A. Trigonometric identity

Fact A.1.

$$\frac{\sin(\beta + \frac{\alpha}{2} - \theta)}{\sin(\alpha/2)} - \frac{\sin(\beta - \theta)}{\sin(\alpha)} = \frac{\sin(\beta - \theta + \alpha)}{\sin(\alpha/2)}$$

Proof.

$$\begin{aligned}
&\frac{\sin(\beta + \frac{\alpha}{2} - \theta)}{\sin(\alpha/2)} - \frac{\sin(\beta - \theta)}{\sin(\alpha)} \\
&= \frac{\sin(\beta + \frac{\alpha}{2} - \theta)\sin(\alpha) - \sin(\beta - \theta)\sin(\alpha/2)}{\sin(\alpha)\sin(\alpha/2)} \\
&= \frac{1}{2\sin(\alpha)\sin(\frac{\alpha}{2})}\Big(\cos(\beta - \theta - \frac{\alpha}{2}) \\
&- \cos(\beta - \theta + \frac{3\alpha}{2}) \\
&- \cos(\beta - \theta - \frac{\alpha}{2}) + \cos(\beta - \theta + \frac{\alpha}{2})\Big) \\
&= \frac{\cos(\beta - \theta + \frac{\alpha}{2}) - \cos(\beta - \theta + \frac{3\alpha}{2})}{2\sin(\alpha)\sin(\alpha/2)} \\
&= \frac{\sin(\beta - \theta + \alpha)\sin(\alpha)}{\sin(\alpha)\sin(\alpha/2)} \\
&= \frac{\sin(\beta - \theta + \alpha)}{\sin(\alpha/2)}
\end{aligned}$$

where we use the identity that $\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$ ∎

B. Low Width Neural Network for Sparse Vectors

B.1. Theorems

Lemma B.1. *Suppose $\alpha < \frac{1}{n^{8k}}$, then with high probability, for all $S, S' \subseteq [n]$ such that $|S| = |S'| = k$,*

$$\|w_S - w_{S'}\|_2 \leq \alpha^{\frac{1}{4}}$$

Proof. Consider a fixed set $S \subseteq [n]$ such that $|S| = k$. Now for any $S' \neq S$ such that $|S| = k$, consider the set $T' = S' \setminus S$.

$$\begin{aligned}
\Pr[\|w_S - w_{S'}\|_2 \leq \alpha^{\frac{1}{4}}] &\leq \Pr[\forall i \in T', w_S \in W_i] \\
&= \prod_{i \in T'} \Pr[w_S \in W_i] \\
&= \alpha^{|T'|/4}
\end{aligned}$$

So, then the probability that there exists a set S' such that w'_S is close is given by:

$$\begin{aligned}
\Pr[\exists S' : \|w_S - w_{S'}\|_2 \leq \alpha^{\frac{1}{4}}] &\leq \sum_{\substack{S' \subseteq [n] \\ |S'| = k, S' \neq S}} \alpha^{|S' \setminus S|/4} \\
&= \sum_{i=1}^{k} \binom{k}{i}\binom{n-k}{i}\alpha^i \\
&\leq (nk\alpha^{\frac{1}{4}})
\end{aligned}$$

where the last inequality follows because $\alpha < 1/nk$. Now, applying a union bound over all choices of S, we get

$$\begin{aligned}
\Pr[\exists S, S' : \|w_S - w_{S'}\|_2 \leq \alpha] &\leq \binom{n}{k} \times (nk\alpha^{\frac{1}{4}}) \\
&\leq 1/n^k
\end{aligned}$$

∎

Lemma B.2. *Suppose $\alpha < \frac{1}{n^{8k}}$, then given $S_1, S_2 \subseteq [n]$, such that $|S_1| = |S_2| = k$ and $|S_1 \cap S_2| = l$,*

$$W_{S_1} \cap W_{S_2} = \emptyset$$

with probability $1 - 1/n^{6k}$

Proof. Let us denote $R = \|w_{S_1} - w_{S_2}\|_2$. We know from Lemma B.1 that with high probability $\|w_{S_1} - w_{S_2}\|_2 \geq \alpha^{1/4}$.

Since $\tan(\alpha) \approx \alpha$ when α is small, we will substitute α in place of $\tan(\alpha)$.

Let V'_S denote a matrix whose rows consist of $\{v'_i \mid i \in S\}$. Observe that $W_{S_1} \cap W_{S_2} = \emptyset$ is equivalent to stating that

$$\nexists x \in \mathbb{S}^k : \left\|V'_{S_1}x\right\|_\infty < \alpha \wedge \left\|V'_{S_2}x\right\|_\infty < \alpha \tag{4}$$

Consider an ϵ-net N over $\mathbb{S}^k$ where $\epsilon = \alpha$. If the above guarantee holds with 2α when restricted to points in N, then for any element $x \in \mathbb{S}^k$, if $p \in N$ is the element closest to x, we have a $b \in \{1, 2\}$ for which we know that $\left\|V'_{S_b}p\right\|_\infty \geq 2\alpha$. Hence

$$\begin{aligned}
\left\|V'_{S_b}x\right\|_\infty &\geq \left\|V'_{S_b}p\right\|_\infty - \left\|V'_{S_b}(x - p)\right\|_\infty \\
&\geq 2\alpha - \epsilon \\
&\geq \alpha
\end{aligned}$$

So, we prove that

$$\nexists x \in N : \left\|V'_{S_1}x\right\|_\infty < \alpha \wedge \left\|V'_{S_2}x\right\|_\infty < \alpha \tag{5}$$

We split this into two cases.

Case 1: Points close to either W_{S_1} or W_{S_2}
Consider the set $T^{(1)} = \{x \in N \mid \|x - w_{S_1}\|_2 \leq$

$R/2\}$. We can partition this into sets $T_i^{(1)} = \{x \in N \mid \|x - w_{S_1}\|_2 \in [2^{i-1}\epsilon, 2^i\epsilon]\}$ for $i \in [1, \log(R/\epsilon)]$, and $T_0^{(1)} = \{x \in N \mid \|x - w_{S_1}\|_2 < \epsilon\}$.

Observe that for any point $x \in T_r^{(1)}$,

$$\begin{aligned}
&\Pr\left[\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&= \Pr\left[\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \middle| \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&\cdot \Pr\left[\left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&\leq \left(\frac{\alpha}{\|x - w_{S_1}\|_2}\right)^{k-l}\left(\frac{\alpha}{\|x - w_{S_2}\|_2}\right)^k \\
&\leq \left(\frac{\alpha}{r}\right)^{k-l}\left(\frac{2\alpha}{R}\right)^k \\
&= \frac{2^k\alpha^{2k-l}}{r^{k-l}R^k}
\end{aligned}$$

Since the $\left|T_r^{(1)}\right| \leq (r/\epsilon)^k$(by a volume argument):

$$\begin{aligned}
&\Pr\Big[\exists x \in T_r^{(1)} : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\qquad\leq \left(\frac{2^k\alpha^{2k-l}}{\epsilon^k R^{k-l}}\right)
\end{aligned}$$

So, if we take a union bound over the $r = 1, \ldots, \log(R/\epsilon)$ values, we get

$$\begin{aligned}
&\Pr\Big[\exists x \in T^{(1)} : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\leq \log(R/\epsilon)\left(\frac{2^k\alpha^{2k-l}}{\epsilon^k R^{k-l}}\right)
\end{aligned}$$

We can similarly show this for every $x \in T^{(2)}$.

$$\begin{aligned}
&\Pr\Big[\exists x \in T^{(2)} : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\qquad\leq \log(R/\epsilon)\left(\frac{2^k\alpha^{2k-l}}{\epsilon^k R^{k-l}}\right)
\end{aligned}$$

Case 2: Points close to neither W_{S_1} nor W_{S_2} Let $T' = \{x \in N \mid \|x - w_{S_1}\|_2 > R/2 \wedge \|x - w_{S_1}\|_2 > R/2\}$. We partition T' into the sets $T'_0, T'_1, \ldots$.

Consider $T'_0 = \{x \in N \mid \|x - w_{S_1}\|_2 \geq R/2 \wedge \|x - w_{S_2}\|_2 \geq R/2 \wedge \|x - ((w_{S_1} + w_{S_2})/2)\|_2 \leq R\}$.

For any point $x \in T'_0$:

$$\begin{aligned}
&\Pr\left[\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&= \Pr\left[\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha\right] \\
&\qquad\cdot \Pr\left[\left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&= \left(\frac{\alpha}{\|x - w_{S_1}\|_2}\right)^k\left(\frac{\alpha}{\|x - w_{S_2}\|_2}\right)^{k-l} \\
&\leq \left(\frac{2\alpha}{R}\right)^k\left(\frac{2\alpha}{R}\right)^{k-l} \\
&= \left(\frac{2\alpha}{R}\right)^{2k-l}
\end{aligned}$$

and since $|T'_0| \precsim (R/\epsilon)^k$, we can conclude

$$\begin{aligned}
&\Pr\Big[\exists x \in T'_0 : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\leq \frac{(2\alpha)^{2k-l}}{\epsilon^k R^{k-l}}
\end{aligned}$$

Define $T'_i = \{x \in \mathbb{S}^k \mid \|x - ((w_{S_1} + w_{S_2})/2)\|_2 \in [2^{i-1}R, 2^iR]\}$. For any $x \in T'_i$,

$$\begin{aligned}
&\Pr\left[\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\right] \\
&\leq \left(\frac{\alpha}{\|x - w_{S_1}\|_2}\right)^k\left(\frac{\alpha}{\|x - w_{S_2}\|_2}\right)^k \\
&\leq \left(\frac{\alpha}{2^{i-1}R}\right)^k\left(\frac{2\alpha}{2^{i-1}R}\right)^{k-l} \\
&= \left(\frac{8\alpha}{2^iR}\right)^{2k-l}
\end{aligned}$$

So, taking a union bound over all points in T'_i, we have:

$$\begin{aligned}
&\Pr\Big[\exists x \in T'_i : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\leq \left(\frac{2^iR}{\epsilon}\right)^k\left(\frac{4\alpha}{2^iR}\right)^{2k-l} \\
&= \frac{(8\alpha)^{2k-l}}{(2^iR)^{k-l}\epsilon^k}
\end{aligned}$$

So, bounding over all partitions of T', we get:

$$\begin{aligned}
&\Pr\Big[\exists x \in T' : \\
&\left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\qquad\leq \sum_{i=0}^{\infty} \frac{(4\alpha)^{2k-l}}{(2^iR)^{k-l}\epsilon^k} \\
&\qquad\leq \frac{(8\alpha)^{2k-l}}{R^{k-l}\epsilon^k}
\end{aligned}$$

So, (5) holds with probability:

$$\begin{aligned}
&\Pr\Big[\exists x \in N : \\
&\quad \left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\qquad \leq \log(R/\epsilon)\Big(\frac{2^k\alpha^{2k-l}}{\epsilon^k R^{k-l}}\Big) \\
&\qquad \leq \log(R/\epsilon)\Big(\frac{2^k\alpha^{2k-l}}{\alpha^k\alpha^{k-l/4}}\Big) \\
&\qquad = \log(R/\epsilon)2^k\alpha^{3(k-l)/4}
\end{aligned}$$

Since $\log(R/\epsilon) \approx k\log(n)$ and $\alpha^{3(k-l)/4} < \frac{1}{n^{6k}}$, this is bounded by $\frac{1}{n^{6k}}$. Further, because of the argument which showed that (5) implies (4), up to a factor 2 scaling of α, we get that:

$$\begin{aligned}
&\Pr\Big[\exists x \in \mathbb{S}^k : \\
&\quad \left\|V'_{S_1}(x - w_{S_1})\right\|_\infty < \alpha \wedge \left\|V'_{S_2}(x - w_{S_2})\right\|_\infty < \alpha\Big] \\
&\qquad \leq 1/n^{6k}
\end{aligned}$$

□

Lemma B.3. *Suppose* $\alpha < \frac{1}{n^{8k}}$*, for all sets,* $S_1, S_2 \subseteq [n]$*, such that* $|S_1| = |S_2| = k$*,*

$$W_{S_1} \cap W_{S_2} = \emptyset$$

with probability $1 - 1/n^{3k}$

Proof. For any two sets S_1 and S_2 such that the $S_1 \backslash S_2 = l$, we know that:

$$W_{S_1} \cap W_{S_2} = \emptyset$$

with probability $\geq 1 - \alpha^{3/4}$ So, for a given set S_1,

$$\begin{aligned}
&\Pr\Big[\exists S \subseteq [n], |S| = k : W_{S_1} \cap W_S \neq \emptyset\Big] \\
&\qquad \leq \sum_{i=1}^{k} \binom{k}{i}\binom{n-k}{i}\alpha^{3/4} \\
&\qquad \leq n^{2k}\alpha^{3/4}
\end{aligned}$$

Further, taking a union bound over all choices of S, we get

$$\begin{aligned}
&\Pr\Big[\exists S_1 \neq S_2 \subseteq [n], |S_1| = |S_2| = k : W_{S_1} \cap W_{S_2} \neq \emptyset\Big] \\
&\qquad \leq \binom{n}{k} n^{2k}\alpha^{3k/4} \\
&\qquad \leq n^{3k}\alpha^{3/4} \\
&\qquad \leq 1/n^{3k}
\end{aligned}$$

□

Proof. of Theorem 2.2 Let $y \in \mathbb{R}^n$ be a k-sparse vector and let $S = \{i \in [n] \mid y_i \neq 0\}$. From Lemma B.1 and Lemma B.3, we know that there exists a point w_S such that $G(w_S)$ is non-zero at exactly the points $\{i \in [n] \mid y_i \neq 0\}$.

Consider the polytope on $\mathbb{S}^k$ defined by W_S which contains w_S. As illustrated in Figure 1, each F_i partitions each W_S into 2 linear regions. So, there exist 2^k polytopes which within W_S such that for each polytope, w_S is a vertex. Consider one such polytope P defined by $\langle x, \alpha v'_i + v_i\rangle > 0$ and $\langle x, v_i\rangle \leq 0$.

Let x_0 be the point in P such that $\langle x_0, v_i + \alpha v'_i\rangle = 0$ for all $i \in S$. Let $\langle x_0, v_i\rangle = -r_i$ for each $i \in S$ and define $r = \frac{1}{2}\min_{i\in S} r_i$.

Now, solve for δ such that $\langle \delta, v_i + \alpha v'_i\rangle = |y_i| / \|y\|_2\, r$ for all $i \in S$. Observe that for such a δ:

$$\begin{aligned}
\langle x + \delta, v_i\rangle &= \langle x, v_i\rangle + \langle \delta, v_i\rangle \\
&= -r_i + \langle \delta, v_i\rangle \\
&\leq -r_i + \|\delta\|_2 \cdot \|v_i\|_2 \\
&\leq -r_i/2
\end{aligned}$$

So, $x + \delta$ lies within P and $G(x+\delta)_i = y_i/\|y\|_2\, r$. So, since $G(a \cdot x) = a \cdot G(x)$, we have:

$$G(\|y\|_2 \cdot r \cdot (x + \delta)) = y$$

□