Supplementary material to the paper "Distributional Robust Policy Evaluation and Learning in Offline Contextual Bandits"

Appendix A Proofs

Appendix A.1 Auxiliary Results

In this section, we give all of the auxiliary results used in the proofs.

Theorem A1 (Hoeffding's inequality, Theorem 2.8 in [3]). *Let $X_1, \ldots, X_n$ be independent random variables such that X_i takes its values in $[a_i, b_i]$ almost surely for all $i < n$. Let*

$$S = \sum_{i=1}^{n} (X_i - \mathbf{E}X_i).$$

Then for every $t > 0$,

$$\mathbf{P}(|S| > t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right).$$

Theorem A2 (Functional central limit theorem, Corollary 7.17 in [1]). *Let S be a compact subspace of $\mathbf{R}^d$ and $\mathcal{C}(S)$ be the space of continuous bounded random functions on S equipped with the sup-norm. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of centered i.i.d. $\mathcal{C}(S)-$valued random functions such that $\mathbf{E}X_n^2(s) < \infty$ for some $s \in S$. Suppose there exists a constant M such that*

$$|X_1(s) - X_2(t)| \leq M \|s - t\| \text{ almost surely,}$$

for all $s, t \in S$. Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_n \Rightarrow Y,$$

where Y is a Gaussian process on $\mathcal{C}(S)$.

Definition 1 (Directional differentiability, Gâteaux directional differentiability and Hadamard directional differentiability). *Let B_1 and B_2 be Banach spaces and $G : B_1 \to B_2$ be a mapping. It is* *said that G is directionally differentiable at a considered point $\mu \in B_1$ if the limits*

$$G'_\mu(d) = \lim_{t \downarrow 0} \frac{G(\mu + td) - G(\mu)}{t}$$

exists for all $d \in B_1$.

Furthermore, it is said that G is Gâteaux directionally differentiable at μ if the directional derivative $G'_\mu(d)$ exists for all $d \in B_1$ and $G'_\mu(d)$ is linear and continuous in d. For ease of notation, we also denote $D_\mu(\mu_0)$ be the operator $G'_{\mu_0}(\cdot)$.

Finally, it is said that G is Hadamard directionally differentiable at μ if the directional derivative $G'_\mu(d)$ exists for all $d \in B_1$ and

$$G'_\mu(d) = \lim_{\substack{t \downarrow 0 \\ d' \to d}} \frac{G(\mu + td') - G(\mu)}{t}.$$

Theorem A3 (Danskin theorem, Theorem 4.13 in [2])**.** *Let $\Theta \in \mathbf{R}^d$ be a nonempty compact set and B be a Banach space. Suppose the mapping $G : B \times \Theta \to \mathbf{R}$ satisfies that $G(\mu, \theta)$ and $D_\mu(\mu, \theta)$ are continuous on $O_{\mu_0} \times \Theta$, where $O_{\mu_0} \subset B$ is a neighborhood around μ_0. Let $\phi : B \to \mathbf{R}$ be the inf-functional $\phi(\mu) = \inf_{\theta \in \Theta} G(\mu, \theta)$ and $\bar{\Theta}(\mu) = \arg\max_{\theta \in \Theta} G(\mu, \theta)$. Then, the functional ϕ is directionally differentiable at μ_0 and*

$$G'_{\mu_0}(d) = \inf_{\theta \in \bar{\Theta}(\mu_0)} D_\mu(\mu_0, \theta)\, d.$$

Theorem A4 (Delta theorem, Theorem 7.59 in [6])**.** *Let B_1 and B_2 be Banach spaces, equipped with their Borel σ-algrebras, Y_N be a sequence of random elements of B_1, $G : B_1 \to B_2$ be a mapping, and τ_N be a sequence of positive numbers tending to infinity as $N \to \infty$. Suppose that the space B_1 is separable, the mapping G is Hadamard directionally differentiable at a point $\mu \in B_1$, and the sequence $X_N = \tau_N(Y_N - \mu)$ converges in distribution to a random element Y of B_1. Then,*

$$\tau_N(G(Y_N) - G(\mu)) \Rightarrow G'_\mu(Y) \text{ in distribution,}$$

and

$$\tau_N(G(Y_N) - G(\mu)) = G'_\mu(X_N) + o_p(1).$$

Proposition A1 (Proposition 7.57 in [6])**.** *Let B_1 and B_2 be Banach spaces, $G : B_1 \to B_2$, and $\mu \in B_1$. Then the following hold: (i) If $G(\cdot)$ is Hadamard directionally differentiable at μ, then the directional derivative $G'_\mu(\cdot)$ is continuous. (ii) If $G(\cdot)$ is Lipschitz continuous in a neighborhood of μ and directionally differentiable at μ, then $G(\cdot)$ is Hadamard directionally differentiable at μ.*

Appendix A.2 Proof of Lemma 3.3

Proof of Lemma 3.3. The closed form expression of $\frac{\partial}{\partial\alpha}\hat{\phi}_n(\pi,\alpha)$ and $\frac{\partial^2}{\partial\alpha^2}\hat{\phi}_n(\pi,\alpha)$ follows from elementary algebra. By the Cauchy Schwartz's inequality, we have

$$\Big(\sum_{i=1}^{n} Y_i(A_i)W_i(\pi,\alpha)\Big)^2 \leq n\hat{W}_n(\pi,\alpha)\sum_{i=1}^{n} Y_i^2(A_i)W_i(\pi,\alpha)$$

Therefore, it follows that $\frac{\partial^2}{\partial\alpha^2}\hat{\phi}_n(\pi,\alpha) \leq 0$. Note that the Cauchy Schwartz's inequality is actually an equality if and only if

$$Y_i^2(A_i)W_i(\pi,\alpha) = cW_i(\pi,\alpha) \quad \text{if} \quad W_i(\pi,\alpha) \neq 0$$

for some constant c independent of i. Since the above condition is violated if $\{Y_i(A_i)\mathbf{1}\{\pi(X_i) = A_i\}\}_{i=1}^N$ has at least two different non-zero values, we have in this case $\hat{\phi}_n(\pi,\alpha)$ is strictly-concave in α. □

Appendix A.3 Proof of Theorem 3.4

We first give the upper and lower bound for α^* and in Lemmas A1 and A3.

Lemma A1 (Upper bound of α^*)**.** *Suppose that Assumption 1 is imposed, we have the optimal dual solution $\alpha^* \leq \overline{\alpha} = M/\delta$.*

Proof. Proof First note that $\inf_{\mathbf{P}\in\mathcal{U}_{\mathbf{P}_0(\delta)}} \mathbf{E}_{\mathbf{P}}\left[Y(\pi(X))\right] \geq \min_{i\in[n]}(Y_i) \geq 0$ and

$$-\alpha \log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X))\,/\alpha\right] - \alpha\delta \leq M - \alpha\delta.\right.$$

$M - \alpha\delta \geq 0$ gives the upper bound $\alpha^* \leq \overline{\alpha} = M/\delta$.

□

To prove the lower bound of α^*, we need the following technical lemma.

Lemma A2. *For $c > 0$ and $c\exp(b) < 1/e$, the smallest root of equation $x\log(x) + bx + c = 0$ is*

$$x^* = -\frac{c}{W_{-1}(-c\exp(b))},$$

where $W_{-1}(z)$ is the root of the equation for w in $w\exp(w) = z$ with $z < -1$. Furthermore, we have

$$x^* \in \left(\frac{c\left(-\log c - b - 1\right)}{2\left(\left(\log\left(c\right) + b + 1\right)^2 + 1\right)}, c\right).$$

Proof. Let $f(x) = x\log(x) + bx + c$ and $x = -c/w$ for $w < 0$. Then,

$$f(x) = 0$$

$$\begin{aligned}&\Leftrightarrow \quad (-c/w)\log(-c/w)+b\,(-c/w)+c=0\\&\Leftrightarrow \quad w=\log(-c/w)+b\\&\Leftrightarrow \quad w\exp(w)=-c\exp(b).\end{aligned}$$

Since $x=-c/w$ is one-to-one mapping, we have roots of $f(x)=0$ have one-to-one correspondence with roots of $w\exp(w)=-c\exp(b)$. Note that when $-c\exp(b)\in(-1/e,0)$, $w\exp(w)=-c\exp(b)$ has 2 roots, one is in $(-1,0)$ and the other one is in $(-1,+\infty)$. Therefore, the small root of $f(x)=0$ is of the form

$$x^*=-\frac{c}{W_{-1}(-c\exp(b))}.$$

[4] shows that

$$-1-\sqrt{2u}-u<W_{-1}(-\exp(-u-1))<-1,\ \text{ for } u>0.$$

Notice that $u+1-\sqrt{2u}\geq\frac{1}{2}u$, we have

$$\frac{cu}{2\left(u^2+1\right)}<-c/W_{-1}(-\exp(-u-1))<c.$$

Let $u=-\log c-b-1$, we have that

$$x^*\in\left(\frac{c\left(-\log c-b-1\right)}{2\left(\left(\log\left(c\right)+b+1\right)^2+1\right)},c\right).$$

□

Lemma A3 (Lower bound of α^*)**.** *Suppose that Assumption 1 is imposed, we have*

$$\alpha^*\geq\underline{\alpha}=\frac{\bar{b}^{-1}\exp(-\delta-1)\left(\log\left(\bar{b}\right)-\log(\underline{b})\right)}{2\left(\left(\log(\underline{b})-\log\left(\bar{b}\right)\right)^2+1\right)}. \tag{A.1}$$

Proof. Denote the density of $Y(\pi(X))$ by f_π. It is easy to see that $\bar{b}\geq f_\pi(y)\geq\underline{b}$ for any $\pi\in\Pi$ and any $y\in[0,M]$. First notice that

$$\alpha^*=\arg\max_{\alpha\geq0}\left\{-\alpha\log\mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X))/\alpha\right)\right]-\alpha\delta\right\}=\arg\min_{\alpha\geq0}\left\{\alpha\log\mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X))/\alpha\right)\right]+\alpha\delta\right\}.$$

Then, we have

$$\begin{aligned}\alpha\log\mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X)/\alpha\right)\right]+\alpha\delta &= \alpha\log\left[\int_0^M\exp\left(-y/\alpha\right)f_\pi(y)dy\right]+\alpha\delta\\&\leq \alpha\log\left[\bar{b}\left(\alpha-\exp\left(-M/\alpha\right)\alpha\right)\right]+\alpha\delta\\&\leq \alpha\log\left(\alpha\right)+\alpha\left(\delta+\log(\bar{b})\right).\end{aligned}$$

Therefore, we have

$$\min_{\alpha\geq 0}\alpha\left\{\log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X)/\alpha)\right]+\alpha\delta\right\}\leq \min_{\alpha\geq 0}\left\{a\log(\alpha)+\alpha\left(\delta+\log(\bar{b})\right)\right\}=-\bar{b}^{-1}\exp(-\delta-1).$$

On the other hand, We have

$$\begin{aligned}
&\alpha\log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X))/\alpha\right)\right]+\alpha\delta\\
=\;&\alpha\log\left[\int_0^M\exp\left(-y/\alpha\right)\left(f_\pi(y)-\underline{b}+\underline{b}\right)dy\right]+\alpha\delta\\
\geq\;&\alpha\log\left[\exp(-M/\alpha)\left(1-\underline{b}M\right)+\underline{b}\left(\alpha-\exp\left(-M/\alpha\right)\alpha\right)\right]+\alpha\delta\\
=\;&\alpha\log\left[\exp(-M/\alpha)\left(1-\underline{b}M-\underline{b}\alpha\right)+\alpha\underline{b}\right]+\alpha\delta
\end{aligned}$$

Since $2\underline{b}M\leq 1$, we have when $\alpha\leq\frac{1}{2\underline{b}}$, $1-\underline{b}M-\underline{b}\alpha\geq 0$ and thus

$$\alpha\log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X)/\alpha)\right]+\alpha\delta\geq\alpha\log(\alpha)+\alpha\left(\delta+\log(\underline{b})\right).$$

Consider the function $f(\alpha)=\alpha\log(\alpha)+\alpha\left(\delta+\log(\underline{b})\right)$. $f(\alpha)$ is decreasing when $\alpha\in[0,(\underline{b})^{-1}\exp(-\delta-1))$. Notice that

$$\bar{b}^{-1}\exp(-\delta-1)\exp\left(\delta+\log(\underline{b})\right)\leq 1/e.$$

By applying Lemma A2, we have the smallest root of $f(\alpha)=-\bar{b}^{-1}\exp(-\delta-1)$, α_0 is in

$$\alpha_0\in\left(\frac{\bar{b}^{-1}\exp(-\delta-1)\left(\log\left(\bar{b}\right)-\log(\underline{b})\right)}{2\left(\left(\log(\underline{b})-\log\left(\bar{b}\right)\right)^2+1\right)},\bar{b}^{-1}\exp(-\delta-1)\right).$$

Notice that

$$\frac{\bar{b}^{-1}\exp(-\delta-1)\left(\log\left(\bar{b}\right)-\log(\underline{b})\right)}{2\left(\left(\log(\bar{b})-\log\left(\underline{b}\right)\right)^2+1\right)}<\frac{1}{2\underline{b}},$$

and thus

$$\alpha\log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X))/\alpha\right)\right]+\alpha\delta>\min_{\alpha\geq 0}\alpha\left\{\log \mathbf{E}_{\mathbf{P}_0}\left[\exp\left(-Y(\pi(X)/\alpha)\right]+\alpha\delta\right\}$$
$$\text{for }\alpha\in\left[0,\frac{\bar{b}^{-1}\exp(-\delta-1)\left(\log\left(\bar{b}\right)-\log(\underline{b})\right)}{2\left(\left(\log(\bar{b})-\log\left(\underline{b}\right)\right)^2+1\right)}\right),$$

which concludes (A.1).

□

Now we are ready to show the proof of Theorem 3.4.

Proof of Theorem 3.4. Notice that

$$\sqrt{n}\left(\hat{W}_n(\pi,\alpha) - \mathbf{E}[W_i(\pi,\alpha)]\right) \Rightarrow Z(\alpha),$$

where

$$Z(\alpha) \sim N\left(0, \mathbf{Var}\left[W_i(\pi,\alpha)\right]\right).$$

Since $W_i(\pi,\alpha)$ is Lipschitz continuous if $\alpha \in [\underline{\alpha}/2, 2\overline{\alpha}]$. We have $W_i(\pi,\alpha)$ is a P-Donsker class (see, for example, Corollary 7.17 in [1] and Chapter 19 in [7]). Therefore,

$$\sqrt{n}\left(\hat{W}_n(\pi,\cdot) - \mathbf{E}[W_i(\pi,\cdot)]\right) \Rightarrow Z(\cdot),$$

in a Banach space $\mathcal{C}([\underline{\alpha}/2, 2\overline{\alpha}])$ of continuous functions $\psi : [\underline{\alpha}/2, 2\overline{\alpha}] \to \mathbf{R}$ equipped with the the sup-norm $\|\psi\| = \sup_{\alpha\in[\underline{\alpha}/2,2\overline{\alpha}]} \psi(\alpha)$. Z is a random element in $\mathcal{C}([\underline{\alpha}/2, 2\overline{\alpha}])$.

Define the functionals

$$G(\psi,\alpha) = \alpha \log\left(\psi(\alpha)\right) + \alpha\delta, \text{ and } V(\psi) = \inf_{\alpha\in[\underline{\alpha}/2,2\overline{\alpha}]} G(\psi,\alpha),$$

for $\psi > 0$. By the Danskin theorem (Theorem 4.13 in [2]), $V(\cdot)$ is directionally differentiable at any $\mu \in \mathcal{C}([\underline{\alpha}/2, 2\overline{\alpha}])$ with $\mu > 0$ and

$$V'_\mu(\nu) = \inf_{\alpha\in\bar{X}(\mu)} \alpha\left(1/\mu(\alpha)\right)\nu(\alpha),\ \forall \nu \in \mathcal{C}([\underline{\alpha}/2, 2\overline{\alpha}])),$$

where $\bar{X}(\mu) = \arg\min_{\alpha\in[\underline{\alpha}/2,2\overline{\alpha}])} \alpha \log\left(\mu(\alpha)\right) + \alpha\delta$ and $V'_\mu(\nu)$ is the directional derivative of $V(\cdot)$ at μ in the direction of ν. On the other hand, $V(\psi)$ is Lipschitz continuous if $\psi(\cdot)$ is bounded away from zero. Notice that

$$\mathbf{E}[W_i(\pi,\alpha)] = \mathbf{E}[\exp\left(-Y(\pi(x))/\alpha\right)] \geq \exp\left(-2M/\underline{\alpha}\right). \tag{A.2}$$

Therefore, $V(\cdot)$ is Hadamard directionally differentiable at $\mu = \mathbf{E}[W_i(\pi,\cdot)]$ (see, for example, Proposition 7.57 in [6]). By the Delta theorem (Theorem 7.59 in [6]), we have

$$\sqrt{n}\left(V(\hat{W}_n(\pi,\cdot)) - V(\mathbf{E}[W_i(\pi,\cdot)])\right) \Rightarrow V'_{\mathbf{E}[W_i(\pi,\cdot)]}(Z).$$

Furthermore, we know that $\log\left(\mathbf{E}\left(\exp\left(-\beta Y\right)\right)\right)$ is strictly convex w.r.t β given $\mathbf{Var}(Y) > 0$ and $xf(1/x)$ is strictly convex if $f(x)$ is strictly convex. Therefore, $\alpha \log\left(\mathbf{E}[W_i(\pi,\alpha)]\right) + \alpha\delta$ is strictly convex for $\alpha > 0$ and thus

$$V'_{\mathbf{E}[W_i(\pi,\cdot)]}(Z) = \alpha^*\left(1/\mathbf{E}[W_i(\pi,\alpha^*)]\right) Z(\alpha^*) \stackrel{d}{=} N\left(0, (\alpha^*)^2\left(\mathbf{E}\left[W_i(\pi,\alpha^*)\right]^{-2}\mathbf{Var}\left[W_i(\pi,\alpha^*)\right]\right)\right).$$

By Lemma 3.1, we have that

$$\hat{Q}_{\text{DRO}}(\pi) = -\inf_{\alpha \geq 0}\left(\alpha \log\left(\hat{W}_n(\pi,\alpha)\right) + \alpha\delta\right),$$

and

$$Q_{\text{DRO}}(\pi) = -\inf_{\alpha \geq 0}\left(\alpha \log\left(\mathbf{E}[W_i(\pi,\alpha)]\right) + \alpha\delta\right) = -V(\mathbf{E}[W_i(\pi,\alpha)]).$$

We remain to show $\mathbf{P}(\hat{Q}_{\text{DRO}}(\pi) \neq -V(\hat{W}_n(\pi,\alpha))) \to 0$, as $n \to \infty$. Since Donsker classes are Glivenko–Cantelli classes, we have the uniform convergence

$$\sup_{\alpha \in [\underline{\alpha}/2, 2\overline{\alpha}])}\left|\hat{W}_n(\pi,\alpha) - \mathbf{E}[W_i(\pi,\alpha)]\right| \to 0 \text{ a.s..}$$

Therefore, we further have

$$\sup_{\alpha \in [\underline{\alpha}/2, 2\overline{\alpha}])}\left|\left(\alpha \log\left(\hat{W}_n(\pi,\alpha)\right) + \alpha\delta\right) - \left(\alpha \log\left(\mathbf{E}[W_i(\pi,\alpha)]\right) + \alpha\delta\right)\right| \to 0 \text{ a.s.}$$

given $\mathbf{E}[W_i(\pi,\alpha)]$ is bounded away from zero in (A.2). Let

$$\epsilon = \min\left\{\underline{\alpha}/2 \log\left(\mathbf{E}[W_i(\pi,\underline{\alpha}/2)]\right) + \underline{\alpha}\delta/2, 2\overline{\alpha}\log\left(\mathbf{E}[W_i(\pi,2\overline{\alpha})]\right) + 2\overline{\alpha}\delta\right\} - \left(\alpha^* \log\left(\mathbf{E}[W_i(\pi,\alpha^*)]\right) + \alpha^*\delta\right) > 0.$$

Then, given the event

$$\left\{\sup_{\alpha \in [\underline{\alpha}/2, 2\overline{\alpha}])}\left|\left(\alpha \log\left(\hat{W}_n(\pi,\alpha)\right) + \alpha\delta\right) - \left(\alpha \log\left(\mathbf{E}[W_i(\pi,\alpha)]\right) + \alpha\delta\right)\right| < \epsilon/2\right\},$$

we have

$$\alpha^* \log\left(\hat{W}_n(\pi,\alpha)\right) + \alpha^*\delta < \min\left\{\underline{\alpha}/2 \log\left(\hat{W}_n(\pi,\underline{\alpha}/2)\right) + \underline{\alpha}\delta/2, 2\overline{\alpha}\log\left(\hat{W}_n(\pi,2\overline{\alpha})\right) + 2\overline{\alpha}\delta\right\},$$

which means $\hat{Q}_{\text{DRO}}(\pi) = -V(\hat{W}_n(\pi,\alpha))$ by the convexity of $\alpha \log\left(\hat{W}_n(\pi,\alpha)\right) + \alpha\delta$.

Finally, we complete the proof by Slutsky's lemma:

$$\begin{aligned}
\sqrt{n}\left(\hat{Q}_{\text{DRO}}(\pi) - Q_{\text{DRO}}(\pi)\right) &= \sqrt{n}\left(\hat{Q}_{\text{DRO}}(\pi) + V(\hat{W}_n(\pi,\alpha))\right) + \sqrt{n}\left(V(\mathbf{E}[W_i(\pi,\alpha)]) - V(\hat{W}_n(\pi,\alpha))\right) \\
&\Rightarrow 0 + N\left(0, (\alpha^*)^2\left(\mathbf{E}\left[W_i(\pi,\alpha^*)\right]^{-2}\mathbf{Var}\left[W_i(\pi,\alpha^*)\right]\right)\right) \\
&= N\left(0, (\alpha^*)^2\left(\mathbf{E}\left[W_i(\pi,\alpha^*)\right]^{-2}\mathbf{Var}\left[W_i(\pi,\alpha^*)\right]\right)\right).
\end{aligned}$$

□

Appendix A.4 Proof of lemma 3.5

We use a similar technique as presented in [5, Lemma 5].

Consider the function $f(\beta) = \log \mathbf{E}_{\mathbf{P}_0}[\exp(-\beta Y(\pi(X))]$. The following equalities hold

$$f(0) = 0, f'(0) = -\mathbf{E}_{\mathbf{P}_0}[Y(\pi(X))] \text{ and, } f''(0) = \mathbf{Var}(Y(\pi(X))).$$

Therefore, by second-order Taylor expansion around 0 and $\beta = 1/\alpha$, we have

$$\phi(\pi, \alpha) = \mathbf{E}[Y(\pi(X))] - \frac{\mathbf{Var}(Y(\pi(X)))}{2\alpha} - \alpha\delta + o(1/\alpha).$$

Then, the optimal solution is

$$\alpha^* = \sqrt{\frac{\mathbf{Var}(Y(\pi(X)))}{2\delta}} + o(1/\delta).$$

Appendix A.5 Proof of Proposition 4.1

We first show the upper bound of empirical optimal dual variable for both direct and stable formulations.

Lemma A4. *Let the empirical optimal dual variables* $\hat{\alpha}_n$ *and* $\hat{\alpha}_n^{\text{stable}}$ *be defined as* $\hat{\alpha}_n = \arg\max_{\alpha \geq 0} \left\{ \hat{\phi}_n(\pi, \alpha) \right\}$ *and* $\hat{\alpha}_n^{\text{stable}} = \arg\max_{\alpha \geq 0} \left\{ \hat{\phi}_n^{\text{stable}}(\pi, \alpha) \right\}$, *respectively. Suppose that Assumption 1 is imposed, we have*

$$\hat{\alpha}_n \leq \frac{M}{\log(S_n^\pi) + \delta} \text{ and } \hat{\alpha}_n^{\text{stable}} \leq \frac{M}{\delta},$$

if $\log(S_n^\pi) + \delta > 0$.

Proof. Notice that we have

$$\lim_{\alpha \to 0} -\alpha \log \hat{W}_n(\pi, \alpha) - \alpha\delta \geq \min_{i \in \{1,2,\ldots,n\}} (Y_i) \geq 0, \text{ and } \lim_{\alpha \to 0} -\alpha \log \hat{W}_n^{\text{stable}}(\pi, \alpha) - \alpha\delta \geq 0.$$

In fact, in view of the following inequalities

$$-\alpha \log \hat{W}_n(\pi, \alpha) - \alpha\delta \leq M - \alpha \log(S_n^\pi) - \alpha\delta, \text{ and } -\alpha \log \hat{W}_n^{\text{stable}}(\pi, \alpha) - \alpha\delta \leq M - \alpha\delta,$$

we have the desired results.

□

Proof of Proposition 4.1. Notice that

$$\hat{Q}_{DRO}^{\text{stable}}(\pi) = \sup_{\alpha \geq 0} \left\{ -\alpha \log \hat{W}_n(\pi, \alpha) - \alpha\delta + \alpha \log(S_n^\pi) \right\}.$$

By Lemma A4, if $S_n^\pi > \exp(-\delta/2)$, we have

$$\hat{\alpha}_n \leq \frac{2M}{\delta} \text{ and } \hat{\alpha}_n^{\text{stable}} \leq \frac{M}{\delta}.$$

and further

$$\left|\hat{Q}^{\text{stable}}_{DRO}(\pi) - \hat{Q}_{DRO}(\pi)\right| \leq \frac{2M}{\delta}\log\left(S^\pi_n\right).$$

Notice that $|\log(x)| \leq 2|x-1|$ when $x \geq \frac{1}{4}$. Hence, we have when $S^\pi_n \geq \frac{1}{4}$.

$$\left|\hat{Q}^{\text{stable}}_{DRO}(\pi) - \hat{Q}_{DRO}(\pi)\right| \leq \frac{4M}{\delta}\left|S^\pi_n - 1\right|. \tag{A.3}$$

Recall that $S^\pi_n = \sum_{i=1}^n \frac{\mathbf{1}\{\pi(X_i)=A_i\}}{\pi_0(A_i|X_i)}$ with $\mathbf{E}\left[\frac{\mathbf{1}\{\pi(X_i)=A_i\}}{\pi_0(A_i|X_i)}\right] = 1$ and $\frac{\mathbf{1}\{\pi(X_i)=A_i\}}{\pi_0(A_i|X_i)} \in [0, 1/\eta]$. By Hoeffding's inequality (see, for example, Theorem 2.8 in [3]), we have

$$\mathbf{P}\left(\left|S^\pi_n - 1\right| > t\right) \leq 2\exp\left(-2\eta^2 t^2 n\right).$$

for $t > 0$. Let $t = \log(2/\epsilon)/\left(\sqrt{2n}\eta\right)$ and $t < 1 - \max\{1/4, \exp(-\delta/2)\}$, which is equivalent to

$$n > \left(\log(2/\epsilon)/\left(1 - \max\{1/4, \exp(-\delta/2)\}\right)/\eta\right)^2/2.$$

Finally, when $n > \frac{1}{2}\left(\frac{\log(2/\epsilon)}{(1-\max\{1/4,\exp(-\delta/2)\})\eta}\right)^2$, with probability $1-\epsilon$, we obtain

$$\left|\hat{Q}^{\text{stable}}_{DRO}(\pi) - \hat{Q}_{DRO}(\pi)\right| \leq \frac{2\sqrt{2}M}{\delta\eta}\frac{\log(2/\epsilon)}{\sqrt{n}}.$$

□

References

[1] Aloisio Araujo and Evarist Giné. *The central limit theorem for real and Banach valued random variables*. John Wiley & Sons, 1980.

[2] J Frédéric Bonnans and Alexander Shapiro. Perturbation analysis of optimization problems. 2000.

[3] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.

[4] Ioannis Chatzigeorgiou. Bounds on the lambert function and their application to the outage analysis of user cooperation. *IEEE Communications Letters*, 17(8):1505–1508, 2013.

[5] Louis Faury, Ugo Tanielian, Flavian Vasile, Elena Smirnova, and Elvis Dohmatob. Distributionally robust counterfactual risk minimization. *arXiv preprint arXiv:1906.06211*, 2019.

[6] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on stochastic programming: modeling and theory*. SIAM, 2009.

[7] Aad W. van der Vaart. *Asymptotic Statistics*. Number 3. Cambridge Univ Pr, 2000.