7. Supplementary material

7.1. Sub-regions corresponding to system (9)

All sub-regions related to (9) can be defined as follows (Monfared & Durstewitz, 2020):

\[
S_{\Omega 1}^1 = \hat{S}_0 = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\
\end{bmatrix}_M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\
\end{bmatrix}_M
\]

\[
S_{\Omega 1}^2 = \hat{S}_1 = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
\end{bmatrix}_M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
\end{bmatrix}_M
\]

\[
S_{\Omega 1}^3 = \hat{S}_2 = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\
\end{bmatrix}_M = \begin{bmatrix} 0 & 0 & \cdots & 1 \\
\end{bmatrix}_M
\]

\[
S_{\Omega 1}^4 = \hat{S}_3 = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\
\end{bmatrix}_M = \begin{bmatrix} 0 & 1 & \cdots & 0 \\
\end{bmatrix}_M
\]

\[
S_{\Omega 1}^5 = \hat{S}_4 = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\
\end{bmatrix}_M = \begin{bmatrix} 0 & 1 & \cdots & 1 \\
\end{bmatrix}_M
\]

\[
S_{\Omega 1}^{2M-1} = \hat{S}_M = \hat{\bar{S}}_{\Omega 0} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\
\end{bmatrix}_M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\
\end{bmatrix}_M
\]

where each subindex \(d\) of \(\hat{S}_0, 0 \leq d \leq 2^M - 1\), is associated with a sequence \(d_M \cdots d_1\) of binary digits. The notation \((d_M \cdots d_1)\) in building each corresponding sequence stands for the mirror image of the binary representation of \(d\) with \(M\) digits. By mirror image here we mean writing digits \(d_1 \cdots d_M\) from right to left, i.e. \(d_M \cdots d_1\). For example, for \(M = 2\) there are 4 sub-regions \(S_{\Omega k}, k = 1, 2, 3, 4\), associated with 4 matrices \(D_{\Omega k} := \text{diag}(d_2, d_1)\), where \(d_2 d_1 = (d_1 d_2)\) and \(d_i \in \{0, 1\}\) (Fig. S1).

Denoting switching boundaries \(\Sigma_{ij} = \hat{\bar{S}}_{\Omega i} \cap \hat{\bar{S}}_{\Omega j}\) between every pair of successive sub-regions \(S_{\Omega j}\) and \(S_{\Omega i}\) with \(i, j \in \{1, 2, \cdots, 2^M\}\), we can rewrite map (7) as:

\[
Z_{t+1} = F(Z_t)
\]

\[
= \begin{cases} 
F_1(Z_t) = W_{12} Z_t + h; & Z_t \in \hat{\bar{S}}_{\Omega 1}^1 \\
F_2(Z_t) = W_{13} Z_t + h; & Z_t \in \hat{\bar{S}}_{\Omega 1}^2 \\
F_3(Z_t) = W_{14} Z_t + h; & Z_t \in \hat{\bar{S}}_{\Omega 1}^3 \\
F_4(Z_t) = W_{15} Z_t + h; & Z_t \in \hat{\bar{S}}_{\Omega 1}^4 \\
\vdots & \vdots \\
F_{2M}(Z_t) = W_{2M} Z_t + h; & Z_t \in \hat{\bar{S}}_{\Omega 1}^{2M} 
\end{cases}
\]

\[
(24)
\]

7.2. Discontinuity boundaries

Consider map (7) and two sub-regions \(S_{\Omega i}\) and \(S_{\Omega j}\) \((i, j \in \{1, 2, \cdots, 2^M\})\) as defined in Section 4 (subsection 4.2). Suppose that subindices \(i - 1\) and \(j - 1\) of \(\hat{\bar{S}}_{\Omega i-1}\) and \(\hat{\bar{S}}_{\Omega j-1}\) are associated with \(i - 1 = i_1 i_2 \cdots i_M\) and \(j - 1 = j_1 j_2 \cdots j_M\). Then \(\hat{\bar{S}}_{\Omega i}\) and \(\hat{\bar{S}}_{\Omega j}\) are two successive sub-regions with the switching boundary \(\Sigma_{ij} = \hat{\bar{S}}_{\Omega i} \cap \hat{\bar{S}}_{\Omega j}\), iff there is exactly one \(1 \leq s \leq M\) such that for all \((z_{1t}, \cdots, z_{mt})^T \in \hat{\bar{S}}_{\Omega i}\) and \((z_{1t}, \cdots, z_{mt})^T \in \hat{\bar{S}}_{\Omega j}\)

\[
\begin{align*}
\{ z_{1t}, \cdots, z_{mt} \} & = 0, & 1 \leq r \leq \zeta - s \\
\{ z_{1t}, \cdots, z_{mt} \} & > 0, & r \neq s
\end{align*}
\]

\[
(25)
\]

Moreover, \(\Sigma_{ij}\) is a closed set \((\hat{\bar{S}}_{\Omega i} = \Sigma_{ij})\) and \(\Sigma_{ij} = \hat{\bar{S}}_{\Omega i} \cap \partial \Sigma_{ij}\) such that

\[
\hat{\bar{S}}_{\Omega i} = \Sigma_r = \begin{cases} 
Z_t \in \mathbb{R}^M; z_{st} = 0, & \text{and } \text{sgn}(z_{rt}) = \\
\text{sgn}(z_{i,r}) = \text{sgn}(z_{j,1}), & 1 \leq r \leq M \\
\end{cases}
\]

\[
(26)
\]

and \(\partial \Sigma_{ij} = \bigcup_{s_m = 1}^{M} \Sigma_{\nu, s_m}^{\nu, s_m}\) where

\[
\Sigma_{\nu, s_m}^{\nu, s_m} = \begin{cases} 
Z_t \in \mathbb{R}^M; z_{smt} = z_{st} = 0, & \text{and } \text{sgn}(z_{rt}) = \\
\text{sgn}(z_{rt}), & 1 \leq \nu \neq \nu', s_m \leq M \\
\end{cases}
\]

\[
(27)
\]

Furthermore, it can be proven that

\[
\bigcup_{i,j=1}^{2^M} \Sigma_{ij} = \bigcup_{l=1}^{M^2-1} \Sigma_l \subset \bigcup_{k=1}^{2^M} S_{\Omega k} = \mathbb{R}^M.
\]

\[
(28)
\]

7.3. Proof of theorem 3

(1) Without loss of generality let \(t_0 = 0\). Assume that there exists an equivalent continuous-time system for (10)

\[
Z_{t+1} = F(Z_t)
\]

\[
(7)
\]
on \([0, \Delta t]\), in the form of equation (12). By equivalency in the sense of equation (13), we must have

\[
\zeta(0) = 0, \quad Z_1 = W_{\Omega k} Z_0 + h = \zeta(\Delta t). \tag{29}
\]

According to theorem 1, the solution of system (10) on \([0, \Delta t]\) is

\[
\zeta(t) = e^{\bar{W}_{\Omega k} t} \zeta(0) + e^{\bar{W}_{\Omega k} t} \int_0^t e^{-\bar{W}_{\Omega k} \tau} \bar{h} d\tau, \quad t \in [0, \Delta t]. \tag{30}
\]

If \(\bar{W}_{\Omega k}\) is invertible, then

\[
\int_0^t e^{-\bar{W}_{\Omega k} \tau} \bar{h} d\tau = -\bar{W}_{\Omega k}^{-1} (e^{-\bar{W}_{\Omega k} t} - I) \bar{h}, \tag{31}
\]

and thus

\[
\zeta(t) = e^{\bar{W}_{\Omega k} t} \zeta(0) + \left[ e^{\bar{W}_{\Omega k} t} (-\bar{W}_{\Omega k}^{-1}) e^{-\bar{W}_{\Omega k} t} - e^{\bar{W}_{\Omega k} t} (-\bar{W}_{\Omega k}^{-1}) \right] \bar{h}. \tag{32}
\]

Furthermore, since

\[
(-\bar{W}_{\Omega k}^{-1}) e^{-\bar{W}_{\Omega k} t} = e^{-\bar{W}_{\Omega k} t} (-\bar{W}_{\Omega k}^{-1}), \tag{33}
\]

we have

\[
\zeta(t) = e^{\bar{W}_{\Omega k} t} \zeta(0) + \left[ I - e^{\bar{W}_{\Omega k} t} \right] (-\bar{W}_{\Omega k}^{-1}) \bar{h}, \quad t \in [0, \Delta t]. \tag{34}
\]

Putting conditions (29) in

\[
\zeta(\Delta t) = e^{\bar{W}_{\Omega k} \Delta t} \zeta(0) + \left[ I - e^{\bar{W}_{\Omega k} \Delta t} \right] (-\bar{W}_{\Omega k}^{-1}) \bar{h}, \tag{35}
\]

yields

\[
W_{\Omega k} Z_0 + h = e^{\bar{W}_{\Omega k} \Delta t} \zeta(0) + \left[ I - e^{\bar{W}_{\Omega k} \Delta t} \right] (-\bar{W}_{\Omega k}^{-1}) \bar{h}. \tag{36}
\]

Equation (36) has to hold for all \(Z_0\) including \(Z_0 = 0\). Hence, it is deduced that

\[
\begin{aligned}
W_{\Omega k} &= e^{\bar{W}_{\Omega k} \Delta t} \\
\bar{h} &= \left[ I - e^{\bar{W}_{\Omega k} \Delta t} \right] (-\bar{W}_{\Omega k}^{-1}) \bar{h}. \tag{37}
\end{aligned}
\]

According to (37), matrix \(W_{\Omega k}\) should be invertible and cannot have any zero eigenvalue. Also, since \(\bar{W}_{\Omega k}\) is invertible, it does not have any zero eigenvalue, which implies that \(W_{\Omega k}\) has no eigenvalue equal to one. Then, \(P_{\Omega k} \neq 0\), which means \(\left[ I - W_{\Omega k} \right]\) is invertible and (37) becomes equivalent to (14).

Now, considering \(\bar{W}_{\Omega k}\) and \(\bar{h}\) as in (14), we can obtain the desired equivalent continuous-time system (12) for (10) on \([0, \Delta t]\). It is just required to prove that every fixed point \(Z^*\) of map (10) is also an equilibrium point of system (12), and (14) is a solution of (36) for all \(Z^*\). For this purpose, let \(Z^*\) be a fixed point of (10), then

\[
F(Z^*) = W_{\Omega k} Z^* + h = Z^*. \tag{38}
\]

\(Z^*\) must be an equilibrium of (12), i.e.

\[
G(Z^*) = \bar{W}_{\Omega k} Z^* + \bar{h} = 0. \tag{39}
\]

From (38) and (39) it is concluded that

\[
h = \left[ I - W_{\Omega k} \right] (-\bar{W}_{\Omega k}^{-1}) \bar{h}. \tag{40}
\]
which shows that (37) or, equivalently, (14) is a solution of (36) for all \( \mathbb{Z} \) satisfying both relations (38) and (39). Finally, let each Jordan block of \( \tilde{W}_{\Omega^k} \) associated with a negative eigenvalue occur an even number of times. Then, by theorem (2), the logarithm of real matrix \( \tilde{W}_{\Omega^k} \), i.e. the matrix \( \tilde{W}_{\Omega^k} \) defined in (14), will be real.

(2) Let \( \tilde{W}_{\Omega^k} \) be diagonalizable, then

\[
\tilde{W}_{\Omega^k} = V E_k V^{-1},
\]

where \( E_k = diag(\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^m) \) and \( V \) is the matrix of eigenvectors of \( \tilde{W}_{\Omega^k} \). Since \( \tilde{W}_{\Omega^k} \) is also invertible, by (14)

\[
\tilde{W}_{\Omega^k} = \frac{1}{\Delta t} \log(\tilde{W}_{\Omega^k}) = \frac{1}{\Delta t} \log(V E_k V^{-1})
\]

\[
= V \frac{1}{\Delta t} \log(E_k) V^{-1},
\]

such that

\[
\log(E_k) = diag(\log(\lambda_1^k), \log(\lambda_2^k), \ldots, \log(\lambda_M^k)),
\]

which completes the proof.

Remark. Due to (37) and (14), one can see that system (12) is homogeneous (\( \tilde{h} = 0 \)) if and only if system (10) is homogeneous (\( h = 0 \)).

7.4. Proof of theorem 4

Again we prove the theorem for \( t_0 = 0 \) without loss of generality. Suppose that there is the equivalent continuous-time system (12) for (10) with non-invertible and diagonalizable matrix \( \tilde{W}_{\Omega^k} \). Similar to the proof of the previous theorem, relations (29) and (30) must hold for (10) and (12). On the other hand, non-invertibility and diagonalizability of \( \tilde{W}_{\Omega^k} \) demand that it has at least one eigenvalue equal to zero and

\[
\tilde{W}_{\Omega^k} = V \begin{pmatrix} O_{n \times n} & 0 \\ 0 & C \end{pmatrix} V^{-1},
\]

where \( O_{n \times n} \) is a zero matrix corresponding to zero eigenvalues (\( n \) denotes the number of zero eigenvalues) and \( C \) is an invertible matrix corresponding to nonzero eigenvalues of \( \tilde{W}_{\Omega^k} \). Therefore, for relation (31) we obtain

\[
\int_0^t e^{-\tilde{W}_{\Omega^k} \tau} \tilde{h} d\tau = \begin{pmatrix} t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ -C^{-1}(e^{-Ct} - I) \end{pmatrix} V^{-1} \tilde{h}.
\]

In this case, relation (35) becomes

\[
\zeta(\Delta t) = e^{\tilde{W}_{\Omega^k} \Delta t} \zeta(0) + e^{\tilde{W}_{\Omega^k} \Delta t} V \times \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n \times n} \begin{pmatrix} 0 \\ \vdots \\ -C^{-1}(e^{-C\Delta t} - I) \end{pmatrix} V^{-1} \tilde{h}.
\]

Inserting conditions (29) into (45) gives

\[
\tilde{W}_{\Omega^k} \mathbf{Z}_0 + \tilde{h} = e^{\tilde{W}_{\Omega^k} \Delta t} \mathbf{Z}_0 + e^{\tilde{W}_{\Omega^k} \Delta t} V \times \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n \times n} \begin{pmatrix} 0 \\ \vdots \\ -C^{-1}(e^{-C\Delta t} - I) \end{pmatrix} V^{-1} \tilde{h}.
\]

Denoting

\[
H = \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n \times n},
\]

and considering equality (46) for all \( \mathbf{Z}_0 \), particularly for \( \mathbf{Z}_0 = 0 \), yields

\[
\begin{cases}
\tilde{W}_{\Omega^k} = e^{\tilde{W}_{\Omega^k} \Delta t} \\
h = e^{\tilde{W}_{\Omega^k} \Delta t} V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}(e^{-C\Delta t} - I) \end{pmatrix} V^{-1} \tilde{h}.
\end{cases}
\]

Since

\[
e^{\tilde{W}_{\Omega^k} \Delta t} = V \begin{pmatrix} I & 0 \\ 0 & e^{C\Delta t} \end{pmatrix} V^{-1},
\]

we can simplify \( h \) in (48) and rewrite it as

\[
\begin{cases}
\tilde{W}_{\Omega^k} = e^{\tilde{W}_{\Omega^k} \Delta t} \\
h = V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}(I - e^{C\Delta t}) \end{pmatrix} V^{-1} \tilde{h}.
\end{cases}
\]

or equivalently

\[
\tilde{W}_{\Omega^k} = \frac{1}{\Delta t} \log(\tilde{W}_{\Omega^k}) \\
\tilde{h} = V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}(I - e^{C\Delta t}) \end{pmatrix}^{-1} V^{-1} \tilde{h}.
\]
In addition, we can write
\[
\hat{h} = V \begin{pmatrix} H & 0 \\ 0 & -C^{-1} (I - e^{C \Delta t}) \end{pmatrix}^{-1} V^{-1} h
\]
\[
= V \begin{pmatrix} \frac{1}{\Delta t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\Delta t} \end{pmatrix}_{n \times n} C (e^{C \Delta t} - I)^{-1} V^{-1} h
\]
\[
= V \begin{pmatrix} \frac{1}{\Delta t} (I_n & 0) + \left( O_{n \times n} & 0 \\ 0 & 0 \end{pmatrix} C \right) \times \begin{pmatrix} \left( I_n & 0 \\ 0 & e^{C \Delta t} \right) - \left( O_{n \times n} & 0 \\ 0 & I \right) \end{pmatrix}^{-1} V^{-1} h
\]
\[
= \frac{1}{\Delta t} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} + \tilde{W}_{\Omega h} \times \left( e^{\tilde{W}_{\Omega h} \Delta t} - \left( O_{n \times n} & 0 \\ 0 & I \right) \right) \hat{h}.
\]

Therefore
\[
\tilde{W}_{\Omega h} = \frac{1}{\Delta t} \log(W_{\Omega h}),
\]
\[
\hat{h} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} - \tilde{W}_{\Omega h} \times \left( O_{n \times n} & 0 \\ 0 & I \right) - e^{\tilde{W}_{\Omega h} \Delta t} \hat{h}.
\]

which is equivalent to (15).

Finally, from \( W_{\Omega h} = e^{\tilde{W}_{\Omega h} \Delta t} \) it is deduced that \( W_{\Omega h} \) is invertible. It is only necessary to prove that for every point \( Z^* \) satisfying both equations (38) and (39), i.e. equations
\[
\begin{cases}
(W_{\Omega h} - I) Z^* = -\hat{h} \\
\tilde{W}_{\Omega h} Z^* = -\hat{h},
\end{cases}
\] (54) fulfills solution (15) or, identically, solution (50). Thus, inserting \( \hat{h} = -\tilde{W}_{\Omega h} Z^* \) in (50), we have
\[
\hat{h} = e^{\tilde{W}_{\Omega h} \Delta t} V \begin{pmatrix} H & 0 \\ 0 & -C^{-1} (e^{C \Delta t} - I) \end{pmatrix} V^{-1} h
\]
\[
= e^{\tilde{W}_{\Omega h} \Delta t} V \begin{pmatrix} H & 0 \\ 0 & -C^{-1} (e^{C \Delta t} - I) \end{pmatrix} V^{-1} \tilde{W}_{\Omega h} Z^*
\]
\[
= e^{\tilde{W}_{\Omega h} \Delta t} V \begin{pmatrix} H & 0 \\ 0 & -C^{-1} (e^{C \Delta t} - I) \end{pmatrix} V^{-1} V
\]
\[
= e^{\tilde{W}_{\Omega h} \Delta t} V \begin{pmatrix} 0 & 0 \\ 0 & -C \end{pmatrix} V^{-1} Z^*
\]
\[
= e^{\tilde{W}_{\Omega h} \Delta t} V \begin{pmatrix} 0 & 0 \\ 0 & -C \end{pmatrix} V^{-1} Z^*
\]
\[
= \begin{pmatrix} I - e^{\tilde{W}_{\Omega h} \Delta t} \end{pmatrix} Z^* = (I - W_{\Omega h}) Z^*,
\] (55)

which demonstrates that (54) meets solution (50).

If every Jordan block of \( W_{\Omega h} \) associated with a negative eigenvalue occurs an even number of times, then theorem (2) guarantees that \( W_{\Omega h} \) will be real. Also, similar to the proof of theorem 3, it is easy to see that \( W_{\Omega h} \) will be diagonalizable when \( W_{\Omega h} \) has no negative real eigenvalues.

7.5. Proof of theorem 5

Let \( t_0 = 0 \) without loss of generality and assume there exists the equivalent continuous-time system (12) for (10), for which matrix \( W_{\Omega h} \) is non-invertible. Then, relations (29) and (30) must hold for (10) and (12), analogously to the proofs of the previous theorems. Also, by similar reasoning we have
\[
\zeta(\Delta t) = e^{\tilde{W}_{\Omega h} \Delta t} \zeta(0) + \left[ e^{\tilde{W}_{\Omega h} \Delta t} \int_0^{\Delta t} e^{-\tilde{W}_{\Omega h} \tau} d\tau \right] \hat{h}.
\] (56)

Inserting conditions (29) in equation (56) and solving the resulting equation for all \( Z_0 \), including \( Z_0 = 0 \), yields
\[
\begin{cases}
W_{\Omega h} = e^{\tilde{W}_{\Omega h} \Delta t} \\
\hat{h} = e^{\tilde{W}_{\Omega h} \Delta t} \int_0^{\Delta t} e^{-\tilde{W}_{\Omega h} \tau} d\tau \hat{h}.
\end{cases}
\] (57)

Now let
\[
\lambda \in \text{Spectrum}(W_{\Omega h}) \Rightarrow \lambda \Delta t \notin 2i\pi Z^*.
\] (58)
Then, by proposition 1, \( f_0^\Delta t e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \) is invertible and so

\[
\begin{cases}
\hat{W}_{\Omega k} = \frac{1}{\Delta t} \log(W_{\Omega k}) \\
\hat{h} = \left( \int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \right)^{-1} e^{-\hat{W}_{\Omega k}^t \Delta t} \, h,
\end{cases}
\]

which is equal to equation (17). The last point which still has to be proven is that equation (54) meets solution (17) or, identically, (57), for every \( Z^* \). Since \( \hat{W}_{\Omega k} \) is non-invertible, it can be written in the following Jordan form:

\[
\hat{W}_{\Omega k} = U \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} U^{-1},
\]

where \( B \) is a strictly upper triangular matrix and \( C \) is an invertible matrix. Then

\[
e^{-\hat{W}_{\Omega k}^t \Delta t} = U \begin{pmatrix} e^{-B \Delta t} & 0 \\ 0 & e^{-C \Delta t} \end{pmatrix} U^{-1},
\]

\[
\int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, \hat{h} \, d\tau = U \begin{pmatrix} \int_0^{\Delta t} e^{-B \tau} \, d\tau & 0 \\ 0 & -C^{-1}(e^{-C \Delta t} - I) \end{pmatrix} U^{-1}.
\]

Now, substituting \( \hat{h} = -\hat{W}_{\Omega k} Z^* \) in (57) we have

\[
h = e^{\hat{W}_{\Omega k}^t \Delta t} \left( \int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \right) \hat{h}
= e^{\hat{W}_{\Omega k}^t \Delta t} U \begin{pmatrix} \int_0^{\Delta t} e^{-B \tau} \, d\tau & 0 \\ 0 & -C^{-1}(e^{-C \Delta t} - I) \end{pmatrix} \times U^{-1}(-\hat{W}_{\Omega k} Z^*)
= e^{\hat{W}_{\Omega k}^t \Delta t} U \begin{pmatrix} \int_0^{\Delta t} e^{-B \tau} \, d\tau & 0 \\ 0 & -(e^{-C \Delta t} - I) C^{-1} \end{pmatrix} \times U^{-1} \begin{pmatrix} -B & 0 \\ 0 & -C \end{pmatrix} U^{-1} Z^*
= e^{\hat{W}_{\Omega k}^t \Delta t} U \begin{pmatrix} \int_0^{\Delta t} -B e^{-B \tau} \, d\tau & 0 \\ 0 & (e^{-C \Delta t} - I) \end{pmatrix} \times U^{-1} Z^*
= e^{\hat{W}_{\Omega k}^t \Delta t} U \begin{pmatrix} (e^{-B \Delta t} - I) & 0 \\ 0 & (e^{-C \Delta t} - I) \end{pmatrix} U^{-1} Z^*
= \left( I - e^{\hat{W}_{\Omega k}^t \Delta t} \right) Z^* = (I - \hat{W}_{\Omega k}) Z^*,
\]

which completes the proof.

Finally, due to theorem (2), \( \hat{W}_{\Omega k} \) will be real, provided that each Jordan block of \( W_{\Omega_k} \) related to a negative eigenvalue occurs an even number of times.

**Remark.** In theorem 5, by (62) we have

\[
\begin{pmatrix} \int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \end{pmatrix}^{-1}
= U \begin{pmatrix} 0 & 0 \\ 0 & (I - e^{-C \Delta t})^{-1} C \end{pmatrix} U^{-1} h.
\]

On the other hand, since \( C \) is invertible, \( \det(I - e^{-C \Delta t}) \neq 0 \). Therefore, \( \int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \) is invertible if and only if \( \int_0^{\Delta t} e^{-B \tau} \, d\tau \) is invertible. Thus, for invertibility of \( \int_0^{\Delta t} e^{-\hat{W}_{\Omega k}^t \tau} \, d\tau \), it is required that relation (58) holds only for any pair of eigenvalues of \( B \).

### 7.6. Grazing bifurcation

Here we investigate a grazing bifurcation of periodic orbits for the continuous PLRNN derived from the van-der-Pol oscillator (Example 1). For this purpose, we consider the converted continuous-time system locally in the neighborhood of only one border

\[
\Sigma = \left\{ \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_{10})^T \in \mathbb{R}^{10} \mid H(\zeta) = \zeta_2 = 0 \right\},
\]

where the scalar function \( H : \mathbb{R}^{10} \to \mathbb{R} \) defines the border and has non-vanishing gradient. According to (di Bernardo & Hogan, 2010; Monfared et al., 2017), a periodic orbit \( \zeta(t) \) undergoes a grazing bifurcation for some critical value of a bifurcation parameter, if it is a grazing orbit for some \( t = t^* \). This means \( \zeta(t) \) hits \( \Sigma \) tangentially at the grazing point \( \zeta^* = \zeta(t^*) \) and satisfies the following conditions:

\[
H(\zeta^*) = \zeta_2^* = 0,
\]

\[
\nabla H(\zeta^*) = (0, 1, 0, \cdots, 0)^T \neq 0,
\]

\[
\langle \nabla H(\zeta^*), \hat{W}_{\Omega_1} \zeta^* + \hat{h}_1 \rangle = \sum_{j=1}^{10} \hat{w}_j(1) \dot{\zeta}_j + \hat{h}_{12} = 0,
\]

\[
\langle \nabla H(\zeta^*), \hat{W}_{\Omega_2} \zeta^* + \hat{h}_2 \rangle = \sum_{j=1}^{10} \hat{w}_j(2) \dot{\zeta}_j + \hat{h}_{22} = 0,
\]
\[
\langle \nabla H(\hat{\zeta}^*) , \tilde{W}_{\Omega_1}^2 \hat{\zeta}^* + \tilde{W}_{\Omega_1} \hat{h}_1 \rangle + \langle \nabla^2 H(\hat{\zeta}^*) (\tilde{W}_{\Omega_1} \hat{\zeta}^* \\
+ \hat{h}_1) , \tilde{W}_{\Omega_1} \hat{\zeta}^* + \hat{h}_1 \rangle = \sum_{j=1}^{10} \tilde{v}^{(1)}_{2j} \hat{\zeta}_j^* + \sum_{j=1}^{10} \tilde{w}^{(1)}_{2j} \hat{h}_{1j} = 0,
\]

\[
\langle \nabla H(\hat{\zeta}^*) , \tilde{W}_{\Omega_2}^2 \hat{\zeta}^* + \tilde{W}_{\Omega_2} \hat{h}_2 \rangle + \langle \nabla^2 H(\hat{\zeta}^*) (\tilde{W}_{\Omega_2} \hat{\zeta}^* \\
+ \hat{h}_2) , \tilde{W}_{\Omega_2} \hat{\zeta}^* + \hat{h}_2 \rangle = \sum_{j=1}^{10} \tilde{v}^{(2)}_{2j} \hat{\zeta}_j^* + \sum_{j=1}^{10} \tilde{w}^{(2)}_{2j} \hat{h}_{2j} = 0,
\]

where \( \tilde{W}_{\Omega_1} = [\tilde{w}^{(1)}_{ij}] \), \( \tilde{W}_{\Omega_2} = [\tilde{w}^{(2)}_{ij}] \), \( \tilde{W}_{\Omega_1}^2 = [\tilde{v}^{(1)}_{ij}] \) and \( \tilde{W}_{\Omega_2}^2 = [\tilde{v}^{(2)}_{ij}] \).

In this case the periodic orbit \( \hat{\zeta}(t) \) crosses \( \Sigma \) transversally as the bifurcation parameter passes through the bifurcation value. The grazing bifurcation leads to a transition or a sudden jump in the system’s response by the dis-/appearance of a tangential intersection between the trajectory and the switching boundary. The occurrence of a grazing bifurcation in the continuous PLRNN is illustrated in Fig. 2.