Supplementary Material - Learning Mixture of Graphs from Epidemic Cascades

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A. Necessary Conditions

A.1. We need at least three edges

Let $G = (V, E_1 \cup E_2)$ be the union of the graphs from both mixtures. In this subsection, we prove it is impossible to learn the weights of E_1 and E_2 if G has less than three edges:

One edge: For a graph on two nodes, we have already seen that the cascade distribution are identical if $p_{12} = \beta = 1 - q_{12}$, for any value of β, which proves the problem is not solvable.

Two edges: When we have two nodes and two edges, we can without loss of generality assume that node 1 is connected to node 2 and node 3. Then, if:

- $p_{12} = \beta$
- $q_{12} = 1 - \beta$
- $p_{13} = \frac{\frac{1}{2} - \frac{\beta}{2} + \frac{1}{4}}{\frac{1}{2} - \beta}$
- $q_{13} = \frac{\frac{1}{4} - \frac{\beta}{2}}{\frac{1}{2} - \beta}$

The cascade distribution is identical for any value of $\beta < \frac{1}{2}$. By simple calculations, we can show the following,

- Fraction of cascades with only node 1 infected: $\frac{1}{12}$.
- Fraction of cascades with only node 2 infected: $\frac{1}{6}$.
- Fraction of cascades with only node 3 infected: $\frac{1}{6}$.
- Fraction of cascades where 3 infected 1, but 1 did not infect 2: $\frac{1}{12}$.
- Fraction of cascades where 3 infected 1, 1 infected 2: $\frac{1}{12}$.
- Fraction of cascades where 1 infected 3, but 1 did not infect 2: $\frac{1}{12}$.
- Fraction of cascades where 1 infected 2, but 1 did not infect 3: $\frac{1}{12}$.
- Fraction of cascades where 1 infected 3 and 2: $\frac{1}{12}$.
- Fraction of cascades where 2 infected 1, but 1 did not infect 3: $\frac{1}{12}$.
- Fraction of cascades where 2 infected 1, then 1 infected 3: $\frac{1}{12}$.

Since the distribution of cascades is the same for any value of $\beta < \frac{1}{2}$, the problem is not solvable.

A.2. We need Δ-separation

Separability is necessary for the existence of sample efficient algorithms. Specifically, we show that there exist (many) graphs where separability is violated, and for which the sample complexity is exponential in the size of the graph.

Indeed, consider a graph G composed of two subgraphs A and B, connected by a path P of length d. Suppose the path has the same weight in both mixtures, and for the edges $e \in P$, $\max_{e\in P} p_e < 1$. Similar to the disconnected graph, we write $A_i = A \cap E_i$, and $B_i = B \cap E_i$. To learn the graph completely we need to differentiate between the mixture on E_1 and E_2, and the mixture on $E_1' = A_1 \cup P \cup B_2$ and $E_2' = A_2 \cup P \cup B_1$.

The path P is not informative in the above differentiation as both the mixture in the path have same weights. Therefore, we need at least one cascade covering *at least one edge in A and one edge in B*. Since P is of length d, this happens with probability at most $e^{-\Omega(d)}$. To see such a cascade, we need at least $e^{\Omega(d)}$ cascades in expectation. Therefore, setting $d = cN$, for some constant $c > 0$, we prove that exponential number of samples are necessary for any algorithm to recover the graph if the Δ-separated Condition is violated.

A.3. Dealing with mixtures which are not Δ-separated

In this section, we show how to detect and deduce the weights of edges which have the same weight across both component of the mixture. We assume both G_1 and G_2 follow Conditions **??** and **??** if we remove all non-distinct edges, and in particular remain connected.

Suppose there exists an edge (i, j) in the graph, such that $p_{ij} = q_{ij} > 0$. Then in particular, there exists another edge connecting i to the rest of the graph G_1 through node k, such that $p_{ik} \neq q_{ik}$. Then:

Lemma 1. *Suppose G_1 and G_2 follow assumption* **??** *after removing all non-distinct edges. We can detect and learn the weights of non-distinct edges the following way:*

If $X_{ij} > 0$, and $\forall k \in V,\ X_{ik} > 0 \implies Y_{ik,ij} - X_{ik}X_{ij} = 0$, then $p_{ij} = q_{ij} = X_{ij}$.

Proof. Since G_1 is connected on three nodes or more even when removing edge (i, j), we know there exists a node l such either l is connected to either i or k. Therefore, either $Y_{ik,il} - X_{ik}X_{il} > 0$ or $Y_{ki,kl} - X_{ki}X_{kl} > 0$. In both these cases, we deduce $p_{ik} \neq q_{ik}$. This in turns allow us to detect that $p_{ij} = q_{ij}$. Once this edge is detected, it is very easy to deduce its weight, since $p_{ij} = X_{ij} = q_{ij}$ by definition. □

B. Proofs for unbalanced mixtures

B.1. Estimators - proofs

Lemma 2. *Under Conditions* **??** *and* **??***, in the setting of infinite samples, the weights of the edges for a line structure are then given by:*

$$p_{ua} = X_{ua} + s_{ua}\sqrt{\frac{(Y^{|}_{ua,ub} - X_{ua}X_{ub})R^{|}}{Y^{|}_{ub,bc} - X_{ua}X_{bc}}},\ q_{ua} = X_{ua} - s_{ua}\sqrt{\frac{(Y^{|}_{ua,ub} - X_{ua}X_{ub})R^{|}}{Y^{|}_{ub,bc} - X_{ua}X_{bc}}},$$

$$p_{bc} = X_{uc} + s_{bc}\sqrt{\frac{(Y^{|}_{ub,bc} - X_{ua}X_{bc})R^{|}}{Y^{|}_{ua,ub} - X_{uc}X_{ua}}},\ q_{bc} = X_{uc} - s_{bc}\sqrt{\frac{(Y^{|}_{ub,bc} - X_{ua}X_{bc})R^{|}}{Y^{|}_{ua,ub} - X_{uc}X_{ua}}}$$

$$p_{ub} = X_{ub} + s_{ub}\sqrt{\frac{(Y^{|}_{uc,ua} - X_{uc}X_{ua})(Y^{|}_{ub,bc} - X_{ua}X_{bc})}{R^{|}}},$$

$$q_{ua} = X_{ua} - s_{ua}\sqrt{\frac{(Y^{|}_{ua,ub} - X_{uc}X_{ua})(Y^{|}_{ub,bc} - X_{ua}X_{bc})}{R^{|}}},$$

where $R^{|} = X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}}$, and for $s_{ua} \in \{-1, 1\}$.

Proof. In this case, there is no edge between u and c, which implies that $p_{uc} = q_{uc} = 0$. Hence, we cannot use a variation of the equation above for finding the edges of a star structure without dividing by zero. Therefore, we need to use $Z^{|}_{ua,ub,bc}$. Let $R^{|} = X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}}$. We notice a remarkable simplification:

$$
\begin{aligned}
R^{|} &= X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}} \\
&= \frac{p_{ua}+q_{ua}}{2} \cdot \frac{p_{bc}+q_{bc}}{2} + \frac{\frac{p_{ua}p_{ub}p_{bc}+q_{ua}q_{ub}q_{bc}}{2} - \frac{p_{ua}+q_{ua}}{2} \cdot \frac{p_{ub}p_{bc}+q_{ub}q_{bc}}{2} - \frac{p_{bc}+q_{bc}}{2} \cdot \frac{p_{ua}p_{ua}+q_{ua}q_{ua}}{2}}{\frac{p_{ub}+q_{ub}}{2}} \\
&= \frac{1}{4}(p_{ua}p_{bc} + p_{ua}q_{bc} + q_{ua}p_{bc} + q_{ua}q_{bc}) + \frac{2}{p_{ub}+q_{ub}} \left[\frac{p_{ua}p_{ub}p_{bc} + q_{ua}q_{ub}q_{bc}}{2} \right. \\
&\quad -\frac{1}{4}(p_{ua}p_{ub}p_{bc} + q_{ua}p_{ub}p_{bc} + p_{ua}q_{ub}q_{bc} + q_{ua}q_{ub}q_{bc}) \\
&\quad \left. -\frac{1}{4}(p_{ua}p_{ub}p_{bc} + p_{ua}p_{ub}q_{bc} + q_{ua}q_{ub}p_{bc} + q_{ua}q_{ub}q_{bc}) \right] \\
&= \frac{1}{4}(p_{ua}p_{bc} + p_{ua}q_{bc} + q_{ua}p_{bc} + q_{ua}q_{bc}) - \frac{1}{2(p_{ub}+q_{ub})} [q_{ua}p_{ub}p_{bc} + p_{ua}q_{ub}q_{bc} + p_{ua}p_{ub}q_{bc} + q_{ua}q_{ub}q_{bc}] \\
&= \frac{1}{4}(p_{ua}p_{bc} + q_{ua}p_{bc} + q_{ua}q_{bc} + p_{ua}q_{bc}) - \frac{1}{2(p_{ub}+q_{ub})} [(p_{ub}+q_{ub})(q_{ua}p_{bc} + p_{ua}q_{bc})] \\
&= \frac{1}{4}(p_{ua}p_{bc} + q_{ua}q_{bc} - p_{ua}q_{bc} - q_{ua}p_{bc}) \\
&= \frac{1}{4}(p_{ua} - q_{ua})(p_{bc} - q_{bc})
\end{aligned}
$$

We can then use the same proof techniques as in Lemma **??**, and finally obtain:

$$
|p_{ua} - q_{ua}| = \sqrt{\frac{(Y^{|}_{ua,ub} - X_{ua}X_{ua}) \left(X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}} \right)}{Y^{|}_{ub,bc} - X_{ua}X_{bc}}}.
$$

This gives us the required result. □

B.2. Resolving Sign Ambiguity across Base Estimators

The following lemma handles the sign ambiguity (s_{ua}) introduced above.

Lemma 3. *Suppose Condition* **??** *and* **??** *are true, in the setting of infinite samples, for edges* $(u,a),(u,b)$ *with* $a \neq b$ *for any vertex* u *with degree* ≥ 2, *the sign pattern* s_{ua}, s_{ub} *satisfy the following relation.*

$$
s_{ua}s_{ub} = \mathsf{sgn}(Y_{ua,ub} - X_{ua}X_{ub}).
$$

Proof. From previous analysis, we have $\mathsf{sgn}(p_{ua} - q_{ua}) = s_{ua}$. Therefore:

$$
\begin{aligned}
\mathsf{sgn}(Y_{ua,ub} - X_{ua}X_{ub}) &= \mathsf{sgn}\left(\frac{(p_{ua}-q_{ua})(p_{ub}-q_{ub})}{4} \right) \\
&= s_{ua}s_{ub}.
\end{aligned}
$$

□

Thus fixing sign of one edge gives us the signs of all the other edges adjacent to a star vertex. A similar relationship can be established among the edges of a line vertex, using $\mathsf{sgn}\left(X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}} \right)$.

B.3. Main algorithm - proofs

Here we will present in detail the sub-routines required by our algorithm and the essential lemmas needed for our main proof.

LearnEdges This procedure detects the edges in the underlying graph using the estimate X_{uv}.

Algorithm 1 LEARNEDGES

Input Vertex set V
Output Edges of the graph
0: Set $E \leftarrow \emptyset$
0: **for** $u < v \in V$ **do**
0: Compute $\hat{X}_{uv}$
0: **if** $\hat{X}_{uv} \geq \epsilon$ **then**
0: $E \leftarrow E \cup \{(u, v)\}$
0: Return E =0

Claim 1. LEARNEDGES(V) *outputs* E *such that* $E = E_1 \cup E_2$.

Proof. For each pair of nodes $u, v \in V$, if $(u, v) \in E_1 \cup E_2$ then $X_{uv} \neq 0$ since $X_{uv} = 0$ if and only if $p_{uv} = q_{uv} = 0$, which is equivalent to the edge (u, v) not belonging in the mixture. □

LearnStar This procedure returns the weights of the outgoing edges of a star vertex using the star primitive discussed before.

Algorithm 2 LEARNSTAR

Input Star vertex $u \in V$, edge set E, weights W
Output Weights of edges adjacent to u
0: Use star primitive with star vertex u and learn all adjacent edges weights W^*.
0: **if** $W = \emptyset$ **then**
0: Fix sign of any edge and ensure sign consistency.
0: **else**
0: Set $v \in V$ such that $(u, v) \in W$.
0: Use s_{uv} to remove sign ambiguity
0: Return W^*. =0

Lemma 4. *If* $deg(u) \geq 3$, LEARNSTAR(u, S, W) *recovers* p_{ua}, q_{ua} *for all* a *such that* $(u, a) \in E$.

Proof. The proof follows from using Lemma **??** on star vertex u (degree of $u \geq 3$) and using Lemma 3 to resolve sign ambiguity through fixing an edge or s_{uv} ($(u, v) \in W$ hence know sign). □

LearnLine This procedure returns the weights of the edges of a line $a - b - c - d$ rooted at vertex b of degree 2 using the line primitive discussed before.

Algorithm 3 LEARNLINE

Input Line $a - b - c - d$ with $deg(b) = 2$, edge set E, weights W
Output Weights of edges $(a, b), (b, c), (c, d)$

0: Use line primitive on $a - b - c - d$ rooted at b and learn all edges weights $W^{|}$.
0: **if** $W = \emptyset$ **then**
0:     Fix sign of any edge and ensure sign consistency.
0: **else**
0:     Find edge $e \in \{(a, b), (b, c), (c, d)\}$ such that $e \in W$.
0:     Use s_e to remove sign ambiguity.
0: Return $W^{|}$. =0

Lemma 5. *If $deg(b) = 2$,* LEARNLINE(a, b, c, d, S, W) *recovers* $p_{ab}, q_{ab}, p_{bc}, q_{bc}, pcd, q_{cd}$.

Proof. The proof follows from using Lemma **??** on line $a - b - c - d$ rooted at vertex b (degree of $b = 2$) and using Lemma 3 to resolve sign ambiguity by fixing an edge or using s_e. □

Learn2Nodes This procedure chooses a pair of connected vertices in our graph and outputs the weights of all outgoing edges of each of the two vertices. We initialize our algorithm using this procedure.

Algorithm 4 LEARN2NODES

Input Vertex set V, Edge Set E
Output Set of 2 vertices V, Weight of all edges adjacent to the vertices W

0: $W = \emptyset$
0: Set $u = \arg\max_{a \in V} deg(a)$
0: Set $v = \arg\min_{a \in V, (u,a) \in E} deg(a)$
0: **if** $deg(u) \geq 3$ **then**
0:     $W \leftarrow$ LEARNSTAR(u, E, W)
0:     **if** $deg(v) = 3$ **then**
0:         $W \leftarrow W \cup$ LEARNSTAR(v, E, W)
0:     **else if** $deg(v) = 2$ **then**
0:         Let $t \in V$ be such that $(t, v) \in E$ and $t \neq u$
0:         Let $w \in V$ be such that $(w, u) \in E$ and $w \neq v, t$
0:         **if** $v = t$ **then**
0:             $W \leftarrow W \cup$ LEARNLINE(t, v, u, w, W)
0: **else**
0:     w be such that $(w, u) \in E$ and $w \neq v$
0:     **if** $deg(v) = 2$ **then**
0:         Let $t \in V$ be such that $(t, v) \in E$ and $t \neq u$
0:         $W \leftarrow$ LEARNLINE(w, u, v, t, W)
0:     **else**
0:         Let $t \in V$ be such that $(t, w) \in E$ and $t \neq w$
0:         $W \leftarrow$ LEARNLINE(v, u, w, t, W)
0: Return $(u, v), W$ =0

Lemma 6. *Under Conditions* **??** *and* **??**, LEARN2NODES(V) *outputs two connected nodes* (u, v) *and weights of all edges adjacent to* u, v.

Proof. We will break the proof down into cases based on the degree of chosen vertices u, v as follows,

- $deg(u) \geq 3$: By Lemma 4, we can recover all the edges of u and fix a sign.
  - $deg(v) \geq 3$: By Lemma 4, we can recover all the edges of v and ensure sign consistency by using the edge (u, v).

  - $deg(v) = 2$: Since $deg(v) = 2$, there exists a vertex $t \neq u$ such that $(t, u) \in E$. Since $deg(u) \geq 3$, there must exist $w \neq t, u$ such that $(u, w) \in E$. Now we have line primitive $t - v - u - w$ with $deg(v) = 2$ and Lemma 5 guarantees recovery of the edge weights.
  - $deg(v) = 1$, then we already know all the edges adjacent to v.

- $deg(u) = 2, deg(v) = 2$: Since the max degree of the graph is 2 and it is connected then it can either be a line or a cycle. There are at least 4 nodes in the graph, thus there exist $w \neq v$ such that $(w, u) \in E$ and $t \neq u, w$ such that $(v, t) \in E$. This gives a path $w - u - v - t$ with $deg(u) = 2$ and Lemma 5 guarantees recovery of all edges.

- $deg(u) = 2, deg(v) = 1$: As in the previous case, the underlying graph is a line. Therefore there exist path $v - u - w - t$ and we can similarly apply Lemma 5 to guarantee recovery of all edges.

□

B.4. Finite sample complexity - proofs

In this section, we provide explicit proof for the sample complexity of our algorithm. To do so, we bound below the number of cascades starting on each node through Bernstein inequality, and use this number to obtain concentration of all the estimators.

Definition 1. *Among M cascades, let M_u be the number of times node u is the source.*

Claim 2. *With M samples, every node is the source of the infection at least $\frac{M}{2N}$ times with probability at least $1 - e^{-\frac{3M}{26N}}$.*

Proof. Among M cascade, the expectation of M_u is $\frac{M}{N}$, since the source is chosen uniformly at random among the N vertices of V. Since M_u can be seen as the sum of Bernoulli variable of parameter $\frac{1}{N}$, we can use Bernstein's inequality to bound it below:

$$
\begin{aligned}
\Pr(M_u < \frac{M}{2N}) &= \Pr\left(\frac{M}{N} - M_u > \frac{M}{2N}\right) \\
&\leq e^{-\frac{\left(\frac{M}{2N}\right)^2}{2M\frac{1}{N}(1-\frac{1}{N}) + \frac{1}{3}\frac{M}{2N}}} \\
&\leq e^{-\frac{3M}{26N}}.
\end{aligned}
$$

Claim 3. *Let u either be a star vertex, with neighbors a, b and c, or be part of a line structure rooted in u, with neighbors a, b, and c neighbor of b. Suppose $M_u \geq \frac{M}{2N}$. Then with $M = \frac{N}{\epsilon^2}\log\left(\frac{12N^2}{\delta}\right)$ samples, with probability at least $1 - \frac{\delta}{6N^2}$, we can guarantee any of the following:*

1. $\forall r \in a, b, c,\ \left|\hat{X}_{ur} - X_{ur}\right| \leq \epsilon_1$.
2. $\forall r \neq s \in \{a, b, c\},\ |\hat{Y}^*_{ur,us} - \hat{Y}^*_{ur,us}| \leq \epsilon_1$.
3. $|\hat{Y}^|_{ua,ub} - \hat{Y}^|_{ua,ub}| \leq \epsilon_1$ *and* $|\hat{Y}^|_{ua,ab} - \hat{Y}^|_{ua,ab}| \leq \epsilon_1$.
4. $|\hat{Z}^|_{ua,ub,bc} - Z^|_{ua,ub,bc}| \leq \epsilon_1$.

Proof. By Hoeffding's inequality:

$$
\begin{aligned}
\Pr(|\hat{X}_{ur} - X_{ur}| > \epsilon_1) &= \Pr\left(\left|\sum_{m=1}^{M_u} \mathbb{1}_{\{u \to r \mid u \in I_0\}} - M_u \cdot X_{ur}\right| > M_u \cdot \epsilon_1\right) \\
&\leq \Pr\left(\left|\sum_{m=1}^{\frac{M}{2N}} \mathbb{1}_{\{u \to r \mid u \in I_0\}} - \frac{M}{2N} \cdot X_{ur}\right| > \frac{M}{2N} \cdot \epsilon_1\right) \\
&\leq 2e^{-2\frac{M}{2N}\epsilon_1^2}.
\end{aligned}
$$

Therefore, the quantity above is smaller than $\frac{\delta}{6N^2}$ for $M \geq \frac{N}{\epsilon_1^2}\log\left(\frac{12N^2}{\delta}\right)$. The proof is almost identical for the other quantities involved.

Claim 4. *If we can estimate $X_{ua}, Y^*_{ua,ub}, Y^{|}_{ua,ab}$ and $Z^{|}_{ua,ub,bc}$ within ϵ_1, we can estimate p_{ua} within precision $\epsilon = \frac{41}{p_{min}^3\Delta^2}\cdot\epsilon_1$.*

Proof. If u is of degree three or more, we use a star primitive to estimate it. Let a, b and c be three of its neigbors:

$$\begin{aligned}
\hat{p}_{ua} &= \hat{X}_{ua} + s_{ua}\sqrt{\frac{(\hat{Y}_{ua,ub}-\hat{X}_{ua}\hat{X}_{ub})(\hat{Y}_{ua,uc}-\hat{X}_{ua}\hat{X}_{uc})}{\hat{Y}_{ub,uc}-\hat{X}_{ub}\hat{X}_{uc}}} \\
&\leq X_{ua}+\epsilon_1 \\
&\quad + s_{ua}\left(\frac{(Y_{ua,ub}-X_{ua}X_{ub}+s_{ua}\left(1+X_{ua}+X_{ub}\right)\epsilon_1)(Y_{ua,uc}-X_{ua}X_{uc}+s_{ua}\left[1+X_{ua}+X_{uc}\right)\epsilon_1)}{Y_{ub,bc}-X_{ub}X_{uc}-s_{ua}\left(1+X_{ub}+X_{uc}\right)\epsilon_1)}\right]^{\frac{1}{2}} \\
&\leq X_{ua}+\epsilon_1+s_{ua}\sqrt{\frac{(Y_{ua,ub}-X_{ua}X_{ub})(Y_{ua,uc}-X_{ua}X_{uc})}{Y_{ub,uc}-X_{ub}X_{uc}}}\left(\frac{(1+s_{ua}\frac{3\epsilon_1}{\frac{\Delta^2}{4}})^2}{1-s_{ua}\frac{3\epsilon_1}{\frac{\Delta^2}{4}}}\right)^{\frac{1}{2}} \\
&\leq p_{ua}+\epsilon_1+p_{ua}\left(\frac{12}{\Delta^2}+\frac{6}{\Delta^2}\right)\cdot\epsilon_1+o(\epsilon_1) \\
&\leq p_{ua}+\frac{19}{\Delta^2}\cdot\epsilon_1+o(\epsilon_1).
\end{aligned}$$

Where we have used $Y_{ur,us}-X_{ur}X_{us}\geq\frac{\Delta^2}{4}$, $s_{ua}^2=1$, $p_{ua}\leq 1$, $1\leq\frac{1}{\Delta^2}$. We then conclude by symmetry.

If u is of degree two, we use a line primitive to estimate it:

$$\begin{aligned}
\hat{p}_{ua} &= \hat{X}_{ua}+s_{ua}\sqrt{\frac{(\hat{Y}^{|}_{ua,ub}-\hat{X}_{ua}\hat{X}_{ub})\left(\hat{X}_{ua}\hat{X}_{bc}+\frac{\hat{Z}^{|}_{ua,ub,bc}-\hat{X}_{ua}\hat{Y}^{|}_{ub,bc}-\hat{X}_{bc}\hat{Y}^{|}_{ua,ub}}{\hat{X}_{ub}}\right)}{\hat{Y}^{|}_{ub,bc}-\hat{X}_{ua}\hat{X}_{bc}}} \\
&\leq X_{ua}+\epsilon_1+s_{ua}\sqrt{\frac{(Y^{|}_{ua,ub}-X_{ua}X_{ub}+3s_{ua}\epsilon_1)\left(X_{ua}X_{bc}+2\epsilon_1+\frac{Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub}+5s_{ua}\epsilon_1}{X_{ub}-s_{ua}\epsilon_1}\right)}{Y^{|}_{ub,bc}-X_{ua}X_{bc}-3s_{ua}\epsilon_1}}.
\end{aligned}$$

As shown in the proof of Lemma **??**, we have:

$$\begin{aligned}
Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub} &= \frac{1}{2}(p_{ub}+q_{ub})(q_{ua}p_{bc}+p_{ua}q_{bc}) \\
&\geq \frac{p_{min}^3}{2} \\
\left(X_{ua}X_{bc}+\frac{Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub}}{X_{ub}}\right) &= \frac{1}{4}(p_{ua}-q_{ua})(p_{bc}-q_{bc}) \\
&\geq \frac{\Delta^2}{4}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub}+5s_{ua}\epsilon_1}{X_{ub}-s_{ua}\epsilon_1} &\leq \frac{Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub}}{X_{ub}}\left[\frac{1+s_{ua}\frac{5\epsilon_1}{\frac{p_{min}^3}{2}}}{1-s_{ua}\frac{\epsilon_1}{\frac{p_{min}}{2}}}\right] \\
&\leq \frac{Z^{|}_{ua,ub,bc}-X_{ua}Y^{|}_{ub,bc}-X_{bc}Y^{|}_{ua,ub}}{X_{ub}}+s_{ua}\left(\frac{12}{p_{min}^3}\right)\epsilon_1+o(\epsilon_1).
\end{aligned}$$

We also have:

$$X_{ua}X_{bc} + 2\epsilon_1 + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub} + 5s_{ua}\epsilon_1}{X_{ub} - s_{ua}\epsilon_1} \leq \left(X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}}\right) \cdot \left(1 + s_{ua}\frac{\frac{14}{p^3_{min}}}{\frac{\Delta^2}{4}}\epsilon_1\right) + o(\epsilon_1).$$

Combining all the above inequalitites:

$$\begin{aligned}
\hat{p}_{ua} &\leq X_{ua} + \epsilon_1 + s_{ua}\sqrt{\frac{(Y^{|}_{ua,ub} - X_{ua}X_{ub})\left(X_{ua}X_{bc} + \frac{Z^{|}_{ua,ub,bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}}{X_{ub}}\right)}{Y^{|}_{ub,bc} - X_{ua}X_{bc}}} \\
&\quad \cdot \left[\frac{\left(1 + \frac{3\epsilon_1}{\frac{\Delta^2}{4}}\right)\left(1 + s_{ua}\frac{\frac{14}{p^3_{min}}}{\frac{\Delta^2}{4}}\epsilon_1\right)}{1 - s_{ua}\frac{3\epsilon_1}{\frac{\Delta^2}{4}}}\right]^{\frac{1}{2}} \\
&\leq p_{ua} + \epsilon_1 + p_{ua}s^2_{ua}\left(\frac{6}{\Delta^2} + \frac{28}{p^3_{min}\Delta^2} + \frac{6}{\Delta^2}\right) \cdot \epsilon_1 + o(\epsilon_1) \\
&\leq p_{ua} + \frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1 + o(\epsilon_1).
\end{aligned}$$

We can conclude by symmetry.

Since $\frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1 \geq \frac{19}{\Delta^2} \cdot \epsilon_1$, we conclude that we can know p_{ua} within precision $\epsilon = \frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1$ regardless of the degree of u.

Theorem 1. *Under Conditions* **??** *and* **??***,, with probability* $1-\delta$*, with* $M = N \cdot \frac{41^2}{p^6_{min}\Delta^4 \cdot \epsilon^2} \log\left(\frac{12N^2}{\delta}\right) = \mathcal{O}\left(\frac{N}{\epsilon^2}\log\left(\frac{N}{\delta}\right)\right)$ *samples, we can learn all the edges of the mixture of the graphs within precision* ϵ*.*

Proof. We pick $\epsilon = \frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1$. We use Claim 2 to bound the quantity $\Pr(M_u < \frac{M}{2N})$, and Claim 3 and 4 to bound $\Pr(|\hat{p}_{ua} - p_{ua}| > \frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1 | M_u \geq \frac{M}{2N})$. For (u, a) edge of the graph:

$$\begin{aligned}
\Pr(|\hat{p}_{ua} - p_{ua}| > \epsilon) &\leq \Pr(|\hat{p}_{ua} - p_{ua}| > \epsilon | M_u < \frac{M}{2n}) \cdot \Pr(M_u < \frac{M}{2N}) \\
&\quad + \Pr(|\hat{p}_{ua} - p_{ua}| > \epsilon | M_u \geq \frac{M}{2n}) \cdot \Pr(M_u \geq \frac{M}{2N}) \\
&\leq 1 \cdot 2e^{-2\frac{M}{2N}} + \Pr(|\hat{p}_{ua} - p_{ua}| > \epsilon | M_u \geq \frac{M}{2N}) \cdot 1 \\
&\leq \frac{\delta}{12N^2} + \Pr(|\hat{p}_{ua} - p_{ua}| > \frac{41}{p^3_{min}\Delta^2} \cdot \epsilon_1 | M_u \geq \frac{M}{2N}) \\
&\leq \frac{\delta}{12N^2} + \frac{\delta}{12N^2} \\
&\leq \frac{\delta}{6N^2}.
\end{aligned}$$

We conclude by union bound on the six estimators involved for all the pairs of nodes in the graph, for a total of at most $6N^2$ estimators.

B.5. Complete graph on three nodes

In this section, we prove it is possible to recover the weights of a mixture on three nodes, as long as there are at least three edges in $E_1 \cup E_2$. Since no node is of degree 3, no node is a star vertex, and since there are less than four nodes, no node is a line vertex, and we can not use the techniques developped above for connected graphs on four vertices or more. However, we can still use very similar proofs techniques. Suppose the vertices of V are 1, 2 and 3.

Definition 2. *We reuse the quantities defined for star vertices:*

- *For i, j distinct in* $\{1, 2, 3\}$, $\hat{X}_{ij} = \frac{\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_{i \to j, i \in I_0^m}}{\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_{i \in I_0^m}} \to_{M\to\infty} X_{ij} = \frac{p_{ij}+q_{ij}}{2}$.
- *For i, j, k distinct in* $\{1, 2, 3\}$, $Y_{ij,ik} = \frac{\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_{i \to j, i \to k, i \in I_0^m}}{\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_{u \in I_0^m}} \to_{M\to\infty} Y_{ij,ik} = \frac{p_{ij}p_{ik}+q_{ij}q_{ik}}{2}$.

Even though neither 1, 2 or 3 is a star vertex, we can write the same kind of system of equations as a star vertex would satisfy. In particular:

$$\frac{|p_{ij} - q_{ij}|}{2} = \sqrt{\frac{(Y_{ij,ik} - X_{ij}ik)(Y_{ji,jk} - X_{ji}X_{jk})}{Y_{ki,kj} - X_{ki}X_{kj}}}.$$

Resolving the sign ambiguity as previoulsy (Lemma 3), this finally yields:

$$p_{ij} = X_{ij} + s_{ij}\sqrt{\frac{(Y_{ij,ik} - X_{ij}ik)(Y_{ji,jk} - X_{ji}X_{jk})}{Y_{ki,kj} - X_{ki}X_{kj}}},$$
$$q_{ij} = X_{ij} + s_{ij}\sqrt{\frac{(Y_{ij,ik} - X_{ij}ik)(Y_{ji,jk} - X_{ji}X_{jk})}{Y_{ki,kj} - X_{ki}X_{kj}}}.$$

C. Lower Bounds

C.1. Directed lower bound

We consider the task of learning all the edges of any mixture of graphs up to precision $\epsilon < \Delta$. To do so, we have to be able to learn a mixture on a specific graph, which we present below.

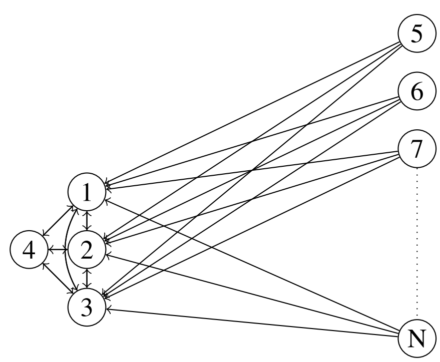


Figure 1: Lower-bound directed graph

The example we focus on is the directed graph of min-degree 3, comprised of a clique on 4 nodes, which we call nodes 1 to 4, and $N - 4$ other nodes with 3 directed edges to nodes 1, 2 and 3. All edges have weight p in E_1, and $p + \Delta$ in E_2.

We define a *valid sample* for edge (i, j) as a cascade during which i became infected when j was not infected. Indeed, in this case, an infection could happen along edge (i, j), and we can therefore gain information about the weight of this edge. We first state a general claim:

Claim 5. *We need at least $\Omega(\frac{1}{\Delta^2})$ valid samples for edge (i, j) to determine the weights of this edge in the mixture.*

Proof. Using Sanov's theorem (**?**), and writing the Kullback–Leibler divergence between p and q as $\mathcal{D}(p||q)$, we know we need at least $\Omega(\mathcal{D}(p||p+\Delta))$ valid samples to determine whether the valid samples came from a random flip of probability p, or a random flip of probability $p+\Delta$, which is an easier task than computing both weights of the mixture.

Then, using standard Kullback–Leibler divergence bounds (**?**), we obtain $\mathcal{D}(p||p+\Delta) \geq \frac{1}{\Delta^2}$, which gives us the desired result.

We now combine this with Coupon collector's result to obtain our lower bound.

Claim 6. *We need at least* $\Omega\left(N\log(N) + \frac{N\log\log(N)}{\Delta^2}\right)$ *cascades to obtain enough valid samples for all the edges in the graph.*

Proof. We notice that if we want to learn all edges in the graph, it implies that we have to learn all the edges from the $N-4$ nodes to node 1. However, if i is not part of the clique, any valid sample for such an edge (i, 1) has to have i as its source. Having enough valid samples for each of these edges is therefore equivalent to collecting $\Omega(\frac{1}{\Delta^2})$ copies of $N-4$ distinct coupons in the standard Coupon collector problem. Using results from (**??**), we need $\Omega((K\log(K) + (d-1)\cdot K\cdot\log\log(K))$ samples to obtain d copies of each coupon when there are K distinct coupons in total, which is here $\Omega\left((N-4)\log(N-4) + (\frac{1}{\Delta^2}-1)\cdot(N-4)\cdot\log\log(N-4)\right)$ cascades. Using standard approximation, we get the desired result.

Combining the results:

Theorem 2. *We need at least* $\Omega\left(N\log(N) + \frac{N\log\log(N)}{\Delta^2}\right)$ *cascades to learn any mixture of directed graphs of minimum out-degree 3.*

C.2. Undirected lower bound

We reuse a lot of the techniques in the previous subsection. This time, we consider a simple line graph on N nodes, where for all $1 \leq i \leq N-1$, node i is connected to node $i+1$. Like in the previous example, the weights are all p in G_1, and all $p+\Delta$ in G_2.

Reusing Claim 5, we now prove:

Claim 7. *We need at least* $\Omega\left(\frac{N}{\Delta^2}\right)$ *cascades to obtain enough valid samples for edge (1,2).*

Proof. To provide a valid sample, either:

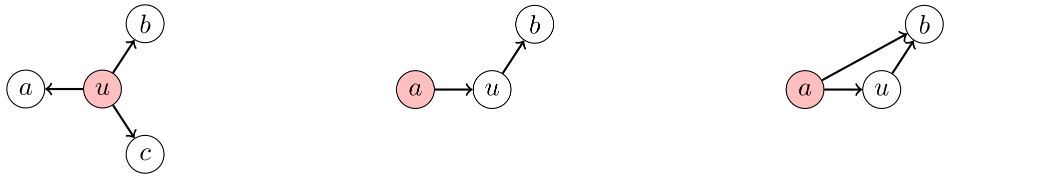
- Node 1 is the source, which happens with probability $\mathcal{P}_1 = \frac{1}{N}$.

- Node 2 was infected, which happens with probability $\mathcal{P}_2 \leq \displaystyle\sum_{i=2}^{N} \frac{1}{N} p_{max}^{i-2} \leq \frac{1}{N}\frac{1}{1-p_{max}}$.

Therefore, the probability of getting a valid sample is smaller than $\mathcal{P}_1 + \mathcal{P}_2 \leq \frac{1}{N}\cdot\frac{2}{1-p_{max}}$. Hence, we need at least $\Omega(\frac{1-p_{max}}{2}\cdot N\cdot\frac{1}{\Delta^2}) = \Omega\left(\frac{N}{\Delta^2}\right)$ cascades to obtain enough valid samples.

Since we need to learn at least edge $(1,2)$ to learn all the edges of this graph:

Theorem 3. *We need at least* $\Omega\left(\frac{N}{\Delta^2}\right)$ *cascades to learn any mixture of undirected graphs.*



(a) A star vertex u for a directed graph. (b) First structure to ensure sign consistency. (c) Second structure to ensure sign consistency.

Figure 2: Structures for directed graphs of minimum out-degree three.

D. Directed graphs

D.1. Structures

Star vertex For directed graph of out-degree at least 3, every vertex is a star vertex. This implies we can reuse the star vertex equations to learn the weights of the whole neighborhood of each node. However, if we learn the neighborhoods of node u in both graphs, which we call $\mathcal{N}_1^u$ and $\mathcal{N}_2^u$, as well as the neighbordhoods of node a, which we call $\mathcal{N}_1^a$ and $\mathcal{N}_2^a$, it is impossible to recover from the star structure alone if $\mathcal{N}_1^u$ and $\mathcal{N}_1^a$ are in the same mixture, or if it is $\mathcal{N}_1^u$ and $\mathcal{N}_2^a$ instead. We therefore use the two other structures in Figure 2 to ensure mixture consistency.

Mixture consistency Suppose we have learned the weights of all the edges stemming from a, as well as all the weighted edges stemming from u, and suppose there is no edge between a and b. The probability that a infected u, which in turn infected b is:

$$\mathbb{P}(a \to u \to b | a \in I_0) = \frac{p_{au}p_{ub} + q_{au}q_{ub}}{2}.$$

This gives us a way to decide whether $\mathcal{N}_1^u$ and $\mathcal{N}_1^a$ are in the same mixture, or if it is $\mathcal{N}_1^u$ and $\mathcal{N}_2^a$ instead. Indeed, if we know $p_{au} \in \mathcal{N}_1^a, q_{au} \in \mathcal{N}_2^a$, and we also know $w_{ub} \in \mathcal{N}_1^u, w'_{ub} \in \mathcal{N}_2^u$, and we have an estimator $\hat{Y}_{au,ub}$ for $\mathbb{P}(a \to u \to b | a \in I_0)$, then we can check whether $\hat{Y}_{au,ub} \approx \frac{p_{au}w_{ub} + q_{au}w'_{ub}}{2}$, in which case $\mathcal{N}_1^u$ belongs with $\mathcal{N}_1^a$, or whether $\hat{Y}_{au,ub} \approx \frac{p_{au}w'_{ub} + q_{au}w_{ub}}{2}$, in which case $\mathcal{N}_2^u$ belongs in the with $\mathcal{N}_1^a$. We call this procedure CHECKPATH.

Similarly, if there is an edge between a and b, then:

$$\mathbb{P}(a \to u \to b | a \in I_0) = \frac{p_{au}(1 - p_{ab})p_{ub} + q_{au}(1 - q_{ab})q_{ub}}{2}.$$

This also allows us to ensure mixture consistency. We call this procedure CHECKTRIANGLE.

Here is the final algorithm:

Algorithm 5 Learn the weights of directed edges

Input Vertex set V
Output Edge weights for the two epidemics graphs
0: $E \leftarrow$ LEARNEDGES(V)
0: Select any first node v
0: $W \leftarrow$ LEARNSTAR(v, E, W)
0: $S = \{v\}$
0: **while** $S \neq V$ **do**
0: Select $a \in S, v \in V \backslash S$ such that $(a, u) \in E$ {v **has out-degree at least 3**}
0: $\mathcal{N}_1, \mathcal{N}_2 \leftarrow$ LEARNSTAR(u, E, W)
0: Select $b \neq a$ neighbor of u {b **exists because u os of degree at least 3.**}
0: **if** $(a, b) \notin E$ **then** {**Use first structure.**}
0: **if** CHECKPATH$(v, u, b, W, \mathcal{N}_1, \mathcal{N}_2)$ **then**
0: $W = \{W_1 \cup \mathcal{N}_1, W_2 \cup \mathcal{N}_2\}$
0: **else**
0: $W = \{W_1 \cup \mathcal{N}_2, W_2 \cup \mathcal{N}_1\}$
0: **else**{**Use second structure.**}
0: **if** CHECKTRIANGLE$(v, u, b, W, \mathcal{N}_1, \mathcal{N}_2)$ **then**
0: $W = \{W_1 \cup \mathcal{N}_1, W_2 \cup \mathcal{N}_2\}$
0: **else**
0: $W = \{W_1 \cup \mathcal{N}_2, W_2 \cup \mathcal{N}_1\}$
0: $S \leftarrow S \cup \{u\}$
return W =0

E. Unbalanced/Unknown Mixtures

In this section we provide the primitives required for LEARNSTAR and LEARNLINE, when the first mixture occurs with probability α and the second mixture with probability $(1 - \alpha)$.

Notations: In this section, to avoid clutter in notation we use i, j and k to be all *distinct* unless mentioned otherwise. Also, let $\sigma(\{a, b, c\}) = \{(a, b, c), (b, c, a), (c, a, b)\}$ denote all the permutations of a, b, and c.

Claim 8. *If a and b are two distinct nodes of $V_1 \cap V_2$ such that $(a, b) \in E_1 \cap E_2$ then under general mixture model $X_{ab} = \alpha p_{ab} + (1 - \alpha) q_{ab}$.*

*Fuither, when the four nodes u, a, b and c forms a star graph (Fig. **??**) with u in the center under general mixture model*

$$1)\ \forall i, j \in \{a, b, c\}, i, j \neq u, Y_{ui,uj} = \alpha p_{ui} p_{uj} + (1 - \alpha) q_{ui} q_{uj},$$
$$2)\ Z_{ua,ub,uc} = \alpha p_{ua} p_{ub} p_{uc} + (1 - \alpha) q_{ua} q_{ub} q_{uc}.$$

*Finally, when the four nodes u, a, b and c forms a line graph (Fig. **??**) under general mixture model*

$$1)\ Y^{|}_{ua,ub} = \alpha p_{ua} p_{ub} + (1 - \alpha) q_{ua} q_{ub},\ 2)\ Y^{|}_{ub,bc} = \alpha p_{ub} p_{bc} + (1 - \alpha) q_{ub} q_{bc},$$
$$3)\ Z^{|}_{ua,ub,bc} = \alpha p_{ua} p_{ub} p_{bc} + (1 - \alpha) q_{ua} q_{ub} q_{bc}.$$

The proof of the above claim is omitted as it follows closely the proofs of Claim **??**, **??**, and **??**.

E.1. Star Graph

We now present the following two lemmas which recover the weights p_{ui},and q_{ui} for all $i \in \{a, b, c\}$ in the star graph (Fig. **??**), and the general mixture parameter α, respectively.

Lemma 7 (Weights of General Star Graph)**.** *Under Conditions* **??** *and* **??***, in the setting of infinite samples, for the* $star$ *structure* (u, a, b, c) *with* u *as the central vertex the weight of any edge* (u, a) *is given by:*

$$p_{ua} = X_{ua} + s_{ua}\sqrt{\tfrac{1-\alpha}{\alpha}}\sqrt{\frac{(Y_{ua,ub} - X_{ua}X_{ub})(Y_{ua,uc} - X_{ua}X_{uc})}{Y_{ub,uc} - X_{ub}X_{uc}}}$$

$$q_{ua} = X_{ua} - s_{ua}\sqrt{\tfrac{\alpha}{1-\alpha}}\sqrt{\frac{(Y_{ua,ub} - X_{ua}X_{ub})(Y_{ua,uc} - X_{ua}X_{uc})}{Y_{ub,uc} - X_{ub}X_{uc}}}$$

where $s_{ua} \in \{-1, 1\}$ *and* $b, c \in N_1(u) \cap N_2(u)$ *such that* $b, c \neq a$*,* $b \neq c$*.*

Proof. We notice that for $r \neq j \in \{a, b, c\}$

$$\begin{aligned}(Y_{ui,uj} - X_{ui}X_{uj}) &= (\alpha p_{ui}p_{uj} + (1-\alpha)q_{ui}q_{uj}) - (\alpha p_{ui} + (1-\alpha)q_{ui})\,(\alpha p_{uj} + (1-\alpha)q_{uj}) \\ &= \alpha(1-\alpha)(p_{ui} - q_{ui})(p_{uj} - q_{uj}).\end{aligned}$$

The rest of the proof follows the same steps as given in the proof of Lemma **??** with the above modification. □

Lemma 8 (Sign Ambiguity Star Graph)**.** *Under Conditions* **??** *and* **??***, in the setting of infinite samples, for edges* $(u, a), (u, b)$ *for the* $star$ *structure* (u, a, b, c) *with* u *as the central vertex, the sign pattern* s_{ua}, s_{ub} *satisfy the following relation.*

$$s_{ub}s_{ua} = \mathsf{sgn}(Y_{ua,ub} - X_{ua}X_{ub}).$$

Proof. The proof of the first statement follows the same logic as the proof of Lemma 3, after noting that $\mathsf{sgn}(\alpha(1-\alpha)) = 1$ for $\alpha \in (0, 1)$. □

E.2. Line Graph

We now present the recovery of parameters in the case of a line graph with knowledge of α

Lemma 9 (Weights of General Line Graph)**.** *Under Conditions* **??** *and* **??***, in the setting of infinite samples, the weights of the edges* (u, a)*, and* (u, b) *for a* $line$ *graph* $a - u - b - c$ *can be learned in closed form (as given in the proof), as a function of*
(1) the mixture parameter α*,*
(2) estimators X_{ua}*,* X_{ub}*,* X_{bc}*,* $Y^{|}_{ua,ub}$*,* $Y^{|}_{ub,bc}$*, and* $Z^{|}_{ua,ub,bc}$*,*
(3) one variable $s_{ub} \in \{-1, +1\}$*.*

Proof. We first note that we have access to the following three relations

$$\begin{aligned}&1)\quad (Y^{|}_{ua,ub} - X_{ua}X_{ub}) = \alpha(1-\alpha)(p_{ua} - q_{ua})(p_{ub} - q_{ub}) \\ &2)\quad (Y^{|}_{ub,bc} - X_{ub}X_{bc}) = \alpha(1-\alpha)(p_{ub} - q_{ub})(p_{bc} - q_{bc}) \\ &3)\quad (Z^{|}_{ua,ub,bc} + X_{ua}X_{ub}X_{bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}) \\ &\qquad = \alpha(1-\alpha)((1-\alpha)p_{ub} + \alpha q_{ub})(p_{ua} - q_{ua})(p_{bc} - q_{bc}).\end{aligned}$$

The first two inequalities follow similar to Lemma **??**. We derive the final equality below.

$$\begin{aligned}&Z^{|}_{ua,ub,bc} + X_{ua}X_{ub}X_{bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub} \\ &= \alpha p_{ua}p_{ub}p_{bc} + (1-\alpha)q_{ua}q_{ub}q_{bc} \\ &- (\alpha p_{ua} + (1-\alpha)q_{ua})((\alpha p_{ub}p_{bc} + (1-\alpha)q_{ub}q_{bc}) - (\alpha p_{bc} + (1-\alpha)q_{bc})((\alpha p_{ua}p_{ub} + (1-\alpha)q_{ua}q_{ub}) \\ &+ (\alpha p_{ua} + (1-\alpha)q_{ua})(\alpha p_{ub} + (1-\alpha)q_{ub})(\alpha p_{bc} + (1-\alpha)q_{bc}) \\ &= \alpha(1-\alpha)^2 p_{ua}p_{ub}p_{bc} + \alpha^2(1-\alpha)q_{ua}q_{ub}q_{bc} \\ &- \alpha(1-\alpha)^2 p_{ua}p_{ub}q_{bc} + \alpha^2(1-\alpha)p_{ua}q_{ub}p_{bc} - \alpha(1-\alpha)^2 q_{ua}p_{ub}p_{bc} \\ &- \alpha^2(1-\alpha)q_{ua}q_{ub}p_{bc} + \alpha(1-\alpha)^2 q_{ua}p_{ub}q_{bc} - \alpha^2(1-\alpha)p_{ua}q_{ub}q_{bc} \\ &= \alpha(1-\alpha)((1-\alpha)p_{ub} + \alpha q_{ub})(p_{ua} - q_{ua})(p_{bc} - q_{bc})\end{aligned}$$

Therefore, we obtain the following quadratic equation in p_{ub} and q_{ub} (unlike the $\alpha = 1/2$ case it cannot be easily reduced to a linear equation),

$$\frac{\alpha(1-\alpha)(p_{ub}-q_{ub})^2}{((1-\alpha)p_{ub}+\alpha q_{ub})} = \frac{(Y^{|}_{ua,ub} - X_{ua}X_{ub})(Y^{|}_{ub,bc} - X_{ub}X_{bc})}{(Z^{|}_{ua,ub,bc} + X_{ua}X_{ub}X_{bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub})} := C^{|}_{ub}$$

Note that $X_{ub} = \alpha p_{ub} + (1-\alpha)q_{ub}$, thus the above can be reduced to

$$\begin{aligned}
&\frac{\alpha(1-\alpha)(p_{ub}-X_{ub})^2/(1-\alpha)^2}{(p_{ub}(1-2\alpha)+\alpha X_{ub})/(1-\alpha)} = C^{|}_{ub} \\
&p_{ub}^2 - 2\left(X_{ub} + \tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ub}\right)p_{ub} = C^{|}_{ub}X_{ub} - X_{ub}^2 \\
&p_{ub} = X_{ub} + \tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ub} + s_{ub}\sqrt{\left(\tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ub}\right)^2 + \tfrac{1-\alpha}{\alpha}C^{|}_{ub}X_{ub}} \\
&q_{ub} = X_{ub} - \tfrac{(1-2\alpha)}{2(1-\alpha)}C^{|}_{ub} - s_{ub}\sqrt{\left(\tfrac{(1-2\alpha)}{2(1-\alpha)}C^{|}_{ub}\right)^2 + \tfrac{\alpha}{1-\alpha}C^{|}_{ub}X_{ub}}
\end{aligned}$$

We substitute in the above two equations θ and s_α as defined below

$$\alpha = \tfrac{1}{2}(1 - s_\alpha\sqrt{\theta}), \qquad (1-\alpha) = \tfrac{1}{2}(1 + s_\alpha\sqrt{\theta}), \qquad (1-2\alpha) = s_\alpha\sqrt{\theta}.$$

From the substitution we obtain,

$$\begin{aligned}
p_{ub} &= X_{ub} + \tfrac{s_\alpha\sqrt{\theta}(1+s_\alpha\sqrt{\theta})C^{|}_{ub}}{(1-\theta)}\left(1 + s_\alpha s_{ub}\sqrt{1 + \tfrac{(1-\theta)X_{ub}}{\theta C^{|}_{ub}}}\right) \\
q_{ub} &= X_{ub} - \tfrac{s_\alpha\sqrt{\theta}(1-s_\alpha\sqrt{\theta})C^{|}_{ub}}{(1-\theta)}\left(1 + s_\alpha s_{ub}\sqrt{1 + \tfrac{(1-\theta)X_{ub}}{\theta C^{|}_{ub}}}\right)
\end{aligned}$$

Next we use p_{ub}, and q_{ub} to obtain p_{ua}, and q_{ua}. Specifically, we have

$$\begin{aligned}
&\alpha(1-\alpha)(p_{ub}-q_{ub})(p_{ua}-q_{ua}) = (Y^{|}_{ua,ub} - X_{ua}X_{ub}) \\
&(p_{ua}-q_{ua}) = \frac{4(Y^{|}_{ua,ub} - X_{ua}X_{ub})}{s_\alpha\sqrt{\theta}\left(1 + s_\alpha s_{ub}\sqrt{1 + \tfrac{(1-\theta)X_{ub}}{\theta C^{|}_{ub}}}\right)}.
\end{aligned}$$

Finally, we use the above relation to arrive at the required result.

$$\begin{aligned}
p_{ua} &= X_{ua} + \frac{2(1+s_\alpha\sqrt{\theta})(Y^{|}_{ua,ub} - X_{ua}X_{ub})}{s_\alpha\sqrt{\theta}\left(1 + s_\alpha s_{ub}\sqrt{1 + \tfrac{(1-\theta)X_{ub}}{\theta C^{|}_{ub}}}\right)} \\
q_{ua} &= X_{ua} - \frac{2(1-s_\alpha\sqrt{\theta})(Y^{|}_{ua,ub} - X_{ua}X_{ub})}{s_\alpha\sqrt{\theta}\left(1 + s_\alpha s_{ub}\sqrt{1 + \tfrac{(1-\theta)X_{ub}}{\theta C^{|}_{ub}}}\right)}
\end{aligned}$$

□

Lemma 10 (Sign Ambiguity Line graph on 5 nodes)**.** *Under Conditions* **??** *and* **??***, in the setting of infinite samples, for a line structure $a-u-b-c-d$ the sign patterns s_{ub} and s_{bc} satisfy the relation, $s_{ub}s_{bc} = \mathsf{sgn}(Y^{|}_{ub,bc} - X_{ub}X_{bc})$.*

Proof. The proof is almost identical to the other sign ambiguity proofs. □

E.3. Finite Sample Complexity

We start by observing that the Claim 2 still holds in the general case.

Claim 9. *If we can estimate $X_{ua}, Y^{*}_{ua,ub}, Y^{|}_{ua,ab}$ and $Z^{|}_{ua,ub,bc}$ within ϵ_1, we can estimate p_{ua} and q_{ua} within precision $\epsilon = \mathcal{O}\left(\epsilon_1 / \min(p_{min}, \Delta)^5 \min(\alpha, 1-\alpha)^4\right)$.*

Proof. The proof proceeds in a very similar manner as Claim4. Following the derivations for $\hat{p}_{ua}$ and $\hat{q}_{ua}$ in the proof of Claim4, we can see that for the star primitive all the computation carry over with a scaling of $\frac{4}{\alpha(1-\alpha)}$ as we have $Y^{*}_{ur,us} - X_{ur}X_{us} \geq \Delta^2\alpha(1-\alpha)$ instead of $\Delta^2/4$.

The line primitive presents with increased difficulty as the estimator is more complex. We first observe that $\alpha(1-\alpha)\Delta^2 \leq C^{|}_{ub} \leq \max(\alpha, (1-\alpha))$. We recall that

$$
\begin{aligned}
&(Z^{|}_{ua,ub,bc} + X_{ua}X_{ub}X_{bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub}) \\
&= \alpha(1-\alpha)((1-\alpha)p_{ub} + \alpha q_{ub})(p_{ua} - q_{ua})(p_{bc} - q_{bc}) \\
&\geq \min(\alpha, 1-\alpha)^2 p_{min} \min(p_{min}, \Delta)^2/2, \\
&(Y^{|}_{ua,ub} - X_{ua}X_{ub}) = \alpha(1-\alpha)(p_{ua} - q_{ua})(p_{ub} - q_{ub}) \geq \min(\alpha, 1-\alpha)\min(p_{min}, \Delta)^2/2.
\end{aligned}
$$

Let us assume the error in $(Z^{|}_{ua,ub,bc} + X_{ua}X_{ub}X_{bc} - X_{ua}Y^{|}_{ub,bc} - X_{bc}Y^{|}_{ua,ub})$ is bounded as ϵ_d and the error in $(Y^{|}_{ua,ub} - X_{ua}X_{ub})(Y^{|}_{ub,bc} - X_{ub}X_{bc})$ is bounded as ϵ_n. We have $\epsilon_n \leq 4\epsilon_1$ and $\epsilon_d \leq 3\epsilon_1$ as all the estimators are assumed to have error bounded by ϵ_1.

Therefore, using $|x/y - \hat{x}/\hat{y}| \leq x/y(\delta_x/x + \delta_y/y) + \mathcal{O}(\delta_x\delta_y)$,

$$
\begin{aligned}
|\hat{C}^{|}_{ub} - C^{|}_{ub}| \leq \epsilon_c &:= \mathcal{O}\left(\tfrac{\epsilon_n}{\min(\alpha,1-\alpha)^2\min(p_{min},\Delta)^4} + \tfrac{\epsilon_d}{\min(\alpha,1-\alpha)^2 p_{min}\min(p_{min},\Delta)^2}\right) \\
&= \mathcal{O}(\epsilon_1/\min(\alpha, 1-\alpha)^2\min(p_{min}, \Delta)^4).
\end{aligned}
$$

Using the above bound in the expression of p_{ua} we can obtain,

$$
\begin{aligned}
&|\hat{p}_{ua} - p_{ua}| \leq |\hat{X}_{ua} - X_{ua}| + \tfrac{(1-2\alpha)}{2\alpha}|\hat{C}^{|}_{ua} - C^{|}_{ua}| + \dots \\
&+ \left|\sqrt{\left(\tfrac{(1-2\alpha)}{2\alpha}\hat{C}^{|}_{ua}\right)^2 + \tfrac{(1-\alpha)}{\alpha}\hat{C}^{|}_{ua}\hat{X}_{ua}} - \sqrt{\left(\tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ua}\right)^2 + \tfrac{(1-\alpha)}{\alpha}C^{|}_{ua}X_{ua}}\right| \\
&\leq |\hat{X}_{ua} - X_{ua}| + \tfrac{(1-2\alpha)}{2\alpha}|\hat{C}^{|}_{ua} - C^{|}_{ua}| + \dots \\
&+ \frac{\left(\tfrac{(1-2\alpha)}{2\alpha}\right)^2 |\hat{C}^{|}_{ua} - C^{|}_{ua}|(\hat{C}^{|}_{ua} + C^{|}_{ua}) + \tfrac{(1-\alpha)}{\alpha}|\hat{C}^{|}_{ua}\hat{X}_{ua} - C^{|}_{ua}X_{ua}|}{\sqrt{\left(\tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ua}\right)^2 + \tfrac{(1-\alpha)}{\alpha}C^{|}_{ua}X_{ua}}} \\
&\leq \epsilon_1 + \tfrac{(1-2\alpha)}{2\alpha}\epsilon_c + \tfrac{2}{(1-\alpha)\min(p_{min},\Delta)}\left(2\left(\tfrac{(1-2\alpha)}{2\alpha}\right)^2\epsilon_c + \tfrac{(1-\alpha)}{\alpha}(\epsilon_1 + \epsilon_c)\right) + o(\epsilon_1) + o(\epsilon_c) \\
&\leq \mathcal{O}(\epsilon_1/\min(p_{min}, \Delta)\alpha(1-\alpha)) + \mathcal{O}(\epsilon_c/\min(p_{min}, \Delta)\alpha^2(1-\alpha)) + o(\epsilon_1) + o(\epsilon_c)
\end{aligned}
$$

Therefore, using the estimate of ϵ_c we obtain,

$$
|\hat{p}_{ua} - p_{ua}| \leq \mathcal{O}\left(\epsilon_1/\min(p_{min}, \Delta)^5\alpha\min(\alpha, 1-\alpha)^3\right).
$$

Switching α and $(1-\alpha)$ gives us the same bounds for $|\hat{q}_{ua} - q_{ua}|$. □

In the above derivation we have used $\sqrt{\left(\frac{(1-2\alpha)}{2\alpha}C^{|}_{ua}\right)^2 + \frac{(1-\alpha)}{\alpha}C^{|}_{ua}X_{ua}} \geq (1-\alpha)\min(p_{min}, \Delta)/2$. We now derive the above inequality.

$$
\begin{aligned}
&|\sqrt{\left(\tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ua}\right)^2+\tfrac{(1-\alpha)}{\alpha}C^{|}_{ua}X_{ua}}| = |p_{ua}-X_{ua}-\tfrac{(1-2\alpha)}{2\alpha}C^{|}_{ua}| \\
&= |(1-\alpha)(p_{ua}-q_{ua}) - \tfrac{(1-2\alpha)(1-\alpha)(p_{ua}-q_{ua})^2}{2((1-\alpha)p_{ua}+\alpha q_{ua})}| \\
&\geq \begin{cases} (1-\alpha)\min(p_{min},\Delta), (\alpha \geq 1/2 \wedge p_{ua} \geq q_{ua}) \vee (\alpha < 1/2 \wedge p_{ua} < q_{ua}) \\ (1-\alpha)\min(p_{min},\Delta)|1-\frac{(1-2\alpha)}{2(1-\alpha)}|, (\alpha < 1/2 \wedge p_{ua} \geq q_{ua}) \\ (1-\alpha)\min(p_{min},\Delta)|1-\frac{(2\alpha-1)}{2\alpha}|, (\alpha \geq 1/2 \wedge p_{ua} < q_{ua}), \end{cases}
\end{aligned}
$$

Finally, using union bound on all the estimators involved accross all possible edges, we can obtain the error bound in the following Theorem 4.

Theorem 4. *Suppose Condition* **??** *and* **??** *are true, there exists an algorithm that runs on epidemic cascades over a mixture of two undirected, weighted graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, and recovers the edge weights corresponding to each graph up to precision ϵ in time $O(N^2)$ and sample complexity $O\left(\frac{N \log N}{\epsilon^2 \Delta^4}\right)$ for $\alpha = 1/2$ and $O\left(\frac{N \log N}{\epsilon^2 \Delta^{10} \min(\alpha, 1-\alpha)^8}\right)$ for general $\alpha \in (0, 1), \alpha \neq 1/2$, where $N = |V|$.*