Supplementary File for “A Markov Decision Process Model for Socio-Economic Systems Impacted by Climate Change”

Anonymous Authors

Proof of Theorem 1

Nondecreasing and Concave:
We will first show that if $V(s_n, \ell_n)$ is nondecreasing and concave in $\ell_n$, then so is
\[
F_m(s_n, \ell_n) = \mathbb{E} \left[ m\alpha - \beta y_n + z_n + a_y V(s_{n-1} + m, \ell_{n-1} + r_n) \right],
\]
for $m = 0, 1, \ldots, q$. Assume
\begin{itemize}
  \item \(\frac{\partial}{\partial \ell_n} V(s_n, \ell_n) \geq 0\) (nondecreasing),
  \item \(\frac{\partial^2}{\partial \ell^2_n} V(s_n, \ell_n) < 0\) (concavity).
\end{itemize}
Using the nondecreasing monotonicity of $V(s_n, \ell_n)$ we can write
\[
\frac{\partial}{\partial \ell_n} F_m(s_n, \ell_n) = \frac{n\alpha_{n-1}}{(1-k)s_{n-1}^b} + a_y \mathbb{E} \left[ \frac{\partial}{\partial \ell_n} V(s_n, \ell_n) \right] \geq 0,
\]
where the derivative can be brought inside the integral due to the monotone convergence theorem. Since $0 < a < 1$, for the second derivative we have
\[
\frac{\partial^2}{\partial \ell^2_n} F_m(s_n, \ell_n) = \frac{n\alpha(a-1)\ell_{n-1}^{-2}}{(1-k)s_{n-1}^b} + a_y \mathbb{E} \left[ \frac{\partial^2}{\partial \ell^2_n} V(s_n, \ell_n) \right] < 0.
\]
Hence, it is sufficient to show that $V(s_n, \ell_n)$ is nondecreasing and concave.

Finding the value function iteratively (i.e., value iteration) is a common approach which is known to converge (Sutton & Barto, 2018): \(\lim_{n \to \infty} V_i(s, \ell) = V(s, \ell)\). For brevity, we drop the time index from now on. We will next prove that $V(s, \ell)$ is nondecreasing and concave iteratively. Initializing all the state values as zero, i.e., $V^0(s, \ell) = 0, \forall s, \ell$, after the first iteration we get
\[
V_1(s, \ell) = \min_x \left\{ \mathbb{E}[\alpha x - \beta y(x, z) + z(s, \ell) + a_y V^0(s + x, \ell + r)] \right\},
\]
\[
= \mathbb{E}[z(s, \ell)] = \theta + \frac{\epsilon}{1-k} = \theta + \frac{\eta R}{(1-k)s^b},
\]
where we used the fact that $\mathbb{E}[y] = 0$ when $x = 0$ for all states. Differentiating with respect to $\ell$, we get
\[
\frac{\partial}{\partial \ell} V_1(s, \ell) = \eta \alpha \frac{\ell_{n-1}}{(1-k)s^b} \geq 0, \ \forall s, \tag{S1}
\]
\[
\frac{\partial^2}{\partial \ell^2} V_1(s, \ell) = \eta \alpha(a-1) \frac{\ell_{n-2}}{(1-k)s^b} < 0, \ \forall s.
\]
As a result, by mathematical induction we can conclude that \( V_1(s, \ell) \) is nondecreasing and concave in \( \ell \) for all \( s \). Next, value function after the second iteration becomes

\[
V_2(s, \ell) = \min_x \left\{ E[ax - \beta y(x, z) + z(s, \ell) + a_g V_1(s + x, \ell + r)] \right\}
\]

\[
= \min_x \left\{ E[ax - \beta y(x, z)] + \theta + \eta \frac{\ell^a}{(1-k)s^b} + a_g \theta + a_g E \left[ \frac{\eta(\ell + r)^a}{(1-k)(s + x)^b} \right] \right\}
\]

Denoting the optimum action with \( \bar{x} \) we will show that \( V_2(s, \ell) \) is nondecreasing and concave for any \( \bar{x} \). Moreover, the pointwise minimum of nondecreasing and concave functions is also nondecreasing and concave. The residents’ probability of support \( E[y(x, z)] \) depends on past values of \( x \) and \( z \), but not \( \ell \) directly, so taking the derivative with respect to \( \ell \) we get

\[
\frac{\partial}{\partial \ell} V_2(s, \ell) = \eta \frac{\ell^{a-1}}{(1-k)s^b} + a_g \eta \frac{E[(\ell + r)^{a-1}]}{(1-k)(s + \bar{x})^b} \geq 0, \ \forall s
\]

\[
\frac{\partial^2}{\partial \ell^2} V_2(s, \ell) = \eta(a-1) \frac{\ell^{a-2}}{(1-k)s^b} + a_g \eta(a-1) \frac{E[(\ell + r)^{a-2}]}{(1-k)(s + \bar{x})^b} < 0, \ \forall s.
\]

Hence, \( V_2(s, \ell) \) is nondecreasing and concave. Now, for any \( i \), given that \( V_{i-1}(s, \ell) \) is nondecreasing and concave, we can write

\[
\frac{\partial}{\partial \ell} V_i(s, \ell) = \eta(a-1) \frac{\ell^{a-1}}{(1-k)s^b} + a_g \left[ \frac{\partial}{\partial \ell} V_{i-1}(s + \bar{x}, \ell) \right] \geq 0, \ \forall s
\]

\[
\frac{\partial^2}{\partial \ell^2} V_i(s, \ell) = \eta(a-1) \frac{\ell^{a-2}}{(1-k)s^b} + a_g \left[ \frac{\partial^2}{\partial \ell^2} V_{i-1}(s + \bar{x}, \ell) \right] < 0, \ \forall s.
\]

Consequently, by mathematical induction, \( V(s, \ell) \) is nondecreasing and concave.

**Comparison of Derivatives:**

Similarly, if we show that \( \frac{\partial}{\partial m} F_m(s, \ell) < \frac{\partial}{\partial m} F_{m-1}(s, \ell) \) since

\[
\frac{\partial}{\partial m} F_m(s, \ell) = \frac{\eta a \ell^{a-1}}{(1-k)s^b} + a_g \left[ \frac{\partial}{\partial m} V_{m-1}(s + \bar{x}, \ell) \right]
\]

\[
\frac{\partial}{\partial m} F_{m-1}(s, \ell) = \frac{\eta a \ell^{a-1}}{(1-k)s^b} + a_g \left[ \frac{\partial}{\partial m} V_{m-1}(s + \bar{x}, \ell) \right].
\]

Starting again with \( V_0(s, \ell) = 0, \forall s, \ell \), from (S1) we can write the following inequality for the first iteration

\[
\frac{\partial}{\partial m} V_1(s + m, \ell) = \eta(a-1) \frac{\ell^{a-1}}{(1-k)(s + m)^b} < \frac{\partial}{\partial m} V_1(s + m - 1, \ell) = \eta(a-1) \frac{\ell^{a-1}}{(1-k)(s + m - 1)^b}.
\]

For any \( i \), given that \( \frac{\partial}{\partial m} V_{i-1}(s + m, \ell) < \frac{\partial}{\partial m} V_{i-1}(s + m - 1, \ell) \), from (S2) we have

\[
\frac{\partial}{\partial m} V_i(s + m, \ell) = \eta(a-1) \frac{\ell^{a-1}}{(1-k)(s + m)^b} + a_g \left[ \frac{\partial}{\partial m} V_{i-1}(s + m + \bar{x}, \ell) \right] < \frac{\partial}{\partial m} V_i(s + m - 1, \ell) = \eta(a-1) \frac{\ell^{a-1}}{(1-k)(s + m - 1)^b} + a_g \left[ \frac{\partial}{\partial m} V_{i-1}(s + m - 1 + \bar{x}, \ell) \right].
\]

As a result, by mathematical induction we can conclude that \( \frac{\partial}{\partial m} V(s + m, \ell) < \frac{\partial}{\partial m} V(s + m - 1, \ell) \).

**References**