A. Convergence under Imperfect Feedback with Absolute Random Noise

In this section, we analyze the convergence of OGD-based learning on $\lambda$-cocoercive games under imperfect feedback with absolute random noise (4). We first establish that OGD under noisy feedback converges almost surely in last-iterate to the set of Nash equilibria of a co-coercive game if $\sigma_t^2 \in (0, \sigma^2)$ for some $\sigma^2 < +\infty$ and the finite-time $O(1/\sqrt{T})$ convergence rate on $(1/T)\mathbb{E}[\sum_{t=0}^{T} \epsilon(x_t)]$ under properly diminishing step-size sequences. We also present a finite-time convergence rate on $\mathbb{E}[\epsilon(x_T)]$ if $\sigma_t^2$ satisfies certain conditions.

A.1. Almost Sure Last-Iterate Convergence

We start by developing a key iterative formula for $\mathbb{E}[\epsilon(x_t)]$ in the following lemma.

Lemma A.1 Fix a $\lambda$-cocoercive game $G$ with a continuous action space, $G = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathbb{R}^{n_i}, \{u_i\})_{i=1}^{N}$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) and letting the OGD-based learning run with a step-size sequence $\eta_t \in (0, \lambda)$, the noisy OGD iterate $x_t$ satisfies

$$\mathbb{E}[\epsilon(x_{t+1})] \leq \mathbb{E}[\epsilon(x_t)] + \frac{\sigma_t^2}{\lambda \eta_{t+1}}.$$

We are now ready to establish last-iterate convergence in a strong, almost sure sense. Note that the conditions imposed on $\sigma_t^2$ and $\eta_t$ are minimal.

Theorem A.2 Fix a $\lambda$-cocoercive game $G$ with a continuous action space, $G = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathbb{R}^{n_i}, \{u_i\})_{i=1}^{N}$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Consider the noisy model (2) with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2)$ for some $\sigma^2 < +\infty$ and letting the OGD-based learning run with a step-size sequence $\eta_t = c/\sqrt{T}$ for some constant $c \in (0, \lambda)$, the noisy OGD iterate $x_t$ satisfies

$$\sum_{t=1}^{\infty} \eta_t = +\infty, \quad \sum_{t=1}^{\infty} \eta_t^2 < +\infty.$$

Then the noisy OGD iterate $x_t$ converges to $\mathcal{X}^*$ almost surely.

A.2. Finite-Time Convergence Rate: Time-Average and Last-Iterate

For completeness, we characterize two types of rates: the time-average and last-iterate convergence rate, as formalized by the following theorems.

Theorem A.3 Fix a $\lambda$-cocoercive game $G$ with a continuous action space, $G = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathbb{R}^{n_i}, \{u_i\})_{i=1}^{N}$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2)$ for some $\sigma^2 < +\infty$ and letting the OGD-based learning run with a step-size sequence $\eta_t = c/\sqrt{T}$ for some constant $c \in (0, \lambda)$, the noisy OGD iterate $x_t$ satisfies

$$\frac{1}{T+1} \left( \mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \right) = O \left( \frac{\log(T)}{\sqrt{T}} \right).$$

Inspired by Lemma A.1, we impose an intuitive condition on the variance of noisy process $\{\sigma_t^2\}_{t \geq 0}$. More specifically, there exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\alpha(t) = o(1)$ and $\alpha(t) = \Omega(1/t)$ such that

$$\frac{1}{T+1} \left( \sum_{t=0}^{T-1} (t+1) \sigma_t^2 \right) = O(\alpha(T)). \quad (15)$$

Under this condition, the noisy iterate generated by the OGD-based learning achieves the finite-time last-iterate convergence rate regardless of a sequence of possibly constant step-sizes $\eta_t$ satisfying $0 < \underline{\eta} \leq \eta_t \leq \bar{\eta} < \lambda$ for all $t \geq 1$.

Theorem A.4 Fix a $\lambda$-cocoercive game $G$ with a continuous action space, $G = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^{N} \mathbb{R}^{n_i}, \{u_i\})_{i=1}^{N}$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) satisfying Eq. (15) and letting the OGD-based learning run with a nonincreasing step-size sequence satisfying $0 < \underline{\eta} \leq \eta_t \leq \bar{\eta} < \lambda$ for all $t \geq 1$, the noisy OGD iterate $x_t$ satisfies

$$\mathbb{E}[\epsilon(x_T)] = O(\alpha(T)).$$
B. Proof of Lemma 3.1

Since $\mathcal{X}_i = \mathbb{R}^{n_i}$, we have

$$
\|x_{i,t+2} - x_{i,t+1}\|^2
= \|x_{i,t+1} - x_{i,t} + \eta(v_i(x_{t+1}) - v_i(x_i))\|^2
= \|x_{i,t+1} - x_{i,t}\|^2 + 2\eta(x_{i,t+1} - x_{i,t})^\top (v_i(x_{t+1}) - v_i(x_i)) + \eta^2\|v_i(x_{t+1}) - v_i(x_i)\|^2.
$$

Expanding the right-hand side of the above inequality and summing up the resulting inequality over $i \in \mathcal{N}$ yields that

$$
\|x_{t+2} - x_{t+1}\|^2 = \sum_{i \in \mathcal{N}} \|x_{i,t+2} - x_{i,t+1}\|^2
\leq \sum_{i \in \mathcal{N}} \left( \|x_{i,t+1} - x_{i,t}\|^2 + \eta^2\|v_i(x_{t+1}) - v_i(x_i)\|^2 + 2\eta(x_{i,t+1} - x_{i,t})^\top (v_i(x_{t+1}) - v_i(x_i)) \right)
= \|x_{t+1} - x_t\|^2 + 2\eta(x_{t+1} - x_t)^\top (v(x_{t+1}) - v(x_t)) + \eta^2\|v(x_{t+1}) - v(x_t)\|^2.
$$

Since $G$ is a $\lambda$-cocoercive game, we have

$$(x_{t+1} - x_t)^\top (v(x_{t+1}) - v(x_t)) \leq -\lambda\|v(x_{t+1}) - v(x_t)\|^2.$$

Plugging the above equation into Eq. (16) together with the condition $\eta \in (0, \lambda]$ yields that

$$
\|x_{t+2} - x_{t+1}\|^2 \leq \|x_{t+1} - x_t\|^2.
$$

Using the update formula in Eq. (1), we have $\|v(x_{t+1})\| \leq \|v(x_t)\|$ for all $t \geq 0$. Then we proceed to bound $\sum_{t=0}^{+\infty} \|v(x_t)\|^2$. Indeed, for any $x_i \in \mathcal{X}_i$, we have

$$(x_i - x_{i,t+1})^\top (x_{i,t+1} - x_{i,t} - \eta v_i(x_i)) = 0.$$

Applying the equality $a^\top b = (\|a + b\|^2 - \|a\|^2 - \|b\|^2)/2$ yields that

$$(x_{i,t+1} - x_i)^\top v_i(x_i) = \frac{1}{2\eta} \left( \|x_{i,t} - x_{i,t+1}\|^2 + \|x_i - x_{i,t+1}\|^2 - \|x_i - x_{i,t}\|^2 \right).$$

Summing up the resulting inequality over $i \in \mathcal{N}$ yields that

$$(x_{t+1} - x_t)^\top v(x_t) = \frac{1}{2\eta} \left( \|x_t - x_{t+1}\|^2 + \|x - x_{t+1}\|^2 - \|x - x_t\|^2 \right), \quad \forall x \in \mathcal{X}.$$

Letting $x = x^* \in \mathcal{X}^*$, we have

$$(x_{t+1} - x^*)^\top v(x_t) = \frac{1}{2\eta} \left( \|x_t - x_{t+1}\|^2 + \|x^* - x_{t+1}\|^2 - \|x^* - x_t\|^2 \right).$$

(17)

Furthermore, we have

$$(x_{t+1} - x^*)^\top v(x_t) = (x_t - x^*)^\top v(x_t) + (x_{t+1} - x_t)^\top v(x_t).$$

Since $G$ is a $\lambda$-cocoercive game and $v(x^*) = 0$, we have

$$(x_t - x^*)^\top v(x_t) = (x_t - x^*)^\top (v(x_t) - v(x^*)) \leq -\lambda\|v(x_t) - v(x^*)\|^2 = -\lambda\|v(x_t)\|^2.$$

By Young’s inequality we have

$$(x_{t+1} - x_t)^\top v(x_t) \leq \frac{\lambda\|v(x_t)\|^2}{2} + \frac{\|x_{t+1} - x_t\|^2}{2\lambda}.$$

Putting these pieces together yields that

$$(x_{t+1} - x^*)^\top v(x_t) \leq \frac{\|x_{t+1} - x_t\|^2}{2\lambda} - \frac{\lambda\|v(x_t)\|^2}{2}.$$

(18)
Plugging Eq. (18) into Eq. (17) together with the condition \( \eta \in (0, \lambda] \) yields that
\[
\lambda \| \nu(x_t) \|^2 \leq \frac{\| x^* - x_t \|^2 - \| x^* - x_{t+1} \|^2}{\eta}.
\]
Summing up the above inequality over \( t = 0, 1, 2, \ldots \) and using the boundedness of \( X \) yields that
\[
\sum_{t=0}^{+\infty} \| \nu(x_t) \|^2 \leq \frac{\| x^* - x_0 \|^2}{\eta \lambda}.
\]
Note that \( x^* \in X^* \) is chosen arbitrarily, we let \( x^* = \Pi_{X^*}(x_0) \) and conclude the desired inequality.

C. Postponed Proofs in Section 4

In this section, we present the missing proofs in Section 4.

C.1. Proof of Lemma 4.1

Using the update formula of \( x_{i,t+1} \) in Eq. (1), we have the following for any \( x_i^* \in X_i^* \):
\[
\| x_{i,t+1} - x_i^* \|^2 = \| x_{i,t} + \eta t \vartheta_i,t+1 - x_i^* \|^2.
\]
which implies that
\[
\| x_{i,t+1} - x_i^* \|^2 = \| x_{i,t} - x_i^* \|^2 + \eta^2 \| \vartheta_i,t+1 \|^2 + 2\eta \lambda (x_{i,t} - x_i^*)^\top \vartheta_i,t+1.
\]
Summing up the above inequality over \( i \in N \) and rearranging yields that
\[
\| x_{t+1} - x^* \|^2 = \| x_t - x^* \|^2 + 2\eta \lambda (x_t - x^*)^\top \vartheta_t + \eta^2 \| \vartheta_t \|^2.
\] (19)

Using Young’s inequality, we have
\[
\| x_{t+1} - x^* \|^2 \leq \| x_t - x^* \|^2 + 2\eta \lambda \| \vartheta_t \|^2 + 2\eta^2 \| \vartheta_t \|^2 + 2\eta \lambda (x_t - x^*)^\top (\nu(x_t) + \xi_t + 1).
\]
Since \( x^* \in X^* \) and \( G \) is a \( \lambda \)-cocoercive game, we have \( \nu(x^*) = 0 \) and
\[
(x_t - x^*)^\top \nu(x_t) = (x_t - x^*)^\top (\nu(x_t) - \nu(x^*)) \leq -\lambda \| \nu(x_t) - \nu(x^*) \|^2 = -\lambda \| \nu(x_t) \|^2.
\]
Putting these pieces yields the desired inequality.

C.2. Proof of Lemma 4.2

Using the same argument as in Lemma 4.1, we have
\[
\mathbb{E}[\epsilon(x_{t+1}) | F_t] - \epsilon(x_t) \leq \frac{\mathbb{E}[\| x_{t+1} \|^2 | F_t]}{\lambda \eta_{t+1}} + \left( 1 - \frac{\lambda}{\eta_{t+1}} \right) \mathbb{E}[\| \nu(x_{t+1}) - \nu(x_t) \|^2 | F_t].
\]
Since the noisy model (2) is with relative random noise (4), we have \( \mathbb{E}[\| \xi_{t+1} \|^2 | F_t] \leq \tau_{t+1} \| \nu(x_t) \|^2 \). Also, \( \eta_t \in (0, \lambda) \) for all \( t \geq 1 \). Therefore, we conclude that
\[
\mathbb{E}[\epsilon(x_{t+1}) | F_t] - \epsilon(x_t) \leq \frac{\tau_{t+1} \| \nu(x_t) \|^2}{\lambda \eta_{t+1}}.
\]
Taking an expectation of both sides yields the desired inequality.
C.3. Proof of Theorem 4.4

Using the same argument as in Theorem 4.3, we obtain that
\[
\mathbb{E}[\|x_{t+1} - x^*\|^2 \mid F_t] \leq \|x_t - x^*\|^2 - 2(\lambda - \eta - \tau \eta)\eta_{t+1}\|v(x_t)\|^2.
\]  
(20)

Taking an expectation of both sides of Eq. (20) and rearranging yields that
\[
\mathbb{E}[\epsilon(x_t)] \leq \frac{1}{2(\lambda - \eta - \tau \eta)\eta_{t+1}} (\mathbb{E}[\|x_t - x^*\|^2] - \mathbb{E}[\|x_{t+1} - x^*\|^2]).
\]  
(21)

Summing up the above inequality over \( t = 0, 1, \ldots, T \) yields that
\[
\mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \leq \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \frac{\mathbb{E}[\|x_t - x^*\|^2]}{2(\lambda - \eta - \tau \eta)} + \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta - \tau \eta)\eta_{t+1}}.
\]

On the other hand, we have \( \mathbb{E}[\|x_{t+1} - x^*\|^2] \leq \mathbb{E}[\|x_t - x^*\|^2] \). This implies that \( \mathbb{E}[\|x_t - x^*\|^2] \leq \|x_0 - x^*\|^2 \) for all \( t \geq 1 \). Therefore, we conclude that
\[
\mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \leq \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta - \tau \eta)\eta_{T+1}} = O(1).
\]

This completes the proof.

C.4. Proof of Theorem 4.7

Since the step-size sequence \( \{\eta_t\}_{t \geq 1} \) is decreasing and converges to zero, we define the first iconic time in our analysis as follows,
\[
t^* = \max \left\{ t \geq 0 \mid \eta_{t+1} > \frac{\lambda}{2(1 + \tau)} \right\} < +\infty.
\]

First, we claim that \( \mathbb{E}[\|x_t - \Pi_{\chi'}(x_t)\|] \leq D \) where \( D = \max_{1 \leq t \leq t^*} \mathbb{E}[\|x_t - \Pi_{\chi'}(x_t)\|] \). Indeed, it suffices to show that \( \mathbb{E}[\|x_t - \Pi_{\chi'}(x_t)\|] \leq \mathbb{E}[\|x_t - \Pi_{\chi'}(x_{t^*})\|] \) holds for all \( t > t^* \). The desired inequality follows from Eq. (21) and the fact that \( \mathbb{E}[\epsilon(x_t)] \geq 0 \) for all \( t > t^* \).

Furthermore, we derive an upper bound for the term \( \sum_{t=0}^{T} \|v(x_t)\|^2 \). Using the update formula (cf. Eq. (1)) to obtain that
\[
(x_{t+1} - x^*)^T(v(x_t) + \xi_{t+1}) = \frac{1}{2\eta_{t+1}} \left( \|x_t - x_{t+1}\|^2 + \|x^* - x_{t+1}\|^2 - \|x^* - x_t\|^2 \right).
\]

Recall that \( \mathcal{G} \) is \( \lambda \)-cocoercive and the noisy model is defined with relative random noise, we have
\[
\mathbb{E}[(x_t - x^*)^T(v(x_t) + \xi_{t+1}) \mid F_t] \geq \lambda \|v(x_t)\|^2.
\]

Using Young’s inequality, we have
\[
\mathbb{E}[(x_{t+1} - x_t)^T(v(x_t) + \xi_{t+1}) \mid F_t] \geq -\lambda \frac{\|v(x_t)\|^2}{2} - \frac{(1 + \tau)\mathbb{E}[\|x_{t+1} - x_t\|^2 \mid F_t]}{\lambda}.
\]

Putting these pieces together and taking an expectation yields that
\[
\lambda \mathbb{E}[\|v(x_t)\|^2] \leq \mathbb{E} \left[ \frac{\|x^*-x_t\|^2 - \|x^*-x_{t+1}\|^2}{\eta_{t+1}} \right] + \mathbb{E} \left[ \left( \frac{2(1 + \tau)}{\lambda} - \frac{1}{\eta_{t+1}} \right) \|x_t - x_{t+1}\|^2 \right].
\]  
(22)
Recall that the step-size sequence \( \{\eta_t\}_{t \geq 1} \) is nonincreasing and \( \|x_t - \Pi_{\mathcal{X}}(x_{t^*})\| \leq D \), we let \( x^* = \Pi_{\mathcal{X}}(x_{t^*}) \) in Eq. (22) and obtain that
\[
\sum_{t=0}^{T} \lambda \mathbb{E}[\|v(x_t)\|^2] \leq \mathbb{E}\left[\frac{D^2}{\eta_{t+1}} + \sum_{t=0}^{T} \mathbb{E}\left[\left(\frac{2(1+\tau)}{\lambda} - \frac{1}{\eta_{t+1}}\right)\|x_t - x_{t+1}\|^2\right]\right].
\]

To proceed, we define the second iconic time as
\[
t^*_1 = \max\left\{t \geq 0 | \eta_{t+1} > \frac{\lambda}{4(1+\tau)D^2 + 2(1+\tau)}\right\} > t^*.
\]

It is clear that \( t^*_1 < +\infty \) and \( \eta_{t+1} \leq \lambda/(2+2\tau) \) for all \( t > t^*_1 \) which implies that \((2+2\tau)/\lambda - 1/\eta_{t+1} \leq 0\). Assume \( T \) sufficiently large without loss of generality, we have
\[
\sum_{t=0}^{T} \lambda \mathbb{E}[\|v(x_t)\|^2] \leq \mathbb{E}\left[\frac{D^2}{\eta_{t+1}} + \frac{2(1+\tau)}{\lambda} \left(\sum_{t=0}^{t^*_1} \mathbb{E}[\|x_t - x_{t+1}\|^2]\right)\right] = I + II.
\]

We also use Lemmas A.1 and A.2 from Bach & Levy (2019) to bound terms I and II. For convenience, we present these two lemmas here:

**Lemma C.1** For a sequence of numbers \( a_0, a_1, \ldots, a_n \in [0, a] \) and \( b \geq 0 \), the following inequality holds:
\[
\sqrt{b + \sum_{i=0}^{n-1} a_i} - \sqrt{b} \leq \frac{n}{\sqrt{b + \sum_{j=0}^{n-1} a_j}} \leq \frac{2a}{\sqrt{b}} + 3\sqrt{a} + 3\sqrt{b + \sum_{i=0}^{n-1} a_i}.
\]

**Bounding I:** We derive from the definition of \( \eta_t \) and Jensen’s inequality that
\[
I \leq D^2 \sqrt{\beta + \log(T+1)} + \sum_{j=0}^{T-1} \mathbb{E}\left[\frac{\|x_j - x_{j+1}\|^2}{\eta^2_{j+1}}\right].
\]

Since \( \mathbb{E}[\|x_t - \Pi_{\mathcal{X}}(x_{t^*})\|] \leq D \) for all \( t \geq 0 \) and the notion of \( \lambda \)-cocoercivity implies the notion of \((1/\lambda)\)-Lipschitz continuity, we have
\[
\mathbb{E}\left[\frac{\|x_t - x_{t+1}\|^2}{\eta^2_{t+1}}\right] \leq (2+2\tau)\mathbb{E}[\|v(x_t)\|^2] \leq \frac{(2+2\tau)\|x_t - \Pi_{\mathcal{X}}(x_{t^*})\|^2}{\lambda^2} \leq \frac{(2+2\tau)D^2}{\lambda^2}.
\]

Using the first inequality in Lemma C.1, we have
\[
I \leq D^2 \sqrt{\beta + \log(T+1)} + \sum_{t=0}^{T} \frac{D^2\mathbb{E}[\|x_t - x_{t+1}\|^2/\eta^2_{t+1}]}{\sqrt{\beta + \log(T+1)} + \sum_{j=0}^{t-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta^2_{j+1}]} \leq D^2 \sqrt{\beta + \log(T+1)} + \sum_{t=0}^{t^*_1} \frac{D^2\mathbb{E}[\|x_t - x_{t+1}\|^2/\eta^2_{t+1}]}{\sqrt{\beta + \log(T+1)} + \sum_{j=0}^{t-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta^2_{j+1}]} + \sum_{t=t^*_1+1}^{T} (2+2\tau)D^2\eta_{t+1}\mathbb{E}[\|v(x_t)\|^2].
\]
Since $\eta_{t+1} \leq \lambda/[4(1+\tau)D^2]$ for all $t > t_1^*$, we have

$$\sum_{t=t_1^*+1}^{T} (2+2\tau)D^2\eta_{t+1}\mathbb{E}[\|v(x_t)\|^2] \leq \sum_{t=t_1^*+1}^{T} \frac{\lambda\mathbb{E}[\|v(x_t)\|^2]}{2}. \quad (24)$$

Using the second inequality in Lemma 3.5, we have

$$\sum_{t=0}^{t_1^*} \frac{D^2\mathbb{E}[\|x_t - x_{t+1}\|^2/\eta_{t+1}^2]}{\sqrt{\beta + \log(T + 1)} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2]} \leq \frac{(4 + 4\tau)D^2}{\lambda^2 \sqrt{\beta + \log(T + 1)} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2]} + 3D\sqrt{2 + 2\tau}\frac{\beta + \log(T + 1)}{\lambda} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2]. \quad (25)$$

By the definition of $\eta_t$, we have

$$\sqrt{\beta + \log(T + 1)} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2] \leq \frac{1}{\eta_{t+1}^*} + \sqrt{\log(T + 1)} < \frac{4(1+\tau)D^2 + 2(1+\tau)}{\lambda} + \sqrt{\log(T + 1)}. \quad (26)$$

Putting Eq. (24)-(26) together yields that

$$I \leq D^2\sqrt{\beta + \log(T + 1)} + \frac{(4 + 4\tau)D^2}{\lambda^2 \sqrt{\beta + \log(T + 1)} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2]} + 3D\sqrt{2 + 2\tau}\frac{\beta + \log(T + 1)}{\lambda} + \frac{12(1+\tau)D^2 + 6(1+\tau)}{\lambda} + \sqrt{\log(T + 1)} + \sum_{t=t_1^*+1}^{T} \frac{\lambda\mathbb{E}[\|v(x_t)\|^2]}{2}.$$

**Bounding II:** Recalling that

$$\mathbb{E}\left[\|x_t - x_{t+1}\|^2/\eta_{t+1}^2\right] \leq \frac{(2+2\tau)D^2}{\lambda^2},$$

and $\eta_t \leq 1/\beta$ for all $t \geq 1$, we have

$$\mathbb{E}\left[\|x_t - x_{t+1}\|^2\right] \leq \frac{(2+2\tau)D^2}{\lambda^2 \beta^2},$$

Putting these pieces together yields that

$$\text{II} = \frac{2(1+\tau)}{\lambda} \left(\sum_{t=0}^{t_1^*} \mathbb{E}[\|x_t - x_{t+1}\|^2]\right) \leq \frac{4(1+\tau)^2D^2 t_1^*}{\lambda^3 \beta^2}.$$

Therefore, we have

$$\sum_{t=0}^{T} \frac{\lambda\mathbb{E}[\|v(x_t)\|^2]}{2} \leq D^2\sqrt{\beta + \log(T + 1)} + \sqrt{\log(T + 1)} + \frac{(4 + 4\tau)D^2}{\lambda^2 \sqrt{\beta + \log(T + 1)} + \sum_{j=0}^{t_1^*-1} \mathbb{E}[\|x_j - x_{j+1}\|^2/\eta_{j+1}^2]} + 3D\sqrt{2 + 2\tau}\frac{\beta + \log(T + 1)}{\lambda} + \frac{12(1+\tau)D^2 + 6(1+\tau)}{\lambda} + \frac{4(1+\tau)^2D^2 t_1^*}{\lambda^3 \beta^2}.$$

By the definition, we have $t_1^* < +\infty$ is uniformly bounded. To this end, we conclude that $\sum_{t=0}^{T} \mathbb{E}[\|v(x_t)\|^2] \leq C_1 + C_2 \sqrt{\log(T + 1)}$, where $C_1 > 0$ and $C_2 > 0$ are universal constants.
Finally, we proceed to bound the term \( \epsilon(x_T) \). Without loss of generality, we can start the sequence at a later index \( t^*_1 \) since \( t^*_1 < \infty \). This implies that \( \eta_{t+1} \leq \lambda/(1+\tau) \). Using the last equation in the proof of Lemma 4.2, we have

\[
\mathbb{E}[\epsilon(x_T)] \leq \mathbb{E}[\epsilon(x_t)] + \frac{T-1}{\lambda t} \sum_{j=t}^{T-1} \tau_j \mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right].
\]

Summing up the above inequality over \( t = t^*_1, \ldots, T+ \) yields

\[
(T - t^*_1 + 1)\mathbb{E}[\epsilon(x_T)] \leq \sum_{t=t^*_1}^{T} \mathbb{E}[\epsilon(x_t)] + \frac{1}{\lambda} \sum_{t=t^*_1}^{T-1} \sum_{j=t}^{T-1} \tau_j \mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right].
\]

Since \( \{\tau_t\}_{t \geq 0} \) is an nonincreasing sequence, we have

\[
\sum_{t=t^*_1}^{T-1} \sum_{j=t}^{T-1} \tau_j \mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right] \leq \left( \sum_{t=0}^{T-1} \tau_t \right) \left( \sum_{t=t^*_1}^{T-1} \mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right] \right).
\]

Using Eq. (20) and \( \eta_{t+1} \leq \lambda/(1+\tau) \) for all \( t > t^*_1 \), we have

\[
\mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right] \leq \mathbb{E} \left[ \frac{\|x_j - x^*\|^2 - \|x_j - x^* - x_t\|^2}{\eta_{j+1}^2} \right].
\]

Note that \( \{\eta_t\}_{t \geq 0} \) is an nonnegative and nonincreasing sequence and \( \mathbb{E}[\|x_{j+1} - x^*\|^2] \leq D^2 \). Putting these pieces together yields that

\[
\sum_{t=t^*_1}^{T-1} \mathbb{E} \left[ \frac{\|v(x_j)\|^2}{\eta_{j+1}} \right] \leq \mathbb{E} \left[ \frac{D^2}{\eta_T^2} \right] \leq D^2 \left( \beta + \log(T) + \sum_{t=0}^{T} \mathbb{E} \left[ \frac{\|x_t - x_{t+1}\|^2}{\eta_{t+1}^2} \right] \right)
\]

\[
\leq D^2 \left( \beta + \log(T) + 2(1+\tau) \sum_{t=0}^{T} \mathbb{E} \left[ \|v(x_t)\|^2 \right] \right)
\]

\[
= O(\log(T)).
\]

Therefore, we conclude that

\[
\mathbb{E}[\epsilon(x_T)] \leq \frac{\sum_{t=t^*_1}^{T} \mathbb{E}[\epsilon(x_t)]}{T-t^*_1+1} + C \log(T+1) \frac{\sum_{t=0}^{T-1} \tau_t}{\lambda(T-t^*_1+1)} \text{ for some } C > 0.
\]

This completes the proof.

**D. Postponed Proofs in Section A**

In this section, we present the missing proofs in Section A.

**D.1. Proof of Lemma A.1**

By the definition of \( \epsilon(x) \), we have

\[
\epsilon(x_{t+1}) - \epsilon(x_t) = \|v(x_{t+1})\|^2 - \|v(x_t)\|^2 = (v(x_{t+1}) - v(x_t))^T (v(x_{t+1}) + v(x_t))
\]

\[
= 2(v(x_{t+1}) - v(x_t))^T v(x_t) + \|v(x_{t+1}) - v(x_t)\|^2.
\]

Using the update formula in Eq. (1), it holds that \( v(x_t) = \eta_{t+1}^{-1}(x_{t+1} - x_t) - \xi_{t+1} \). Therefore, we have

\[
\epsilon(x_{t+1}) - \epsilon(x_t) = \frac{2}{\eta_{t+1}} ((v(x_{t+1}) - v(x_t))^T (x_{t+1} - x_t) - (v(x_{t+1}) - v(x_t))^T \xi_{t+1}) + \|v(x_{t+1}) - v(x_t)\|^2.
\]
Since $G$ is a $\lambda$-cocoercive game, we have
\[ (v(x_{t+1}) - v(x_t))^T (x_{t+1} - x_t) \leq -\lambda \|v(x_{t+1}) - v(x_t)\|^2. \]

Using Young’s inequality, we have
\[ -(v(x_{t+1}) - v(x_t))^T \xi_{t+1} \leq \frac{\lambda \|v(x_{t+1}) - v(x_t)\|^2}{2} + \|\xi_{t+1}\|^2. \]

Putting these pieces together yields that
\[ \epsilon(x_{t+1}) - \epsilon(x_t) \leq \frac{\|\xi_{t+1}\|^2}{\lambda \eta_{t+1}} + \left(1 - \frac{\lambda}{\eta_{t+1}}\right) \|v(x_{t+1}) - v(x_t)\|^2. \] (27)

Taking an expectation of Eq. (27) conditioned on $F_t$ yields that
\[ \mathbb{E}[\epsilon(x_{t+1}) | F_t] - \epsilon(x_t) \leq \frac{\mathbb{E}[\|\xi_{t+1}\|^2 | F_t]}{\lambda \eta_{t+1}} + \left(1 - \frac{\lambda}{\eta_{t+1}}\right) \mathbb{E}[\|v(x_{t+1}) - v(x_t)\|^2 | F_t]. \]

Since the noisy model (2) is with absolute random noise (4), we have $\mathbb{E}[\|\xi_{t+1}\|^2 | F_t] \leq \sigma_t^2$. Also, $\eta_t \in (0, \lambda)$ for all $t \geq 1$. Therefore, we conclude that
\[ \mathbb{E}[\epsilon(x_{t+1}) | F_t] - \epsilon(x_t) \leq \frac{\sigma_t^2}{\lambda \eta_{t+1}}. \]

Taking the expectation of both sides yields the desired inequality.

D.2. Proof of Theorem A.2

Recalling Eq. (11) (cf. Lemma 4.1), we take the expectation of both sides conditioned on $F_t$ to obtain
\[ \mathbb{E}[\|x_{t+1} - x^*\|^2 | F_t] \leq \|x_t - x^*\|^2 - (2\lambda - 2\eta_{t+1})\eta_{t+1}\|v(x_t)\|^2 + 2\eta_{t+1}^2 \mathbb{E}[\|\xi_{t+1}\|^2 | F_t] + 2\eta_{t+1} \mathbb{E}[(x_t - x^*)^T \xi_{t+1} | F_t]. \]

Since the noisy model (2) is with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2)$ for some $\sigma^2 < +\infty$, we have $\mathbb{E}[(x_t - x^*)^T \xi_{t+1} | F_t] = 0$ and $\mathbb{E}[\|\xi_{t+1}\|^2 | F_t] \leq \sigma^2$. Therefore, we have
\[ \mathbb{E}[\|x_{t+1} - x^*\|^2 | F_t] \leq \|x_t - x^*\|^2 - (2\lambda - 2\eta_{t+1})\eta_{t+1}\|v(x_t)\|^2 + 2\eta_{t+1}^2 \sigma^2. \]

Since $\sum_{t=1}^{\infty} \eta_t^2 < \infty$, we have $\eta_t \to 0$ as $t \to +\infty$. Without loss of generality, we assume $\eta_t \leq \lambda$ for all $t$. Then we have
\[ \mathbb{E}[\|x_{t+1} - x^*\|^2 | F_t] \leq \|x_t - x^*\|^2 + 2\eta_{t+1}^2 \sigma^2. \] (28)

We let $M_t = \|x_t - x^*\|^2 + 2\sigma^2(\sum_{j=t+1}^{\infty} \eta_j^2)$ and obtain that $M_t$ is an nonnegative supermartingale. Then Doob’s martingale convergence theorem shows that $M_n$ converges to an nonnegative and integrable random variable almost surely. Let $M_\infty = \lim_{t \to +\infty} M_t$, it suffices to show that $M_\infty = 0$ almost surely.

We first claim that every neighborhood $U$ of $\mathcal{X}^*$ is recurrent: there exists a subsequence $x_{t_k}$ of $x_t$ such that $x_{t_k} \to \mathcal{X}^*$ almost surely. Equivalently, there exists a Nash equilibria $x^* \in \mathcal{X}^*$ such that $\|x_{t_k} - x^*\|^2 \to 0$ almost surely. To this end, we can define $M_t$ with such Nash equilibria. Since $\sum_{t=1}^{\infty} \eta_t^2 < \infty$, we have $\sum_{j=t+1}^{\infty} \eta_j^2 \to 0$ as $t \to +\infty$ and the following statement holds almost surely:
\[ \lim_{k \to +\infty} M_{t_k} = \lim_{k \to +\infty} \|x_{t_k} - x^*\|^2 = 0. \]

Since the whole sequence converges to $M_\infty$ almost surely, we conclude that $M_\infty = 0$ almost surely.

Proof of the recurrence claim: Let $U$ be a neighborhood of $\mathcal{X}^*$ and assume to the contrary that, $x_t \notin U$ for sufficiently large $t$ with positive probability. By starting the sequence at a later index if necessary and noting that $\sum_{t=1}^{\infty} \eta_t^2 < \infty$, we may assume that $x_t \notin U$ and $\eta_t \leq \lambda/2$ for all $t$ without loss of generality. Thus, there exists some $c > 0$ such that $\|v(x_t)\|^2 \geq c$ for all $t$. As a result, for all $x^* \in \mathcal{X}^*$, we let $\psi_{t+1} = (x_t - x^*)^T \xi_{t+1}$ and have
\[ \|x_{t+1} - x^*\|^2 \leq \|x_t - x^*\|^2 - \lambda c \eta_{t+1} + 2\eta_{t+1} \psi_{t+1} + 2\eta_{t+1}^2 \|\xi_{t+1}\|^2. \]
Summing up the above inequality over $t = 0, 1, \ldots, T$ together with $\theta_t = \sum_{j=1}^{t} \eta_j$ yields that

$$
\|x_{t+1} - x^*\|^2 \leq \|x_0 - x^*\|^2 - \lambda c \theta_{t+1} + 2 \theta_{t+1} \left[ \sum_{j=1}^{t+1} \eta_j \psi_j + \sum_{j=1}^{T+1} \eta_j^2 \xi_j^2 \right].
$$

(29)

Since the noisy model (2) is with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2]$ for some $\sigma^2 < +\infty$, we have $E[\psi_{t+1} | F_t] = 0$. Furthermore, we obtain by taking the expectation of both sides of Eq. (28) that

$$
E[\|x_{t+1} - x^*\|^2] \leq E[\|x_t - x^*\|^2] + 2 \eta_{t+1}^2 \sigma^2,
$$

and the following inequality holds true for all $t \geq 1$:

$$
E[\|x_t - x^*\|^2] \leq \|x_0 - x^*\|^2 + 2 \sigma^2 \sum_{j=1}^{\infty} \eta_j^2 < +\infty.
$$

Since $\|x_t - x^*\|^2 \geq 0$, we have $\|x_t - x^*\|^2 < +\infty$ almost surely. Therefore, $E[\|x_{t+1}\|^2 | F_t] \leq \sigma^2 \|x_t - x^*\|^2 < +\infty$. Then the law of large numbers for martingale differences yields that $\bar{\eta}_{T+1} (\sum_{t=1}^{T+1} \eta_t \psi_t) \rightarrow 0$ almost surely (Hall & Heyde, 2014, Theorem 2.18). Furthermore, let $R_t = \sum_{j=1}^{t} \eta_j^2 \xi_j^2$, then $R_t$ is a submartingale and

$$
E[R_t] \leq \sigma^2 \sum_{j=1}^{t} \eta_j^2 < \sigma^2 \sum_{j=1}^{\infty} \eta_j^2 < +\infty.
$$

From Doob’s martingale convergence theorem, $R_t$ converges to some random, finite value almost surely (Hall & Heyde, 2014, Theorem 2.5). Putting these pieces together with Eq. (29) yields that $\|x_t - x^*\|^2 \sim -\lambda c \tau_t \rightarrow -\infty$ almost surely, a contradiction. Therefore, we conclude that every neighborhood of $x^*$ is recurrent.

### D.3. Proof of Theorem A.3

Since $\eta_t = c/\sqrt{t}$ for all $t \geq 1$, we have $\eta_t \rightarrow 0$ and $\eta_t \leq c$ for all $t \geq 1$. This implies that

$$
\lambda \eta_{t+1} - \eta_t^2 \geq (\lambda - c) \eta_{t+1}.
$$

(30)

Plugging Eq. (30) into Eq. (11) (cf. Lemma 4.1) yields that

$$
\|x_{t+1} - x^*\|^2 \leq \|x_t - x^*\|^2 - 2(\lambda - c) \eta_{t+1} \|v(x_t)\|^2 + 2 \eta_{t+1} (x_t - x^*)^T \xi_{t+1} + 2 \eta_{t+1}^2 \xi_{t+1}^2.
$$

Using the same argument as in Theorem A.2, we have

$$
E[\|x_{t+1} - x^*\|^2 | F_t] \leq \|x_t - x^*\|^2 - 2(\lambda - c) \eta_{t+1} \|v(x_t)\|^2 + 2 \eta_{t+1}^2 \sigma^2.
$$

(31)

Taking the expectation of both sides of Eq. (31) and rearranging yields that

$$
E[\varepsilon(x_t)] \leq \frac{1}{2(\lambda - c) \eta_{t+1}} \left( E[\|x_t - x^*\|^2] - E[\|x_{t+1} - x^*\|^2] \right) + \frac{\eta_{t+1} \sigma^2}{\lambda - c}.
$$

Summing up the above inequality over $t = 0, 1, \ldots, T$ and using $\eta_t = c/\sqrt{T+1}$ yields that

$$
E \sum_{t=0}^{T} \varepsilon(x_t) \leq \frac{1}{2(\lambda - c) \eta_t} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) E[\|x_t - x^*\|^2] + \frac{\sigma^2}{\lambda - c} \left( \sum_{t=1}^{T+1} \eta_t \right).
$$

On the other hand, we have

$$
E[\|x_{t+1} - x^*\|^2] \leq E[\|x_t - x^*\|^2] + 2 \eta_{t+1}^2 \sigma^2.
$$

This implies that the following inequality holds for all $t \geq 1$:

$$
E[\|x_t - x^*\|^2] \leq \|x_0 - x^*\|^2 + 2 \sigma^2 \left( \sum_{j=1}^{t} \eta_j^2 \right) \leq \|x_0 - x^*\|^2 + 2 \sigma^2 \sigma^2 \log(t + 1).
$$
Therefore, we conclude that
\[
\mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \leq \frac{\|x_0 - x^*\|^2 + 2\sigma^2 e^2 \log(T + 1)}{2(\lambda - c)} \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \leq \frac{\|x_0 - x^*\|^2}{2(\lambda - c)\eta_1} + \frac{\sigma^2}{\lambda - c} \sum_{t=1}^{T+1} \eta_t \\
\leq \sqrt{T + 1} \left(\frac{\|x_0 - x^*\|^2}{2c(\lambda - c)} + \frac{\sigma^2 c \sqrt{T + 1}}{\lambda - c}\right) \\
= O \left(\sqrt{T + 1} \log(T + 1)\right).
\]
This completes the proof.

D.4. Proof of Theorem A.4

Using Lemma A.1, we have
\[
\mathbb{E}[\epsilon(x_T)] \leq \mathbb{E}[\epsilon(x_t)] + \frac{1}{\lambda} \left( \sum_{j=t}^{T-1} \frac{\sigma_j^2}{\eta_{j+1}} \right) \leq \mathbb{E}[\epsilon(x_t)] + \frac{1}{\lambda \eta} \left( \sum_{j=t}^{T-1} \sigma_j^2 \right).
\]
Summing up the above inequality over \( t = 0, 1, \ldots, T \) yields that
\[
(T + 1)\mathbb{E}[\epsilon(x_T)] \leq \sum_{t=0}^{T} \mathbb{E}[\epsilon(x_t)] + \frac{1}{\lambda} \left( \sum_{t=0}^{T-1} \sum_{j=t}^{T-1} \frac{\sigma_j^2}{\eta_{j+1}} \right) \leq \sum_{t=0}^{T} \mathbb{E}[\epsilon(x_t)] + \frac{1}{\lambda \eta} \left( \sum_{t=0}^{T-1} (t + 1) \sigma_t^2 \right).
\]
On the other hand, the derivation in Theorem A.3 implies that
\[
\mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \leq \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta)\eta_1} + \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \mathbb{E}[\|x_t - x^*\|^2] \leq \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta)\eta_1} + \frac{1}{\lambda - \eta} \left( \sum_{t=1}^{T+1} \eta_t \sigma_t^2 \right).
\]
On the other hand, we have
\[
\mathbb{E}[\|x_{t+1} - x^*\|^2] \leq \mathbb{E}[\|x_t - x^*\|^2] + 2\eta_{t+1}^2 \sigma_t^2.
\]
This implies that the following inequality holds for all \( t \geq 1 \):
\[
\mathbb{E}[\|x_t - x^*\|^2] \leq \|x_0 - x^*\|^2 + 2\eta_t^2 \left( \sum_{j=1}^{t} \sigma_j^2 \right).
\]
Therefore, we conclude that
\[
\mathbb{E} \left[ \sum_{t=0}^{T} \epsilon(x_t) \right] \leq \frac{\|x_0 - x^*\|^2 + 2\eta_t^2 (\sum_{j=1}^{t} \sigma_j^2)}{2(\lambda - \eta)} \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) + \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta)\eta_1} + \frac{\eta}{\lambda - \eta} \left( \sum_{t=1}^{T+1} \sigma_t^2 \right) \\
\leq \frac{\|x_0 - x^*\|^2}{2(\lambda - \eta)\eta_1} + \frac{\eta}{1 + \frac{\eta}{\lambda - \eta}} \left( \sum_{t=1}^{T+1} \sigma_t^2 \right) \\
\leq \frac{\|x_0 - x^*\|^2}{(\lambda - \eta)\eta} + \frac{\eta}{1 + \frac{\eta}{\lambda - \eta}} \left( \sum_{t=1}^{T+1} (t + 1) \sigma_t^2 \right).
\]
Putting these pieces together yields that

\[
E[\epsilon(x_T)] \leq \frac{1}{T+1} \left[ \sum_{t=0}^{T} E[\epsilon(x_t)] + \frac{1}{\lambda \eta} \left( \sum_{t=0}^{T-1} (t+1)\sigma^2_t \right) \right]
\]

\[
\leq \frac{1}{T+1} \left( \frac{\|x_0 - x^*\|^2}{(\lambda - \eta)\eta} + \left( \frac{1}{\lambda \eta} + \left( 1 + \frac{\eta}{\lambda} \frac{\eta}{\lambda - \eta} \right) \left( \sum_{t=1}^{T+1} (t+1)\sigma^2_t \right) \right) \right)
\]

Eq. (15) \(= O(a(T)) \).

This completes the proof.