover the signed neighborhood of i.

To proceed, we define $G^M_i = -\nabla \ell_i(w^*_i; D)$ and rewrite (22) as

$$\nabla \ell_i(\hat{w}_i; D) - \nabla \ell_i(w^*_i; D) = G^M_i - \lambda_M \hat{\kappa}_i\ . \tag{27}$$

Applying the mean value theorem component-wise, we can write (27) as

$$\nabla^2 \ell_i(w^*_i; D)(\hat{w}_i - w^*_i) = G^M_i - \lambda_M \hat{\kappa}_i - R^M_i\ , \tag{28}$$

where

$$R^M_{i,j} = \left(\nabla^2 \ell_i(\overline{w}^{(j)}_i; D) - \nabla^2 \ell_i(w^*_i; D)\right)^\top_j (\hat{w}_i - w^*_i)\ ,$$

for some vector $\overline{w}^{(j)}_i = t_j \hat{w}_i + (1 - t_j) w^*_i$, $t_j \in [0, 1]$. Here, $(A)^\top_j$ denotes row j of matrix A.

We will now prove some auxiliary results that we will use in the proof of Theorem 2.

Lemma 1. *We have that*

$$\mathbb{P}\left(||G^M_i||_\infty \geq \frac{\lambda_M}{4}\frac{\gamma}{2-\gamma}\right) \leq 2\exp\left(-\frac{\gamma^2\lambda^2_M}{32\alpha^2(2-\gamma)^2}M + \log(n)\right)\ ,$$

which converges to zero at rate $\exp(-C_{\alpha,\gamma}\lambda^2_M M)$ *(where constant* $C_{\alpha,\gamma}$ *depends on* α *and* γ*) whenever*

$$\lambda_M \geq \frac{8\alpha(2-\gamma)}{\gamma}\sqrt{\frac{\log(n)}{M}}\ .$$

Proof. We note that

$$G^M_i = -\nabla \ell_i(w^*_i; D) = \frac{1}{M}\sum_{m=1}^{M} \underbrace{-\alpha(\sigma^{*(m)}_i - a^{*(m)}_i)\,\Phi^{(m)}_{-i}}_{Z_{i,m}}\ ,$$

where $|Z^u_{i,m}| \leq \alpha$ for each component $Z^m_{i,u}$ of random vector Z^m_i. Moreover, $\mathbb{E}(Z^m_{i,u}) = 0$ under w^*_i, and $Z^1_{i,u}, \ldots, Z^M_{i,u}$ are independent. Invoking the Hoeffding's inequality, we have that for any $\delta > 0$,

$$\mathbb{P}(|G^M_{i,u}| \geq \delta) \leq 2\exp\left(-\frac{M\delta^2}{2\alpha^2}\right)\ ,$$

where $G^M_{i,u}$ denotes the component at index u of vector G^M_i. Setting $\delta = \dfrac{\gamma\lambda_M}{4(2-\gamma)}$, we get

$$\mathbb{P}\left(|G^M_{i,u}| \geq \frac{\gamma\lambda_M}{4(2-\gamma)}\right) \;\leq\; 2\exp\left(-\frac{M}{2\alpha^2}\frac{\gamma^2\lambda_M^2}{16(2-\gamma)^2}\right) .$$

Then, applying a union bound over indices $u \in [n-1]$, we get

$$\begin{aligned}\mathbb{P}\left(||G^M_i||_\infty \geq \frac{\gamma\lambda_M}{4(2-\gamma)}\right) &\leq 2(n-1)\exp\left(-\frac{M}{2\alpha^2}\frac{\gamma^2\lambda_M^2}{16(2-\gamma)^2}\right)\\ &< 2\exp\left(-\frac{M}{2\alpha^2}\frac{\gamma^2\lambda_M^2}{16(2-\gamma)^2} + \log(n)\right) .\end{aligned}$$

□

Lemma 2. *Let* $\lambda_M d \;\leq\; \dfrac{\alpha C^2_{\min}}{10 C_{\max}}$ *and* $||G^M_i||_\infty \;\leq\; \dfrac{\lambda_M}{4}$*. Then,*

$$||\hat{w}_{i,S} - w^*_{i,S}||_2 \;\leq\; \frac{5}{\alpha^2 C_{\min}}\lambda_M\sqrt{d} .$$

Proof. We define a function $F : \mathbb{R}^d \to \mathbb{R}$ that quantifies the change in optimization objective at a distance $\Delta_{i,S}$ from the true parameters $w^*_{i,S}$. Specifically,

$$F(\Delta_{i,S}) \;\triangleq\; \ell_i(w^*_{i,S} + \Delta_{i,S}; D) - \ell_i(w^*_{i,S}; D) + \lambda_M(||w^*_{i,S} + \Delta_{i,S}||_1 - ||w^*_{i,S}||_1) .$$

Note that F is convex and $F(\mathbf{0}) \;=\; 0$. Moreover, F is minimized for $\hat{\Delta}_{i,S} = \hat{w}_{i,S} - w^*_{i,S}$. Therefore, $F(\hat{\Delta}_{i,S}) \leq 0$. We show that the function F is strictly positive on the surface of a Euclidean ball of radius B for some $B > 0$. Then, the vector $\hat{\Delta}_{i,S}$ lies inside the ball, i.e.,

$$||\hat{w}_{i,S} - w^*_{i,S}||_2 \leq B .$$

This follows since otherwise, the convex combination $t\hat{\Delta}_{i,S} + (1-t)\mathbf{0}$ would lie on boundary of the ball for some $t \in (0,1)$, which would imply the contradiction

$$F(t\hat{\Delta}_{i,S} + (1-t)\mathbf{0}) \leq tF(\hat{\Delta}_{i,S}) + (1-t)F(\mathbf{0}) \leq 0.$$

Therefore, let $\Delta \in \mathbb{R}^d$ be an arbitrary vector such that $||\Delta||_2 = B$. We then have from Taylor's series

$$F(\Delta) = \nabla\ell_i(w^*_{i,S}; D)^\top\Delta + \Delta^\top\nabla^2\ell(w^*_{i,S} + \theta\Delta; D)\Delta + \lambda_M(||w^*_{i,S} + \Delta||_1 - ||w^*_{i,S}||_1) , \tag{29}$$

for some $\theta \in [0,1]$. We lower bound $F(\Delta)$ by bounding each term on the right side of (29).

We let $B = O\lambda_M\sqrt{d}$ where we will choose $O > 0$ later. From Cauchy-Schwartz inequality,

$$\nabla\ell_i(w^*_{i,S}; D)^\top\Delta \;\geq\; -||\nabla\ell_i(w^*_{i,S}; D)||_\infty||\Delta||_1 \tag{30}$$

$$\geq\; -||\nabla\ell_i(w^*_{i,S}; D)||_\infty\sqrt{d}||\Delta||_2 \tag{31}$$

$$\geq\; -(\lambda_M\sqrt{d})^2\frac{O}{4} , \tag{32}$$

where in the last inequality we have used $||\Delta||_2 = B = O\lambda_M\sqrt{d}$, and

$$-||\nabla\ell_i(w^*_{i,S}; D)||_\infty \geq -||\nabla\ell_i(w^*_i; D)||_\infty \;=\; -||-\nabla\ell_i(w^*_i; D)||_\infty \;=\; -||G^M_i||_\infty \;\geq\; -\frac{\lambda_M}{4}$$

by our assumption on $||G^M_i||_\infty$ in the lemma statement. Next, by triangle inequality, we have

$$\lambda_M(||w^*_{i,S} + \Delta||_1 - ||w^*_{i,S}||_1) \;\geq\; -\lambda_M||\Delta||_1 \;\geq\; -\lambda_M\sqrt{d}||\Delta||_2 \;\geq\; -(\lambda_M\sqrt{d})^2 O . \tag{33}$$

We now bound the quantity $\Delta^\top \nabla^2 \ell(w^*_{i,S} + \theta\Delta; D)\Delta$. We note that

$$\begin{aligned}
\Delta^\top \nabla^2 \ell(w^*_{i,S} + \theta\Delta; D)\Delta &\geq \min_{||\tilde{\Delta}||_2 = B} \tilde{\Delta}^\top \nabla^2 \ell(w^*_{i,S} + \theta\Delta; D)\tilde{\Delta} \\
&\geq \min_{\tilde{\theta}\in[0,1]} B^2 \Lambda_{\min}(\nabla^2 \ell(w^*_{i,S} + \tilde{\theta}\Delta; D)) \\
&= B^2 \min_{\tilde{\theta}\in[0,1]} \Lambda_{\min}\left(\frac{1}{M}\sum_{m=1}^{M} \eta_i(w^*_{i,S} + \tilde{\theta}\Delta; m)\, \Phi^{(m)}_{-i}\Phi^{(m)^\top}_{-i}\right) .
\end{aligned}$$

Applying Taylor's series expansion, we note that $\Delta^\top \nabla^2 \ell(w^*_{i,S} + \theta\Delta; D)\Delta$

$$\begin{aligned}
&\geq B^2 \Lambda_{\min}\left(\frac{1}{M}\sum_{m=1}^{M} \eta_i(w^*_{i,S}; m)\Phi^{(m)}_{-i}\Phi^{(m)^\top}_{-i}\right) \\
&\quad - B^2 \max_{\tilde{\theta}\in[0,1]} \left|\left|\left|\frac{1}{M}\sum_{m=1}^{M} \eta'_i(w^*_{i,S} + \overline{\theta}\Delta; m)(\Phi^{(m)^\top}_{-i}\tilde{\theta}\Delta)\, \Phi^{(m)}_{-i}\Phi^{(m)^\top}_{-i}\right|\right|\right|_2 \\
&= B^2 \Lambda_{\min}(H^{*M}_{i,SS}) - B^2 \max_{\tilde{\theta}\in[0,1]} \left|\left|\left|\frac{1}{M}\sum_{m=1}^{M} \eta'_i(w^*_{i,S} + \overline{\theta}\Delta; m)(\Phi^{(m)^\top}_{-i}\tilde{\theta}\Delta)\, \Phi^{(m)}_{-i}\Phi^{(m)^\top}_{-i}\right|\right|\right|_2 \\
&= B^2 \alpha^2 C_{\min} - B^2 \max_{\tilde{\theta}\in[0,1]} \left|\left|\left|\frac{1}{M}\sum_{m=1}^{M} \eta'_i(w^*_{i,S} + \overline{\theta}\Delta; m)(\Phi^{(m)^\top}_{-i}\tilde{\theta}\Delta)\, \Phi^{(m)}_{-i}\Phi^{(m)^\top}_{-i}\right|\right|\right|_2 .
\end{aligned}$$

Now, a simple calculation shows that $|\eta'_i(\cdot)| \leq \alpha^3$. Moreover, we note for $\tilde{\theta} \in [0, 1]$,

$$|\Phi^{(m)^\top}_{-i}\tilde{\theta}\Delta| \leq ||\Phi^{(m)}_{-i}||_\infty ||\tilde{\theta}\Delta||_1 \leq ||\Phi^{(m)}_{-i}||_\infty ||\Delta||_1 \leq ||\Delta||_1 \leq \sqrt{d}||\Delta||_2 = O\lambda_M d\,.$$

Putting all these facts together, along with our assumption (19), we get

$$\Delta^\top \nabla^2 \ell(w^*_{i,S} + \theta\Delta; D)\Delta \geq B^2\alpha^2 C_{\min} - B^2\alpha^3(O\lambda_M d)C_{\max} \geq B^2\alpha^2\frac{C_{\min}}{2} \tag{34}$$

when $\lambda_M \leq \dfrac{C_{\min}}{2\alpha C_{\max} O d}$. Therefore, plugging the lower bounds from (30), (33), and (34) in (29),

$$F(\Delta) \geq \lambda_M^2 d\left(-\frac{O}{4} - O + \frac{O^2\alpha^2 C_{\min}}{2}\right) > 0\,,$$

for $O = \dfrac{5}{\alpha^2 C_{\min}}$. Thus, for $\lambda_M \leq \dfrac{C_{\min}}{2\alpha C_{\max} O d} = \dfrac{\alpha C^2_{\min}}{10 C_{\max} d}$, we must have

$$||\hat{w}_{i,S} - w^*_{i,S}||_2 \leq B = O\lambda_M\sqrt{d} = \frac{5}{\alpha^2 C_{\min}}\lambda_M\sqrt{d}\,.$$

□

Lemma 3. *Let* $\lambda_M d \leq \dfrac{\alpha C^2_{\min}}{100 C_{\max}}\dfrac{\gamma}{2-\gamma}$ *and* $||G^M_i||_\infty \leq \dfrac{\lambda_M}{4}$. *Then,*

$$\frac{||R^M_i||_\infty}{\lambda_M} \leq \frac{25 C_{\max}}{\alpha C^2_{\min}}\lambda_M d \leq \frac{1}{4}\left(\frac{\gamma}{2-\gamma}\right) \leq \frac{\gamma}{4}\,.$$

Proof. We have for $j \in [n] \setminus \{i\}$ and some $\overline{w}_i^{(j)} = t_j \hat{w}_i + (1-t_j) w_i^*$, $t_j \in [0,1]$,

$$
\begin{aligned}
R_{i,j}^M &= \left(\nabla^2 \ell_i(\overline{w}_i^{(j)}; D) - \nabla^2 \ell_i(w_i^*; D)\right)_j^\top (\hat{w}_i - w_i^*) \\
&= \frac{1}{M} \sum_{m=1}^{M} \left(\left(\eta_i(\overline{w}_i^{(j)}; m) - \eta_i(w_i^*; m)\right) \Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top}\right)_j^\top (\hat{w}_i - w_i^*) \\
&= \frac{1}{M} \sum_{m=1}^{M} \left(\eta_i'(\overline{\overline{w}}_i^{(j)}; m) \left(\Phi_{-i}^{(m)^\top} (\overline{w}_i^{(j)} - w_i^*)\right) \Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top}\right)_j^\top (\hat{w}_i - w_i^*) \,,
\end{aligned}
$$

where $\overline{\overline{w}}_i^{(j)}$ is a point on the line between $\overline{w}_i^{(j)}$ and w_i^*, by the mean value theorem. We note that

$$
\left(\Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top}\right)_j^\top = \phi_j^{(m)} \Phi_{-i}^{(m)^\top} \,.
$$

We thus write

$$
\begin{aligned}
R_{i,j}^M &= \frac{1}{M} \sum_{m=1}^{M} \eta_i'(\overline{\overline{w}}_i^{(j)}; m) \phi_j^{(m)} \left((\overline{w}_i^{(j)} - w_i^*)^\top \Phi_{-i}^{(m)}\right) \Phi_{-i}^{(m)^\top} (\hat{w}_i - w_i^*) \\
&= \frac{1}{M} \sum_{m=1}^{M} \eta_i'(\overline{\overline{w}}_i^{(j)}; m) \phi_j^{(m)} \left((\overline{w}_i^{(j)} - w_i^*)^\top \Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top} (\hat{w}_i - w_i^*)\right) \\
&= \frac{1}{M} \sum_{m=1}^{M} \underbrace{\eta_i'(\overline{\overline{w}}_i^{(j)}; m) \phi_j^{(m)}}_{p^{(m)}} \underbrace{\left(t_j (\hat{w}_i - w_i^*)^\top \Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top} (\hat{w}_i - w_i^*)\right)}_{q^{(m)}} \,,
\end{aligned}
$$

which is of the form $\frac{1}{M} p^\top q$, where $p, q \in \mathbb{R}^M$. Thus, we have by Cauchy-Schwartz inequality,

$$
|R_{i,j}^M| = \frac{1}{M} |p^\top q| \le \frac{1}{M} ||p||_\infty ||q||_1 \,.
$$

It can be shown that $p^{(m)} = \alpha^3 \overline{\overline{\sigma}}_i^{(m)} (1 - \overline{\overline{\sigma}}_i^{(m)})(1 - 2\overline{\overline{\sigma}}_i^{(m)})$, whereby $||p||_\infty \le \alpha^3$.

Finally, we see that $q^{(m)} = t_j \left|\left|\Phi_{-i}^{(m)^\top} (\hat{w}_i - w_i^*)\right|\right|_2^2 \ge 0$ since $t_j \in [0,1]$. Therefore $||q||_1 = q^\top \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^M$ is a vector of all ones. Moreover, since $\hat{w}_{i,S^c} = w_{i,S^c}^* = \mathbf{0}$, we note that

$$
\begin{aligned}
\frac{1}{M} ||q||_1 &= t_j (\hat{w}_i - w_i^*)^\top \left(\frac{1}{M} \sum_{m=1}^{M} \Phi_{-i}^{(m)} \Phi_{-i}^{(m)^\top}\right) (\hat{w}_i - w_i^*) \\
&= t_j (\hat{w}_{i,S} - w_{i,S}^*)^\top \left(\frac{1}{M} \sum_{m=1}^{M} \Phi_{-i,S}^{(m)} \Phi_{-i,S}^{(m)^\top}\right) (\hat{w}_{i,S} - w_{i,S}^*) \\
&\le C_{\max} \left|\left|\hat{w}_{i,S} - w_{i,S}^*\right|\right|_2^2 \,.
\end{aligned}
$$

Since $\gamma \in (0,1]$, so

$$
\lambda_M d \le \frac{\alpha C_{\min}^2}{100 C_{\max}} \frac{\gamma}{2-\gamma} \le \frac{\alpha C_{\min}^2}{100 C_{\max}} \le \frac{\alpha C_{\min}^2}{10 C_{\max}} \,.
$$

Therefore, we can invoke Lemma 2 when $||G_i^M||_\infty \le \frac{\lambda_M}{4}$. Specifically, we then have for each j,

$$
|R_{i,j}^M| \le \alpha^3 C_{\max} \left|\left|\hat{w}_{i,S} - w_{i,S}^*\right|\right|_2^2 \le \alpha^3 C_{\max} \left(\frac{5}{\alpha^2 C_{\min}} \lambda_M \sqrt{d}\right)^2 = \frac{25 C_{\max}}{\alpha C_{\min}^2} \lambda_M^2 d \,.
$$

This immediately yields $\frac{||R_i^M||_\infty}{\lambda_M} \le \frac{25 C_{\max}}{\alpha C_{\min}^2} \lambda_M d \,.$ □

We are now ready to prove our main result.

Theorem 1. *Let* $M > \frac{80^2 C_{\max}^2}{C_{\min}^4}\left(\frac{2-\gamma}{\gamma}\right)^4 d^2\log(n)$, *and* $\lambda_M \geq \frac{8\alpha(2-\gamma)}{\gamma}\sqrt{\frac{\log(n)}{M}}$. *Suppose the sample satisfies assumptions* (18), (19), *and* (20). *Define* $C_{\alpha,\gamma} = \frac{\gamma^2}{32\alpha^2(2-\gamma)^2}$. *Consider any player* $i \in [n]$. *The following results hold with probability at least* $1 - 2\exp(-C_{\alpha,\gamma}\lambda_M^2 M) \to 1$ *for* i.

1. *The corresponding* L_1*-regularized optimization problem has a unique solution, i.e., a unique set of neighbors for* i.
2. *The set of predicted neighbors of* i *is a subset of the true neighbors. Additionally, the predicted set contains all true neighbors* j *for which* $|w_{ij}^*| \geq \frac{10}{\alpha^2 C_{\min}}\sqrt{d}\lambda_M$. *In particular, the set of true neighbors of* i *is exactly recovered if*

$$\min_{j\in S_i}|w_{ij}^*| \geq \frac{10}{\alpha^2 C_{\min}}\sqrt{d}\lambda_M \ .$$

Taking a union bound over players, our results imply that we recover the true signed neighborhoods for all players in the game with probability at least $1 - 2n\exp(-C_{\alpha,\gamma}\lambda_M^2 M)$.

Proof. Since $\lambda_M \geq \frac{8\alpha(2-\gamma)}{\gamma}\sqrt{\frac{\log(n)}{M}}$, Lemma 1 holds. Thus, with high probability (as stated in the theorem statement), we obtain

$$||G_i^M||_\infty \ \leq \ \frac{\lambda_M}{4}\frac{\gamma}{2-\gamma} \ \leq \ \frac{\gamma\lambda_M}{4} \ \leq \ \frac{\lambda}{4}\,, \tag{35}$$

since $\gamma \in (0,1]$. Moreover, for the specified lower bound on sample size M, a simple computation shows

$$\lambda_M d \ \leq \ \frac{\alpha C_{\min}^2}{10 C_{\max}}\frac{\gamma}{2-\gamma} \ . \tag{36}$$

Thus the conditions required for both Lemma 2 and Lemma 3 are satisfied. By our primal-dual construction, $\hat{w}_{i,S^c} = \mathbf{0}$. Furthermore, using (18), $\Lambda_{\min}(H_{i,SS}^{*M}) > 0$, and so $H_{i,SS}^{*M}$ is invertible. Separating the rows in the support of i and others, we write (28) as

$$\begin{aligned} H_{i,S^cS}^{*M}(\hat{w}_{iS} - w_{iS}^*) &= G_{i,S^c}^M - \lambda_M\hat{\kappa}_{i,S^c} - R_{i,S^c}^M \\ H_{i,SS}^{*M}(\hat{w}_{iS} - w_{iS}^*) &= G_{i,S}^M - \lambda_M\hat{\kappa}_{i,S} - R_{i,S}^M \ . \end{aligned}$$

These two equations can be combined into one as

$$H_{i,S^cS}^{*M}(H_{i,SS}^{*M})^{-1}\left(G_{i,S}^M - \lambda_M\hat{\kappa}_{i,S} - R_{i,S}^M\right) \ = \ G_{i,S^c}^M - \lambda_M\hat{\kappa}_{i,S^c} - R_{i,S^c}^M \ .$$

Recalling that $||\hat{\kappa}_{i,S}||_\infty < 1$, we immediately get that $\lambda_M||\hat{\kappa}_{i,S^c}||_\infty$

$$\begin{aligned} &\leq \ |||H_{i,S^cS}^{*M}(H_{i,SS}^{*M})^{-1}|||_\infty\left(||G_{i,S}^M||_\infty \ + \ ||R_{i,S}^M||_\infty + \lambda_M\right) \ + \ ||G_{i,S^c}^M||_\infty \ + \ ||R_{i,S^c}^M||_\infty \\ &\leq \ (1-\gamma)\left(||G_{i,S}^M||_\infty \ + \ ||R_{i,S}^M||_\infty + \lambda_M\right) \ + \ ||G_{i,S^c}^M||_\infty \ + \ ||R_{i,S^c}^M||_\infty \\ &\leq \ (1-\gamma)\lambda_M \ + \ ||G_i^M||_\infty \ + \ ||R_i^M||_\infty \\ &\leq \ \lambda_M\left(1 - \gamma + \frac{\gamma}{4} + \frac{\gamma}{4}\right) \\ &= \ \lambda_M\left(1 - \frac{\gamma}{2}\right) \ . \end{aligned}$$

Since $\gamma \in (0,1]$ and $\lambda_M > 0$, we immediately get $||\hat{\kappa}_{i,S^c}||_\infty < 1$. Therefore, strict dual feasibility is established and (23) is verified. Then, using Lemma 1 of (Ravikumar et al., 2010), we note that any optimal solution $\tilde{w}_i$ of (21) must have $\tilde{w}_{i,S^c} = \mathbf{0}$. In particular, we have $\hat{w}_{i,S^c} = \mathbf{0}$ as desired. Thus, we can focus on $\hat{w}_{i,S}$. We now prove uniqueness of $\hat{w}_i$ by showing that $\Lambda_{\min}\left(\hat{H}_{i,SS}^M\right) \ > \ 0$. Let $\Delta = \hat{w}_{i,S} - w_{i,S}^* \in \mathbb{R}^d$. Then, using Lemma 2, we have

$$||\Delta||_2 \ \leq \ \frac{5}{\alpha^2 C_{\min}}\lambda_M\sqrt{d}\,.$$

Note that

$$\begin{aligned}\Lambda_{\min}\left(\hat{H}^M_{i,SS}\right) &= \Lambda_{\min}\left(\frac{1}{M}\sum_{m=1}^{M}\eta_i(\hat{w}_i;m)\,\Phi^{(m)}_{-i,S}\Phi^{(m)^\top}_{-i,S}\right)\\ &= \Lambda_{\min}\left(\frac{1}{M}\sum_{m=1}^{M}\eta_i(\hat{w}_{i,S};m)\,\Phi^{(m)}_{-i,S}\Phi^{(m)^\top}_{-i,S}\right)\\ &= \Lambda_{\min}\left(\frac{1}{M}\sum_{m=1}^{M}\eta_i(w^*_{i,S}+\Delta;m)\,\Phi^{(m)}_{-i,S}\Phi^{(m)^\top}_{-i,S}\right)\,.\end{aligned}$$

Performing a Taylor expansion around $w^*_{i,S}$, and making arguments similar to the proof segment between (33) and (34) in Lemma 2, we can show that

$$\begin{aligned}\Lambda_{\min}\left(\hat{H}^M_{i,SS}\right) &\geq \alpha^2 C_{\min} - \alpha^3\sqrt{d}||\Delta||_2 C_{\max}\\ &\geq \alpha^2 C_{\min} - \left(\frac{5\alpha C_{\max}}{C_{\min}}\right)\lambda_M d\\ &\geq \alpha^2 C_{\min} - \alpha^2\frac{C_{\min}}{2}\frac{\gamma}{2-\gamma}\\ &\geq \alpha^2\frac{C_{\min}}{2}\,,\end{aligned}$$

which is greater than 0. Therefore, $\hat{H}^M_{i,SS}$ is positive definite, and Lemma 1 of (Ravikumar et al., 2010) guarantees that $\hat{w}_i$ is the unique optimal primal solution for (21).

We finally argue about the only remaining condition (24). In order for neighbor j to be correctly recovered with sign, i.e., $\text{sign}(\hat{w}_{ij}) = \text{sign}(w^*_{ij})$, it suffices to have

$$|\hat{w}_{ij} - w^*_{ij}| \leq \frac{|w^*_{ij}|}{2}\,. \tag{37}$$

Moreover to recover the neighborhood of i exactly, it is sufficient to show

$$\min_{j\in S_i}|w^*_{ij}| \geq 2||\hat{w}_{i,S} - w^*_{i,S}||_\infty\,, \tag{38}$$

which implies (37). We note that

$$||\hat{w}_{i,S} - w^*_{i,S}||_\infty \leq ||\hat{w}_{i,S} - w^*_{i,S}||_2 \leq \frac{5}{\alpha^2 C_{\min}}\lambda_M\sqrt{d}\,.$$

Using (38), it immediately follows that the neighborhood of i is recovered with correct sign if

$$\min_{j\in S_i}|w^*_{ij}| \geq \frac{10}{\alpha^2 C_{\min}}\lambda_M\sqrt{d}\,.$$

□

B. General game dynamics and convergence

In this section we provide an in-depth look at the game dynamics along with associated convergence guarantees.

Recall that in of our protocols, players take actions stochastically according to σ_i^k and the best response mapping g_i is unique for each k. Assuming that the error sequence in updating $\{\sigma_i^k\}$ is a martingale, our updates satisfy the conditions outlined in (section 2.1 of (Borkar, 2008)) and we can analyze the stochastic evolution of each setting as a noisy discretization of a limiting ordinary differential equation (ODE). In particular, Lipschitz condition is satisfied since g_i and h_i are both Lipschitz continuous, step size condition is fulfilled since the sequence $b^{k-1} = 1/k$ satisfies $\sum_k b^{k-1} = \infty$ and $\sum_k (b^{k-1})^2 < \infty$,

$$\begin{aligned}
(\textbf{SAP-FP/AA}) \quad \dot{q}_i &= \beta_i^{\tau}(\mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}), z_i) - q_i, & \dot{r}_i = \lambda(q_i - r_i) \quad (39)\\
(\textbf{SAP-FP/ PA}) \quad \dot{q}_i &= \beta_i^{\tau}(\mathcal{A}_i(q_{-i}) + \gamma \dot{r}_i, z_i) - q_i, & \dot{r}_i = \lambda(\mathcal{A}_i(q_{-i}) - r_i) \quad (40)\\
(\textbf{SAP-GP/AA}) \quad \dot{q}_i &= \Pi_{\Delta}[q_i + \mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i] - q_i, & \dot{r}_i = \lambda(q_i - r_i) \quad (41)\\
(\textbf{SAP-GP/PA}) \quad \dot{q}_i &= \Pi_{\Delta}[q_i + \mathcal{A}_i(q_{-i}) + \gamma \dot{r}_i - z_i] - q_i, & \dot{r}_i = \lambda(\mathcal{A}_i(q_{-i}) - r_i) \quad (42)
\end{aligned}$$

and our iterates remain bounded since they remain confined to $\Delta(A)$. Thus, we can investigate the conditions under which the fixed points of the limiting ODE are *locally asymptotically stable* (*l.a.s.*), and as a consequence, our discrete updates would converge to a Nash equilibrium with positive probability (Shamma and Arslan, 2005). An equilibrium point s is said to be *l.a.s.* if every ODE trajectory that starts at a point in a small neighborhood of s remains forever in that neighborhood and eventually converges to s.

Our updates in (9) lead to the implicit ODEs (39)-(42) for SAP-FP and SAP-GP under AA and PA settings, where $\lambda > 0$, $\dot{r}_i$ is an estimate for $\dot{q}_i$, and $\dot{r}_{-i} \triangleq \{\dot{r}_j | j \neq i, w_{ij} \neq 0\}$. We will call a matrix *stable* if all its eigenvalues have strictly negative real parts. Let I denote the identity matrix. We now state results that characterize conditions under which different dynamics lead to asymptotically stable equilibria.

Theorem 2. (SAP-FP/AA convergence to NE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a NE under the dynamics in* (39)*. There exists a matrix $\mathcal{D}$ such that the linearization of* (39) *with $\gamma > 0$ is l.a.s. for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} -I + (1+\gamma\lambda)\mathcal{D} & -\gamma\lambda\mathcal{D} \\ \lambda I & -\lambda I \end{bmatrix}.$$

Theorem 3. (SAP-FP/PA convergence to NE) *Let the weight matrix W be stochastic. Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a NE under the dynamics in* (40)*. There exists a matrix $\mathcal{D}_1$ with zero diagonal, and a block diagonal matrix $\mathcal{D}_2$ such that the linearization of* (40) *with $\gamma > 0$ is l.a.s. for $\lambda > 0$ if and only if if the following matrix is stable*

$$\begin{bmatrix} -I + (1+\gamma\lambda)\mathcal{D}_1 & -\gamma\lambda\mathcal{D}_2 \\ \lambda W & -\lambda I \end{bmatrix}.$$

Theorem 4. (SAP-GP/AA convergence to CMNE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a completely mixed NE under the dynamics in* (41)*. Then the linearization of* (41) *with $\gamma > 0$ is l.a.s. for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda I & -\lambda I \end{bmatrix}.$$

Theorem 5. (SAP-GP/AA convergence to SNE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a strict NE under the dynamics in* (41)*. The associated equilibrium point $(q_i = q_i^*, q_{-i} = q_{-i}^*, r_i = q_i^*, r_{-i} = q_{-i}^*)$ is l.a.s. for any $\gamma > 0$ and $\lambda > 0$.*

Theorem 6. (SAP-GP/PA convergence to CMNE) *Let the weight matrix W be stochastic. Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a completely mixed NE under the dynamics in* (42)*. Then the linearization of* (42) *with $\gamma > 0$ is l.a.s. for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda W & -\lambda I \end{bmatrix}.$$

Theorem 7. (SAP-GP/PA convergence to SNE) *Let the weight matrix W be doubly stochastic. Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a strict NE under the dynamics in* (42)*. The equilibrium point $(q_i = q_i^*, r_i = A_i(q_{-i}^*))_{i \in [n]}$ is l.a.s. for sufficiently small $\gamma\lambda$, where $\gamma > 0$ and $\lambda > 0$.*

We now provide some insight into our proof techniques. We follow the general proof structure of (Shamma and Arslan, 2005). Specifically, we prove convergence to SNE via carefully crafted *Lyapunov functions* $\mathcal{V}$ that are locally positive definite and have a locally negative semidefinite time derivative, and thus satisfy the *Lyapunov stability* criterion. The other proofs track the evolution of game dynamics around an equilibrium, where $\dot{q}_i = 0$ and $\dot{r}_i = 0$. Specifically, we analyze conditions under which the Jacobian matrix of the linearization is *Hurwitz stable*, i.e., all the eigenvalues have negative real roots, and exploit the fact that the behavior of the ODE near equilibrium is same as its linear approximation when the real parts of all eigenvalues are non-zero. Our discrete updates would then converge to a Nash equilibrium with positive probability (Shamma and Arslan, 2005).

Recall that AA reveals more information about the evolution of neighbors' strategy. As a result, the PA settings, i.e. (40) and (42), require additional subtle reasoning since at equilibrium r_i^* converges only to $\mathcal{A}_i(q_{-i}^*)$ and not to q_i^*. Since q_i evolves within $\Delta(A)$, stochasticity assumptions are required to ensure r_i stays within the probability simplex as well. Note that the SAP-FP updates to strategies are smooth due to the entropy term (since $\tau > 0$), unlike SAP-GP. Consequently, the results for SAP-GP require a separate treatment of completely mixed NE and strict NE, unlike SAP-FP where they can be analyzed without distinction. Note that $\tau > 0$ ensures that best response is a singleton set and therefore we could leverage the ODE formulations. Differential inclusions (Benaïm et al., 2005; 2006) could possibly be used to handle $\tau = 0$.

We now provide detailed proofs on convergence of dynamics. We restructure the theorem statements to have the results for the active aggregator setting precede those for the passive aggregator setting. We use AA1, AA2 etc. to indicate that the result pertains to convergence in an active aggregator setting. Likewise, we will use PA1 etc. for the passive aggregator setting. We start with the active aggregator.

Theorem AA1. (Convergence under SAP-FP/AA to NE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a NE under the dynamics in* (39). *There exists a matrix $\mathcal{D}$ such that the linearization of* (39) *with $\gamma > 0$ is locally asymptotically stable for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} -I + (1+\gamma\lambda)\mathcal{D} & -\gamma\lambda\mathcal{D} \\ \lambda I & -\lambda I \end{bmatrix}.$$

Proof. Since $\tau > 0$, best response is a singleton set, and the unique best response σ_i^* can be obtained by setting the gradients of the payoff functions to 0. In particular, we have the best response

$$\beta_i^\tau(\mathcal{A}_i(\sigma_{-i}), z_i) = \zeta\left(\frac{\sum_{j\neq i} w_{ij}\sigma_j - z_i}{\tau}\right) = \zeta\left(\frac{\mathcal{A}_i(\sigma_{-i}) - z_i}{\tau}\right), \tag{43}$$

where ζ is the softmax function with output coordinate ℓ given by

$$(\zeta(x))_\ell = \exp(x_\ell) \Big/ \sum_k \exp(x_k).$$

Now recall from (39) that we have the following ODE:

$$\dot{q}_i = \underbrace{\beta_i^\tau(\mathcal{A}_i(q_{-i} + \gamma\dot{r}_{-i}), z_i^*) - q_i}_{\triangleq F_i(q_i, q_{-i}, r_{-i})} \tag{44}$$

$$\dot{r}_i = \lambda(q_i - r_i). \tag{45}$$

Since β_i^τ maps it input to the simplex $\Delta(A)$, we note that the right side of (44) is a difference between two probability distributions. Therefore this difference must sum to zero. Moreover, since $|A| = m$, we have $m-1$ degrees of freedom that can be used to express this difference. Therefore, we can investigate the evolution of q_i via a matrix $N \in \mathbb{R}^{m\times(m-1)}$ of $(m-1)$ orthonormal columns such that

$$N^\top N = I_{m-1}, \text{ and } 1_m^\top N = 0_{m-1},$$

where I_{m-1} is the identity matrix of order $m-1$, and 1_m and 0_m are m-dimensional vectors with all coordinates set to 1 and 0 respectively. We will sometimes omit the subscripts for I_m, 1_m, and 0_m when the size will be clear from the context. The equilibrium (q_i^*, q_{-i}^*) corresponds to a point $(q_i(t) = q_i^*, q_{-i}(t) = q_{-i}^*, r_i(t) = q_i^*, r_{-i}(t) = q_{-i}^*)$ of the dynamics. It will be convenient to investigate the dynamics as the evolution of deviations around this point. Since q_i is confined to $\Delta(A)$, we can express

$$q_i(t) = q_i^* + N\delta x_{q_i}(t),$$

where $\delta x_{q_i}(t) \in \mathbb{R}^{m-1}$ is uniquely specified, and likewise $r_i = q_i^* + \delta x_{r_i}(t)$ for some $\delta x_{r_i}(t)$. Thus, we can define a block diagonal matrix $\mathcal{N} \in \mathbb{R}^{2nm\times 2n(m-1)}$, with each diagonal block set to N and all other elements set to 0, such that

$$(q_1(t) - q_1^*, \ldots, q_n(t) - q_n^*, \quad r_1(t) - q_1^*, \ldots, r_n(t) - q_n^*)^\top = \mathcal{N}\delta x(t), \tag{46}$$

where

$$\delta x(t) = (\delta x_{q_1}(t), \ldots, \delta x_{q_n}(t), \delta x_{r_1}(t), \ldots, \delta x_{r_n}(t))^\top \in \mathbb{R}^{2n(m-1)}$$

is formed by stacking together the deviations at time t in a column vector. Then, the following is immediate from (46):

$$\mathcal{N}^\top(q_1(t)-q_1^*,\ldots,q_n(t)-q_n^*,r_1(t)-q_1^*,\ldots,r_n(t)-q_n^*)^\top \;=\; \mathcal{N}^\top\mathcal{N}\delta x(t) \;=\; \delta x(t). \tag{47}$$

Denote the Jacobian matrix obtained by taking derivatives of vector y with respect to vector x by $J_x y$. We will linearize $\dot{q}_i = F_i(q_i, q_{-i}, r_{-i})$ in (44) around $\triangleq (q_1^*, q_{-1}^*, q_1^*, q_{-1}^*)$ using first order Taylor series. Then, since $\dot{q}_i^* = 0$, we note from (44) and (47) that

$$\dot{\delta x}_{q_i} = N^\top(\dot{q}_i - \dot{q}_i^*) = N^\top \dot{q}_i(t) = N^\top F_i(q_i, q_{-i}, r_{-i}). \tag{48}$$

Now, at equilibrium, we have $\dot{q}_i = 0$ for all $i \in [n]$, and therefore we have from (44) that

$$F_i(q_i^*, q_{-i}^*, r_{-i}^*) = 0_m.$$

Let $\operatorname{diag}(b)$ be a diagonal matrix with vector b on the diagonal and all other elements set to 0. Ignoring the second order and higher terms, we therefore have by the Taylor series approximation that

$$\begin{aligned}
& F_i(q_i, q_{-i}, r_{-i}) \\
\approx\;& \sum_{k=1}^{n} J_{q_k}F_i(q_k, q_{-k}^*, q_{-k}^*)\Big|_{q_k=q_k^*}(q_k - q_k^*) \;+\; \sum_{k\neq i} J_{r_k}F_i(q_k^*, q_{-k}^*, q_{-ki}^*, r_k)\Big|_{r_k=q_k^*}(r_k - q_k^*) \\
=\;& \sum_{k=1}^{n} J_{q_k}F_i(q_k, q_{-k}^*, q_{-k}^*)\Big|_{q_k=q_k^*} N\delta x_{q_k} \;+\; \sum_{k\neq i} J_{r_k}F_i(q_k^*, q_{-k}^*, q_{-ki}^*, r_k)\Big|_{r_k=q_k^*} N\delta x_{r_k} \\
=\;& -N\delta x_{q_i} + \sum_{k\neq i} J_{q_k}F_i(q_k, q_{-k}^*, q_{-k}^*)\Big|_{q_k=q_k^*} N\delta x_{q_k} \;+\; \sum_{k\neq i} J_{r_k}F_i(q_k^*, q_{-k}^*, q_{-ki}^*, r_k)\Big|_{r_k=q_k^*} N\delta x_{r_k} \\
=\;& -N\delta x_{q_i} \;+\; (1+\gamma\lambda)\sum_{k\neq i}\tilde{D}_{ik}N\delta x_{q_k} \;-\; \gamma\lambda\sum_{k\neq i}\tilde{D}_{ik}N\delta x_{r_k},
\end{aligned}$$

where $\tilde{D}_{ik} \triangleq \dfrac{w_{ik}}{\tau}\nabla\zeta\left(\dfrac{A_i(q_{-i}^*) - z_i}{\tau}\right)$, and $\nabla\zeta(b) \triangleq \operatorname{diag}(\zeta(b)) - \zeta(b)\zeta^\top(b)$.

Define $D_{ik} = N^\top \tilde{D}_{ik} N$. Since $N^\top N = I_{m-1}$, it follows immediately from (48) that

$$\dot{\delta x}_{q_i} = -\delta x_{q_i} \;+\; (1+\gamma\lambda)\sum_{k\neq i} D_{ik}\delta x_{q_k} \;-\; \gamma\lambda\sum_{k\neq i} D_{ik}\delta x_{r_k}. \tag{49}$$

Linearizing (45), we see that the Taylor approximation results in

$$\dot{\delta x}_{r_i} = \lambda(\delta x_{q_i} - \delta x_{r_i}). \tag{50}$$

We define

$$\mathcal{D} = \begin{bmatrix} 0 & D_{12} & D_{13} & \ldots & D_{1n} \\ D_{21} & 0 & D_{23} & \ldots & D_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \ldots & 0 \end{bmatrix}.$$

Combining (49) and (50) together, we can write

$$\dot{\delta x} = \begin{bmatrix} -I + (1+\gamma\lambda)\mathcal{D} & -\gamma\lambda\mathcal{D} \\ \lambda I & -\lambda I \end{bmatrix}\delta x.$$

The statement of the theorem now follows immediately from the Hurwitz stability criterion. □

Theorem AA2. (Convergence under SAP-GP/AA to CMNE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a completely mixed NE under the dynamics in* (41)*. Then the linearization of* (41) *with $\gamma > 0$ is locally asymptotically stable for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda I & -\lambda I \end{bmatrix}.$$

Proof. Recall the ODE from (41):

$$\begin{aligned} \dot{q}_i &= \Pi_\Delta[q_i + \mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i] - q_i && (51) \\ \dot{r}_i &= \lambda(q_i - r_i). && (52) \end{aligned}$$

At equilibrium $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$, $\dot{q}_i = 0$ and $\dot{r}_i = 0$. Therefore, using (56), we have:

$$q_i^* = \Pi_\Delta[q_i^* + \mathcal{A}_i(q_{-i}^*) - z_i].$$

Since the equilibrium is completely mixed, q_i^* is in the interior of $\Delta(A)$. We invoke Lemma 4.1 in (Shamma and Arslan, 2005) to get the following:

$$\begin{aligned} NN^\top(\mathcal{A}_i(q_{-i}^*) - z_i) &= 0 && (53) \\ \Pi_\Delta[q_i^* + \mathcal{A}_i(q_{-i}^*) - z_i + \delta y] - q_i^* &= NN^\top\Big(\mathcal{A}_i(q_{-i}^*) \ - \ z_i + \delta y\Big), \end{aligned}$$

for δy sufficiently small, and N as defined in the proof of Theorem AA1. Then, for a sufficiently small deviation δx, where δx is as defined in Theorem 1, we get the following dynamics:

$$\begin{aligned} \dot{q}_i &= NN^\top[\mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i] && (54) \\ \dot{r}_i &= \lambda(q_i - r_i). && (55) \end{aligned}$$

Linearizing these equations and noting that $N^\top N = I$, we get

$$\begin{aligned} \dot{\delta x}_{q_i} &= N^\top \left(NN^\top (1+\gamma\lambda) \sum_{k \neq i} w_{ik} N \delta x_{q_k} \right) \ - \ N^\top \left(NN^\top \gamma\lambda \sum_{k \neq i} w_{ik} N \delta x_{r_k} \right) \\ &= (1+\gamma\lambda) N^\top \sum_{k \neq i} w_{ik} N \delta x_{q_k} \ - \ \gamma\lambda N^\top \sum_{k \neq i} w_{ik} N \delta x_{r_k} \\ &= (1+\gamma\lambda) \sum_{k \neq i} w_{ik} \delta x_{q_k} \ - \ \gamma\lambda \sum_{k \neq i} w_{ik} \delta x_{r_k}, \end{aligned}$$

and

$$\dot{\delta x}_{r_i} = \lambda(\delta x_{q_i} - \delta x_{r_i}).$$

It follows immediately that

$$\dot{\delta x} = \begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda I & -\lambda I \end{bmatrix} \delta x,$$

where the weight matrix

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} & \ldots & w_{1n} \\ w_{21} & 0 & w_{23} & \ldots & w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \ldots & 0 \end{bmatrix}.$$

□

Theorem AA3. (Convergence under SAP-GP/AA to SNE) *Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a strict NE under the dynamics in* (41)*. The associated equilibrium point $(q_i = q_i^*, q_{-i} = q_{-i}^*, r_i = q_i^*, r_{-i} = q_{-i}^*)$ is locally asymptotically stable for any $\gamma > 0$ and $\lambda > 0$.*

Proof. Recall the ODE from (41):

$$\begin{aligned} \dot{q}_i &= \Pi_\Delta[q_i + \mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i] - q_i && (56) \\ \dot{r}_i &= \lambda(q_i - r_i). && (57) \end{aligned}$$

To prove the local asymptotic stability of the ODE dynamics, we will define a Lyapunov function $\mathcal{V}$ that is locally positive definite and has locally negative semi-definite time derivative. Consider

$$\mathcal{V}(q_i, q_{-i}, r_i, r_{-i}) \triangleq \frac{1}{2}\sum_{i=1}^{n}\left((q_i - q_i^*)^\top (q_i - q_i^*) + \lambda(r_i - q_i)^\top (r_i - q_i)\right). \quad (58)$$

We define the shorthand $d_i \triangleq q_i + \mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i$. Applying the chain rule, we see that the time derivative of $\mathcal{V}$,

$$\begin{aligned} \dot{\mathcal{V}} &= \sum_{i=1}^{n}\left(\frac{\partial \mathcal{V}}{\partial q_i}\right)^\top \dot{q}_i + \sum_{i=1}^{n}\left(\frac{\partial \mathcal{V}}{\partial r_i}\right)^\top \dot{r}_i \\ &= \sum_{i=1}^{n}\left[(q_i - q_i^*) + \lambda(q_i - r_i)\right]^\top \dot{q}_i - \lambda^2 \sum_{i=1}^{n}(r_i - q_i)^\top (r_i - q_i) \\ &= \sum_{i=1}^{n}(q_i - q_i^*)^\top \dot{q}_i + \lambda \sum_{i=1}^{n}(q_i - r_i)^\top \dot{q}_i \ - \ \lambda^2 \sum_{i=1}^{n}||r_i - q_i||^2 \\ &= \sum_{i=1}^{n}(q_i - q_i^*)^\top \Pi_\Delta(d_i) - \sum_{i=1}^{n}(q_i - q_i^*)^\top q_i \ + \ \lambda \sum_{i=1}^{n}(q_i - r_i)^\top \dot{q}_i - \lambda^2 \sum_{i=1}^{n}||r_i - q_i||^2. \end{aligned}$$

Also, we note that

$$\begin{aligned} \sum_{i=1}^{n}||\dot{q}_i||^2 &= \sum_{i=1}^{n}||\Pi_\Delta(d_i) - q_i||^2 \\ &= \sum_{i=1}^{n}||\Pi_\Delta(d_i)||^2 + \sum_{i=1}^{n} q_i^\top q_i - 2\sum_{i=1}^{n} q_i^\top \Pi_\Delta(d_i). \end{aligned}$$

This immediately implies

$$\begin{aligned} \dot{\mathcal{V}} \ + \ \sum_{i=1}^{n}||\dot{q}_i||^2 &= \sum_{i=1}^{n}\underbrace{\left(\Pi_\Delta(d_i) - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i\right)}_{(B)} \\ &\quad + \ \lambda \sum_{i=1}^{n}(q_i - r_i)^\top \dot{q}_i - \lambda^2 \sum_{i=1}^{n}||r_i - q_i||^2. \end{aligned} \quad (59)$$

Consider $(B) = \left(\Pi_\Delta(d_i) - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i\right)$. Since $\Delta(A)$ is a convex set, the projection property implies

$$\left[\Pi_\Delta(d_i)\right]^\top \left(\Pi_\Delta(d_i) - q_i\right) \le d_i^\top \left(\Pi_\Delta(d_i) - q_i\right),$$

whence we note

$$\begin{aligned} (B) &= \left(\Pi_\Delta(d_i) - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i\right) \\ &= \left[\Pi_\Delta(d_i)\right]^\top \left(\Pi_\Delta(d_i) - q_i\right) - \left(\Pi_\Delta(d_i) - q_i\right)^\top q_i^* \\ &\le d_i^\top \left(\Pi_\Delta(d_i) - q_i\right) - \left(\Pi_\Delta(d_i) - q_i\right)^\top q_i^* \\ &= \left(d_i - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i\right) \\ &= \left(q_i + \mathcal{A}_i(q_{-i} + \gamma \dot{r}_{-i}) - z_i - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i\right). \end{aligned}$$

Now, we note from the definition of $\mathcal{V}$ in (58) that by decreasing the distances $(q_i - r_i)$ and $(q_i - q_i^*)$, we can make $\mathcal{V}(q_i, q_{-i}, r_i, r_{-i})$ arbitrarily close to 0 from the right. In other words, we can consider a sufficiently small neighborhood around the equilibrium such that as $r_i, q_i \to q_i^*$, (B) tends to

$$\begin{aligned} & \left(q_i^* + \mathcal{A}_i(q_{-i}^* + \delta y) - z_i - q_i^*\right)^\top \left(\Pi_\Delta(d_i) - q_i^*\right) \\ = & \quad \left(\mathcal{A}_i(q_{-i}^* + \delta y) - z_i\right)^\top \left(\Pi_\Delta(d_i) - q_i^*\right), \\ = & \quad \left(\Pi_\Delta(d_i) - q_i^*\right)^\top \frac{\partial U_i(q_i, q_{-i}^* + \delta y, z_i)}{\partial q_i}\bigg|_{q_i = q_i^*} < 0 \end{aligned}$$

for some sufficiently small δy and $\Pi_\Delta(d_i) \neq q_i^*$. The last inequality follows since $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ is a strict Nash equilibrium, whereby (a) (q_i^*, q_{-i}^*) is a pure strategy Nash equilibrium (since $\mathcal{A}_i(\cdot)$ is a linear transformation and the payoff maximization happens at the vertex), and (b) q_i^* is a (strictly) best response to q_{-i}^* and nearby strategies. Therefore, we see from (59) that for a sufficiently small neighborhood around the equilibrium point,

$$\begin{aligned} \dot{\mathcal{V}} &\leq -\sum_{i=1}^n ||\dot{q}_i||^2 + \lambda \sum_{i=1}^n (q_i - r_i)^\top \dot{q}_i - \lambda^2 \sum_{i=1}^n ||r_i - q_i||^2 \\ &= -\sum_{i=1}^n \left(||\dot{q}_i||^2 + \lambda^2 ||r_i - q_i||^2\right) + \lambda \sum_{i=1}^n (q_i - r_i)^\top \dot{q}_i \\ &\leq -\sum_{i=1}^n \left(||\dot{q}_i||^2 + \lambda^2 ||r_i - q_i||^2\right) + \frac{1}{2} \sum_{i=1}^n \left(||\dot{q}_i||^2 + \lambda^2 ||r_i - q_i||^2\right) \\ &= -\frac{1}{2} \sum_{i=1}^n \left(||\dot{q}_i||^2 + \lambda^2 ||r_i - q_i||^2\right), \end{aligned}$$

where we have invoked the Cauchy-Schwarz inequality in the penultimate line. Since this quantity is non-positive, we see that $\dot{\mathcal{V}}$ is locally negative semi-definite. Finally, it is clear from (58) that $\mathcal{V}(q_i, q_{-i}, r_i, r_{-i}) > 0$ in the neighborhood $(q_i, q_{-i}, r_i, r_{-i})$ of the equilibrium point $(q_i = q_i^*, q_{-i} = q_{-i}^*, r_i = q_i^*, r_{-i} = q_{-i}^*)$, and $\mathcal{V}(q_i^*, q_{-i}^*, q_i^*, q_{-i}^*) = 0$. Thus, $\mathcal{V}$ is locally positive definite, and the statement of the theorem follows. □

We will now characterize conditions for convergence in the passive aggregator setting.

Theorem PA1. (Convergence under SAP-FP/PA to NE) *Let the weight matrix W be stochastic. Let $(q_1^*, \ldots, q_n^*, z_1, \ldots, z_n)$ be a NE under the dynamics in* (40). *There exists a matrix $\mathcal{D}_1$ with zero diagonal, and a block diagonal matrix $\mathcal{D}_2$ such that the linearization of* (40) *with $\gamma > 0$ is locally asymptotically stable for $\lambda > 0$ if and only if if the following matrix is stable*

$$\begin{bmatrix} -I + (1 + \gamma\lambda)\mathcal{D}_1 & -\gamma\lambda\mathcal{D}_2 \\ \lambda W & -\lambda I \end{bmatrix}.$$

Proof. We reproduce the ODE from (40):

$$\dot{q}_i = \beta_i^\tau(\mathcal{A}_i(q_{-i}) + \gamma\dot{r}_i), z_i) - q_i \tag{60}$$

$$\dot{r}_i = \lambda(\mathcal{A}_i(q_{-i}) - r_i). \tag{61}$$

Note that at equilibrium $\dot{r}_i = 0$, but unlike Theorem AA1, r_i does not converge to q_i^*. Specifically, we note that the equilibrium (q_i^*, q_{-i}^*) corresponds to a point $(q_i(t) = q_i^*, q_{-i}(t) = q_{-i}^*, r_i(t) = \mathcal{A}_i(q_{-i}^*)), i \in [n]$, of the dynamics. Therefore, we will instead linearize around this point. Since the weight matrix W is stochastic, we must have $\mathcal{A}_i(q_{-i}^*) \in \Delta(A)$. Therefore, we can investigate the deviation of r_i around $\mathcal{A}_i(q_{-i}^*)$ with the help of matrix N defined in Theorem AA1. In particular, we can express the deviation vector $\delta x = (\delta x_{q_1}, \ldots, \delta x_{q_n}, \delta x_{r_1}, \ldots, \delta x_{r_n})^\top$ as:

$$\left(q_1(t) - q_1^*, \ldots, q_n(t) - q_n^*,\ r_1(t) - \mathcal{A}_1(q_{-1}^*), \ldots, r_n(t) - \mathcal{A}_n(q_{-n}^*)\right)^\top = \mathcal{N}\delta x(t), \tag{62}$$

where the block diagonal matrix $\mathcal{N}$ is as defined in Theorem AA1. Linearizing around our equilibrium point and proceeding similarly to Theorem AA1, we get

$$\dot{\delta x}_{q_i} = -\delta x_{q_i} + (1 + \gamma\lambda) \sum_{k \neq i} D_{ik} \delta x_{q_k} - \gamma\lambda C_i \delta x_{r_i}. \tag{63}$$

where

$$D_{ik} \triangleq \frac{w_{ik}}{\tau} N^\top \nabla\zeta\left(\frac{A_i(q^*_{-i}) - z_i}{\tau}\right) N,$$

$$C_i \triangleq \frac{1}{\tau} N^\top \nabla\zeta\left(\frac{A_i(q^*_{-i}) - z_i}{\tau}\right) N,$$

and

$$\nabla\zeta(b) \triangleq \text{diag}(\zeta(b)) - \zeta(b)\zeta^\top(b),$$

with $\zeta(b)$ the same as in Theorem AA1. Additionally, we have

$$\dot{\delta x}_{r_i} = \lambda \sum_{k \neq i} w_{ik} \delta x_{q_i} - \lambda \delta x_{r_i}. \tag{64}$$

Recall that the weight matrix

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & 0 & w_{23} & \dots & w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \dots & 0 \end{bmatrix}.$$

Define

$$\mathcal{D}_1 \triangleq \begin{bmatrix} 0 & D_{12} & D_{13} & \dots & D_{1n} \\ D_{21} & 0 & D_{23} & \dots & D_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{n1} & D_{n2} & D_{n3} & \dots & 0 \end{bmatrix}, \quad \text{and}$$

$$\mathcal{D}_2 \triangleq \begin{bmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_n \end{bmatrix}.$$

Then, the proof follows by combining (63) and (64), since we can express the deviations as

$$\dot{\delta x} = \begin{bmatrix} -I + (1+\gamma\lambda)\mathcal{D}_1 & -\gamma\lambda\mathcal{D}_2 \\ \lambda W & -\lambda I \end{bmatrix} \delta x.$$

□

Theorem PA2. (Convergence under SAP-GP/PA to CMNE) *Let the weight matrix W be stochastic. Let $(q^*_1, \dots, q^*_n, z_1, \dots, z_n)$ be a completely mixed NE under the dynamics in* (42). *Then the linearization of* (42) *with $\gamma > 0$ is locally asymptotically stable for $\lambda > 0$ if and only if the following matrix is stable*

$$\begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda W & -\lambda I \end{bmatrix}.$$

Proof. Recall the ODE from (42):

$$\dot{q}_i = \Pi_\Delta[q_i + \mathcal{A}_i(q_{-i}) + \gamma\dot{r}_i - z_i] - q_i \tag{65}$$

$$\dot{r}_i = \lambda(\mathcal{A}_i(q_{-i}) - r_i). \tag{66}$$

At equilibrium $(q^*_1, \dots, q^*_n, z_1, \dots, z_n)$, $\dot{q}_i = 0$ and $\dot{r}_i = 0$. Therefore, using (65), we have:

$$q^*_i = \Pi_\Delta[q^*_i + \mathcal{A}_i(q^*_{-i}) - z_i].$$

Proceeding along the lines of proof of Theorem AA2, for a sufficiently small deviation δx as defined in Theorem PA1, we can equivalently analyze the following dynamics:

$$\dot{q}_i = NN^\top[\mathcal{A}_i(q_{-i}) + \gamma\dot{r}_{-i} - z_i]$$

$$\dot{r}_i = \lambda(q_i - r_i).$$

Linearizing these equations and noting $N^\top N = I$, we get

$$\begin{aligned}
\dot{\delta x}_{q_i} &= N^\top \left(NN^\top (1+\gamma\lambda) \sum_{k\neq i} w_{ik} N \delta x_{q_k} \right) \; - \; N^\top \left(NN^\top \gamma\lambda \sum_{k \neq i} N \delta x_{r_k} \right) \\
&= (1+\gamma\lambda) N^\top \sum_{k\neq i} w_{ik} N\delta x_{q_k} - \gamma\lambda N^\top \sum_{k\neq i} w_{ik} N \delta x_{r_k} \\
&= (1+\gamma\lambda) \sum_{k\neq i} w_{ik} \delta x_{q_k} - \gamma\lambda \sum_{k\neq i} w_{ik} \delta x_{r_k},
\end{aligned}$$

and

$$\dot{\delta x}_{r_i} = \lambda \sum_{k\neq i} w_{ik} \delta x_{q_i} - \lambda \delta x_{r_i}.$$

It follows immediately that

$$\dot{\delta x} = \begin{bmatrix} (1+\gamma\lambda)W & -\gamma\lambda W \\ \lambda W & -\lambda I \end{bmatrix} \delta x,$$

where the weight matrix

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} & \dots & w_{1n} \\ w_{21} & 0 & w_{23} & \dots & w_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \dots & 0 \end{bmatrix}.$$

□

Theorem PA3. (Convergence under SAP-GP/PA to SNE) *Let the weight matrix W be doubly stochastic. Let $(q_1^*, \dots, q_n^*, z_1, \dots, z_n)$ be a strict NE under the dynamics in (42). The associated equilibrium point $(q_i = q_i^*, r_i = A_i(q_{-i}^*))_{i\in[n]}$ is locally asymptotically stable for sufficiently small $\gamma\lambda$, where $\gamma > 0$ and $\lambda > 0$.*

Proof. Recall the ODE from (42):

$$\begin{aligned}
\dot{q}_i &= \Pi_\Delta[q_i + \mathcal{A}_i(q_{-i}) + \gamma \dot{r}_i - z_i] - q_i \\
\dot{r}_i &= \lambda(\mathcal{A}_i(q_{-i}) - r_i).
\end{aligned}$$

We will prove local asymptotic stability via a Lyapunov function $\mathcal{V}$ that is locally positive definite and has locally negative semi-definite time derivative. Consider

$$\mathcal{V}(q_i, q_{-i}, r_i, r_{-i}) \triangleq \frac{1}{2} \sum_{i=1}^{n} \Big((q_i - q_i^*)^\top (q_i - q_i^*) \; + \; \lambda \left(r_i - \mathcal{A}_i(q_{-i})\right)^\top \left(r_i - \mathcal{A}_i(q_{-i})\right) \Big). \tag{67}$$

We define the shorthand $\tilde{d}_i \triangleq q_i + \mathcal{A}_i(q_{-i}) + \gamma \dot{r}_i - z_i$. Applying the chain rule, we see that the time derivative of $\mathcal{V}$,

$$\begin{aligned}
\dot{\mathcal{V}} &= \sum_{i=1}^{n} \left(\frac{\partial \mathcal{V}}{\partial q_i} \right)^\top \dot{q}_i + \sum_{i=1}^{n} \left(\frac{\partial \mathcal{V}}{\partial r_i} \right)^\top \dot{r}_i \\
&= \sum_{i=1}^{n} \left[(q_i - q_i^*) - \lambda \sum_{k\neq i} w_{ki} (r_k - \mathcal{A}_k(q_{-k})) \right]^\top \dot{q}_i \; - \; \lambda^2 \sum_{i=1}^{n} \left(r_i - \mathcal{A}_i(q_{-i})\right)^\top \left(r_i - \mathcal{A}_i(q_{-i})\right)) \\
&= \sum_{i=1}^{n} (q_i - q_i^*)^\top \dot{q}_i - \lambda^2 \sum_{i=1}^{n} ||r_i - \mathcal{A}_i(q_{-i})||^2 \; - \; \lambda \sum_{i=1}^{n} \left(\sum_{k\neq i} w_{ki} (r_k - \mathcal{A}_k(q_{-k})) \right)^\top \dot{q}_i \; .
\end{aligned}$$

Also, we note that

$$\begin{aligned}\sum_{i=1}^{n}||\dot{q}_i||^2 &= \sum_{i=1}^{n}||\Pi_\Delta(\tilde{d}_i)-q_i||^2\\ &= \sum_{i=1}^{n}||\Pi_\Delta(\tilde{d}_i)||^2+\sum_{i=1}^{n}q_i^\top q_i-2\sum_{i=1}^{n}q_i^\top\Pi_\Delta(\tilde{d}_i).\end{aligned}$$

It can be shown that

$$\begin{aligned}(q_i-q_i^*)^\top\dot{q}_i+||\dot{q}_i||^2 &= (\Pi_\Delta(\tilde{d}_i)-q_i^*)^\top(\Pi_\Delta(\tilde{d}_i)-q_i)\\ &\leq (\tilde{d}_i-q_i^*)^\top(\Pi_\Delta(\tilde{d}_i)-q_i)\\ &= (q_i+\mathcal{A}_i(q_{-i})+\gamma\dot{r}_i-z_i-q_i^*)^\top(\Pi_\Delta(\tilde{d}_i)-q_i)\ .\end{aligned}$$

As $r_i, q_i\to q_i^*$, this quantity tends to

$$\begin{aligned}&\left(q_i^*+\mathcal{A}_i(q_{-i}^*)+\gamma\lambda(\mathcal{A}_i(q_{-i}^*)-q_i^*)-z_i-q_i^*\right)^\top\left(\Pi_\Delta(\tilde{d}_i)-q_i^*\right)\\ =\ &\left(\mathcal{A}_i(q_{-i}^*)+\gamma\lambda(\mathcal{A}_i(q_{-i}^*)-q_i^*)-z_i\right)^\top\left(\Pi_\Delta(\tilde{d}_i)-q_i^*\right)\\ =\ &\left((1+\gamma\lambda)\mathcal{A}_i(q_{-i}^*)-\gamma\lambda q_i^*-z_i\right)^\top\left(\Pi_\Delta(\tilde{d}_i)-q_i^*\right)\\ =\ &\left(\mathcal{A}_i((1+\gamma\lambda)q_{-i}^*)-\gamma\lambda q_i^*-z_i\right)^\top\left(\Pi_\Delta(\tilde{d}_i)-q_i^*\right)\end{aligned}$$

which can be expressed in the form

$$\left(\Pi_\Delta(\tilde{d}_i)-q_i^*\right)^\top\left.\frac{\partial U_i(q_i,q_{-i}^*+\delta y,z_i)}{\partial q_i}\right|_{q_i=q_i^*} < 0$$

when $\gamma\lambda$ is sufficiently small and $\Pi_\Delta(\tilde{d}_i)\neq q_i^*$, by arguing along the lines of proof for Theorem AA3. Therefore,

$$\begin{aligned}\dot{\mathcal{V}} &\leq -\sum_{i=1}^{n}||\dot{q}_i||^2+\lambda\sum_{i=1}^{n}\left(\sum_{k\neq i}w_{ki}(\mathcal{A}_k\left(q_{-k}\right)-r_k)\right)^\top\dot{q}_i\ -\ \lambda^2\sum_{i=1}^{n}||r_i-\mathcal{A}_i(q_{-i})||^2\\ &= -\sum_{i=1}^{n}\left(||\dot{q}_i||^2+\lambda^2||r_i-\mathcal{A}_i(q_{-i})||^2\right)\ +\ \sum_{i=1}^{n}\sum_{k\neq i}w_{ki}\left(\lambda(\mathcal{A}_k\left(q_{-k}\right)-r_k)^\top\dot{q}_i\right)\\ &\leq -\sum_{i=1}^{n}\left(||\dot{q}_i||^2+\lambda^2||r_i-\mathcal{A}_i(q_{-i})||^2\right)\ +\ \frac{1}{2}\sum_{i=1}^{n}\sum_{k\neq i}w_{ki}\left(\lambda^2||r_k-\mathcal{A}_k(q_{-k})||^2+||\dot{q}_i||^2\right)\end{aligned}$$

by noting that $w_{ki}\geq 0$ for all $i\in[n]$, $k\neq i$ and invoking Cauchy-Schwarz. Furthermore, since W is doubly stochastic, we have $\sum_{k\neq i}w_{ki}=1$ and $\sum_{k\neq i}w_{ik}=1$ for all $i\in[n]$. Thus, we may decompose the second term on the right in the last equation as

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\sum_{k\neq i}w_{ki}\left(\lambda^2||r_k-\mathcal{A}_k(q_{-k})||^2+||\dot{q}_i||^2\right)\\ =\ &\frac{\lambda^2}{2}\sum_{i=1}^{n}\sum_{k\neq i}w_{ki}||r_k-\mathcal{A}_k(q_{-k})||^2+\frac{1}{2}\sum_{i=1}^{n}||\dot{q}_i||^2\sum_{k\neq i}w_{ki}\\ =\ &\frac{\lambda^2}{2}\sum_{i=1}^{n}\sum_{k\neq i}w_{ki}||r_k-\mathcal{A}_k(q_{-k})||^2+\frac{1}{2}\sum_{i=1}^{n}||\dot{q}_i||^2.\end{aligned}$$

The first term in the last equation may be interpreted as a weighted outgoing flow from player i to player $k \neq i$. Now viewing this from the equivalent perspective of total incoming flow, we have

$$\begin{aligned}
\dot{\mathcal{V}} &\leq -\sum_{i=1}^{n}\left(||\dot{q}_i||^2 + \lambda^2||r_i - \mathcal{A}_i(q_{-i})||^2\right) \;+\; \frac{\lambda^2}{2}\sum_{i=1}^{n}\sum_{k\neq i} w_{ik}||r_i - \mathcal{A}_i(q_{-i})||^2 + \frac{1}{2}\sum_{i=1}^{n}||\dot{q}_i||^2 \\
&= -\sum_{i=1}^{n}\left(||\dot{q}_i||^2 + \lambda^2||r_i - \mathcal{A}_i(q_{-i})||^2\right) \;+\; \frac{\lambda^2}{2}\sum_{i=1}^{n}||r_i - \mathcal{A}_i(q_{-i})||^2\sum_{k\neq i} w_{ik} + \frac{1}{2}\sum_{i=1}^{n}||\dot{q}_i||^2 \\
&= -\sum_{i=1}^{n}\left(||\dot{q}_i||^2 + \lambda^2||r_i - \mathcal{A}_i(q_{-i})||^2\right) \;+\; \frac{1}{2}\sum_{i=1}^{n}\left(\lambda^2||r_i - \mathcal{A}_i(q_{-i})||^2 + ||\dot{q}_i||^2\right) \\
&= -\frac{1}{2}\sum_{i=1}^{n}\left(\lambda^2||r_i - \mathcal{A}_i(q_{-i})||^2 + ||\dot{q}_i||^2\right) \\
&\leq 0,
\end{aligned}$$

which implies that $\dot{\mathcal{V}}$ is locally negative semi-definite. The local positive definiteness of $\mathcal{V}$ may be argued similarly to the proof of Theorem AA3 and we are done. □