Appendix A  Proofs

Appendix A.1  Auxiliary Results

In this section, we give all of the auxiliary results used in the proofs.

**Theorem A1** (Hoeffding’s inequality, Theorem 2.8 in [3]). Let $X_1, \ldots, X_n$ be independent random variables such that $X_i$ takes its values in $[a_i, b_i]$ almost surely for all $i < n$. Let

$$S = \sum_{i=1}^{n} (X_i - E X_i).$$

Then for every $t > 0$,

$$P(|S| > t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right).$$

**Theorem A2** (Functional central limit theorem, Corollary 7.17 in [1]). Let $S$ be a compact subspace of $\mathbb{R}^d$ and $\mathcal{C}(S)$ be the space of continuous bounded random functions on $S$ equipped with the sup-norm. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of centered i.i.d. $\mathcal{C}(S)$—valued random functions such that $E X_n^2(s) < \infty$ for some $s \in S$. Suppose there exists a constant $M$ such that

$$|X_1(s) - X_2(t)| \leq M \|s - t\| \text{ almost surely},$$

for all $s, t \in S$. Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_n \Rightarrow Y,$$

where $Y$ is a Gaussian process on $\mathcal{C}(S)$.

**Definition 1** (Directional differentiability, Gâteaux directional differentiability and Hadamard directional differentiability). Let $B_1$ and $B_2$ be Banach spaces and $G: B_1 \to B_2$ be a mapping. It is...
said that $G$ is directionally differentiable at a considered point $\mu \in B_1$ if the limits

$$G'_\mu(d) = \lim_{t \downarrow 0} \frac{G(\mu + td) - G(\mu)}{t}$$

exists for all $d \in B_1$.

Furthermore, it is said that $G$ is Gâteaux directionally differentiable at $\mu$ if the directional derivative $G'_\mu(d)$ exists for all $d \in B_1$ and $G'_\mu(d)$ is linear and continuous in $d$. For ease of notation, we also denote $D_{\mu}(\mu_0)$ be the operator $G'_{\mu_0}(\cdot)$.

Finally, it is said that $G$ is Hadamard directionally differentiable at $\mu$ if the directional derivative $G'_\mu(d)$ exists for all $d \in B_1$ and

$$G'_\mu(d) = \lim_{d' \to d} \frac{G(\mu + td') - G(\mu)}{t}.$$  

Theorem A3 (Danskin theorem, Theorem 4.13 in [2]). Let $\Theta \in \mathbb{R}^d$ be a nonempty compact set and $B$ be a Banach space. Suppose the mapping $G : B \times \Theta \to \mathbb{R}$ satisfies that $G(\mu, \theta)$ and $D_{\mu}(\mu, \theta)$ are continuous on $O_{\mu_0} \times \Theta$, where $O_{\mu_0} \subset B$ is a neighborhood around $\mu_0$. Let $\phi : B \to \mathbb{R}$ be the inf-functional $\phi(\mu) = \inf_{\theta \in \Theta} G(\mu, \theta)$ and $\Theta(\mu) = \arg \max_{\theta \in \Theta} G(\mu, \theta)$. Then, the functional $\phi$ is directionally differentiable at $\mu_0$ and

$$G'_{\mu_0}(d) = \inf_{\theta \in \Theta(\mu_0)} D_{\mu}(\mu_0, \theta) d.$$  

Theorem A4 (Delta theorem, Theorem 7.59 in [6]). Let $B_1$ and $B_2$ be Banach spaces, equipped with their Borel $\sigma$-algebras, $Y_N$ be a sequence of random elements of $B_1$, $G : B_1 \to B_2$ be a mapping, and $\tau_N$ be a sequence of positive numbers tending to infinity as $N \to \infty$. Suppose that the space $B_1$ is separable, the mapping $G$ is Hadamard directionally differentiable at a point $\mu \in B_1$, and the sequence $X_N = \tau_N (Y_N - \mu)$ converges in distribution to a random element $Y$ of $B_1$. Then,

$$\tau_N (G(Y_N) - G(\mu)) \Rightarrow G'_\mu(Y) \text{ in distribution},$$

and

$$\tau_N (G(Y_N) - G(\mu)) = G'_\mu(X_N) + o_p(1).$$

Proposition A1 (Proposition 7.57 in [6]). Let $B_1$ and $B_2$ be Banach spaces, $G : B_1 \to B_2$, and $\mu \in B_1$. Then the following hold: (i) If $G(\cdot)$ is Hadamard directionally differentiable at $\mu$, then the directional derivative $G'_{\mu}(\cdot)$ is continuous. (ii) If $G(\cdot)$ is Lipschitz continuous in a neighborhood of $\mu$ and directionally differentiable at $\mu$, then $G(\cdot)$ is Hadamard directionally differentiable at $\mu$.  

2
Appendix A.2 Proof of Lemma 3.3

Proof of Lemma 3.3. The closed form expression of $\frac{\partial}{\partial \alpha} \hat{\phi}_n(\pi, \alpha)$ and $\frac{\partial^2}{\partial \alpha^2} \hat{\phi}_n(\pi, \alpha)$ follows from elementary algebra. By the Cauchy Schwartz’s inequality, we have

$$\left( \sum_{i=1}^{n} Y_i(A_i) W_i(\pi, \alpha) \right)^2 \leq n \hat{W}_n(\pi, \alpha) \sum_{i=1}^{n} Y_i^2(A_i) W_i(\pi, \alpha)$$

Therefore, it follows that $\frac{\partial^2}{\partial \alpha^2} \hat{\phi}_n(\pi, \alpha) \leq 0$. Note that the Cauchy Schwartz’s inequality is actually an equality if and only if $Y_i^2(A_i) W_i(\pi, \alpha) = c W_i(\pi, \alpha)$ if $W_i(\pi, \alpha) \neq 0$ for some constant $c$ independent of $i$. Since the above condition is violated if $\{Y_i(A_i)1\{\pi(X_i) = A_i\}\}_{i=1}^{N}$ has at least two different non-zero values, we have in this case $\hat{\phi}_n(\pi, \alpha)$ is strictly-concave in $\alpha$. □

Appendix A.3 Proof of Theorem 3.4

We first give the upper and lower bound for $\alpha^*$ and in Lemmas A1 and A3.

Lemma A1 (Upper bound of $\alpha^*$). Suppose that Assumption 1 is imposed, we have the optimal dual solution $\alpha^* \leq \tilde{\alpha} = M/\delta$.

Proof. First note that $\inf_{P \in \mathcal{U}} P_0(\delta) \mathbb{E}_P[Y(\pi(X))] \geq \min_{i \in [n]}(Y_i) \geq 0$ and

$$-\alpha \log \mathbb{E}_{P_0}[\exp \left(-Y(\pi(X))/\alpha\right)] - \alpha \delta \leq M - \alpha \delta.$$ 

$M - \alpha \delta \geq 0$ gives the upper bound $\alpha^* \leq \tilde{\alpha} = M/\delta$. □

To prove the lower bound of $\alpha^*$, we need the following technical lemma.

Lemma A2. For $c > 0$ and $c \exp(b) < 1/e$, the smallest root of equation $x \log(x) + bx + c = 0$ is

$$x^* = -\frac{c}{W_{-1}(-c \exp(b))},$$

where $W_{-1}(z)$ is the root of the equation for $w$ in $w \exp(w) = z$ with $z < -1$. Furthermore, we have

$$x^* \in \left( -\frac{c(-\log(c-b-1))}{2 \left((\log(c+b+1)^2 + 1\right)}, \frac{c}{c}\right).$$

Proof. Let $f(x) = x \log(x) + bx + c$ and $x = -c/w$ for $w < 0$. Then,

$$f(x) = 0$$
\[ (-c/w) \log(-c/w) + b(-c/w) + c = 0 \]
\[ w = \log(-c/w) + b \]
\[ w \exp(w) = -c \exp(b). \]

Since \( x = -c/w \) is one-to-one mapping, we have roots of \( f(x) = 0 \) have one-to-one correspondence with roots of \( w \exp(w) = -c \exp(b) \). Note that when \( -c \exp(b) \in (-1/e, 0) \), \( w \exp(w) = -c \exp(b) \) has 2 roots, one is in \((-1, 0)\) and the other one is in \((-1, +\infty)\). Therefore, the small root of \( f(x) = 0 \) is of the form
\[ x^* = -\frac{c}{W_{-1}(-c \exp(b))}. \]

[4] shows that
\[ -1 - \sqrt{2u} - u < W_{-1}(-\exp(-u - 1)) < -1, \text{ for } u > 0. \]

Notice that \( u + 1 - \sqrt{2u} \geq \frac{1}{2}u \), we have
\[ \frac{cu}{2(u^2 + 1)} < -c/W_{-1}(-\exp(-u - 1)) < c. \]

Let \( u = -\log c - b - 1 \), we have that
\[ x^* \in \left( \frac{c(-\log c - b - 1)}{2((\log(c) + b + 1)^2 + 1)}, c \right). \]

**Lemma A3** (Lower bound of \( \alpha^* \)). *Suppose that Assumption [7] is imposed, we have*
\[ \alpha^* \geq \alpha = \frac{\bar{b}^{-1} \exp(-\delta - 1) (\log(\bar{b}) - \log(b))}{2 \left( (\log(b) - \log(b))^2 + 1 \right)}. \]  \hfill (A.1)

*Proof.* Denote the density of \( Y(\pi(X)) \) by \( f_\pi \). It is easy to see that \( \bar{b} \geq f_\pi(y) \geq b \) for any \( \pi \in \Pi \) and any \( y \in [0, M] \). First notice that
\[ \alpha^* = \arg\max_{\alpha \geq 0} \left\{-\alpha \log \mathbb{E}_{P_0} \left[ \exp(-Y(\pi(X))/\alpha) \right] - \alpha \delta \right\} = \arg\min_{\alpha \geq 0} \left\{ \alpha \log \mathbb{E}_{P_0} \left[ \exp(-Y(\pi(X))/\alpha) \right] + \alpha \delta \right\}. \]

Then, we have
\[ \alpha \log \mathbb{E}_{P_0} \left[ \exp(-Y(\pi(X))/\alpha) \right] + \alpha \delta = \alpha \log \left[ \int_0^M \exp(-y/\alpha) f_\pi(y) \, dy \right] + \alpha \delta \]
\[ \leq \alpha \log \left[ \bar{b} (\alpha - \exp(-M/\alpha) \alpha) \right] + \alpha \delta \]
\[ \leq \alpha \log (\alpha) + \alpha (\delta + \log(b)). \]
Therefore, we have

$$\min_{\alpha \geq 0} \{ \log \mathbf{E}_{\mathbf{p}_0} [\exp (-Y(\pi(X)/\alpha))] + \alpha \delta \} \leq \min_{\alpha \geq 0} \{ \alpha \log (\alpha) + \alpha (\delta + \log(b)) \} = -\bar{b}^{-1} \exp(-\delta - 1).$$

On the other hand, we have

$$\alpha \log \mathbf{E}_{\mathbf{p}_0} [\exp (-Y(\pi(X)/\alpha))] + \alpha \delta = \alpha \log \left[ \int_0^{\bar{M}} \exp (-y/\alpha) (f_\pi(y) - b + \bar{b}) \, dy \right] + \alpha \delta \geq \alpha \log [\exp(-M/\alpha) (1 - bM) + \bar{b} (\alpha - \exp(-M/\alpha) \alpha)] + \alpha \delta$$

$$= \alpha \log [\exp(-M/\alpha) (1 - bM - b\alpha) + \bar{b} \alpha] + \alpha \delta$$

Since $2bM \leq 1$, we have when $\alpha \leq \frac{1}{2\bar{b}}$, $1 - bM - b\alpha \geq 0$ and thus

$$\alpha \log \mathbf{E}_{\mathbf{p}_0} [\exp (-Y(\pi(X)/\alpha))] + \alpha \delta \geq \alpha \log (\alpha) + \alpha (\delta + \log(b)).$$

Consider the function $f(\alpha) = \alpha \log (\alpha) + \alpha (\delta + \log(b))$. $f(\alpha)$ is decreasing when $\alpha \in [0, (\bar{b})^{-1} \exp(-\delta - 1)]$. Notice that

$$\bar{b}^{-1} \exp(-\delta - 1) \exp (\delta + \log(b)) \leq 1/e.$$

By applying Lemma A2, we have the smallest root of $f(\alpha) = -\bar{b}^{-1} \exp(-\delta - 1)$, $\alpha_0$ is in

$$\alpha_0 \in \left( \frac{\bar{b}^{-1} \exp(-\delta - 1) (\log(b) - \log(\bar{b}))}{2 \left( (\log(b) - \log(\bar{b}))^2 + 1 \right)}, \bar{b}^{-1} \exp(-\delta - 1) \right).$$

Notice that

$$\frac{\bar{b}^{-1} \exp(-\delta - 1) (\log(b) - \log(\bar{b}))}{2 \left( (\log(b) - \log(\bar{b}))^2 + 1 \right)} < \frac{1}{2\bar{b}},$$

and thus

$$\alpha \log \mathbf{E}_{\mathbf{p}_0} [\exp (-Y(\pi(X)/\alpha))] + \alpha \delta \geq \min_{\alpha \geq 0} \{ \log \mathbf{E}_{\mathbf{p}_0} [\exp (-Y(\pi(X)/\alpha))] + \alpha \delta \}$$

for $\alpha \in \left[ 0, \frac{\bar{b}^{-1} \exp(-\delta - 1) (\log(b) - \log(\bar{b}))}{2 \left( (\log(b) - \log(\bar{b}))^2 + 1 \right)} \right],$

which concludes (A.1).

Now we are ready to show the proof of Theorem 3.4.
Proof of Theorem 3.4. Notice that

\[ \sqrt{n} \left( \hat{W}_n(\pi, \alpha) - \mathbb{E}[W_i(\pi, \alpha)] \right) \Rightarrow Z(\alpha), \]

where

\[ Z(\alpha) \sim N(0, \text{Var}[W_i(\pi, \alpha)]). \]

Since \( W_i(\pi, \alpha) \) is Lipschitz continuous if \( \alpha \in [\alpha/2, 2\alpha] \), we have \( W_i(\pi, \alpha) \) is a \( P \)-Donsker class (see, for example, Corollary 7.17 in [1] and Chapter 19 in [7]). Therefore,

\[ \sqrt{n} \left( \hat{W}_n(\pi, \cdot) - \mathbb{E}[W_i(\pi, \cdot)] \right) \Rightarrow Z(\cdot), \]

in a Banach space \( \mathcal{C}([\alpha/2, 2\alpha]) \) of continuous functions \( \psi : [\alpha/2, 2\alpha] \rightarrow \mathbb{R} \) equipped with the the sup-norm \( \| \psi \| = \sup_{\alpha \in [\alpha/2, 2\alpha]} \psi(\alpha) \). \( Z \) is a random element in \( \mathcal{C}([\alpha/2, 2\alpha]) \).

Define the functionals

\[ G(\psi, \alpha) = \alpha \log(\psi(\alpha)) + \alpha \delta, \quad \text{and} \quad V(\psi) = \inf_{\alpha \in [\alpha/2, 2\alpha]} G(\psi, \alpha), \]

for \( \psi > 0 \). By the Danskin theorem (Theorem 4.13 in [2]), \( V(\cdot) \) is directionally differentiable at any \( \mu \in \mathcal{C}([\alpha/2, 2\alpha]) \) with \( \mu > 0 \) and

\[ V'_\mu(\nu) = \inf_{\alpha \in \tilde{X}(\mu)} \alpha (1/\mu(\alpha)) \nu(\alpha), \quad \forall \nu \in \mathcal{C}([\alpha/2, 2\alpha]), \]

where \( \tilde{X}(\mu) = \arg\min_{\alpha \in [\alpha/2, 2\alpha]} \alpha \log(\mu(\alpha)) + \alpha \delta \) and \( V'_\mu(\nu) \) is the directional derivative of \( V(\cdot) \) at \( \mu \) in the direction of \( \nu \). On the other hand, \( V(\psi) \) is Lipschitz continuous if \( \psi(\cdot) \) is bounded away from zero. Notice that

\[ \mathbb{E}[W_i(\pi, \alpha)] = \mathbb{E}[\exp(-Y(\pi(x))/\alpha)] \geq \exp(-2M/\alpha). \quad (A.2) \]

Therefore, \( V(\cdot) \) is Hadamard directionally differentiable at \( \mu = \mathbb{E}[W_i(\pi, \cdot)] \) (see, for example, Proposition 7.57 in [3]). By the Delta theorem (Theorem 7.59 in [6]), we have

\[ \sqrt{n} \left( V(\hat{W}_n(\pi, \cdot)) - V(\mathbb{E}[W_i(\pi, \cdot)]) \right) \Rightarrow V'_{\mathbb{E}[W_i(\pi, \cdot)]}(Z). \]

Furthermore, we know that \( \log(\mathbb{E}(\exp(-\beta Y))) \) is strictly convex w.r.t \( \beta \) given \( \text{Var}(Y) > 0 \) and \( xf(1/x) \) is strictly convex if \( f(x) \) is strictly convex. Therefore, \( \alpha \log(\mathbb{E}[W_i(\pi, \alpha)]) + \alpha \delta \) is strictly convex for \( \alpha > 0 \) and thus

\[ V'_{\mathbb{E}[W_i(\pi, \cdot)]}(Z) = \alpha^* (1/\mathbb{E}[W_i(\pi, \alpha^*)]) Z(\alpha^*) \overset{d}{=} N \left( 0, (\alpha^*)^2 \left( \mathbb{E}[W_i(\pi, \alpha^*)]^{-2} \text{Var}[W_i(\pi, \alpha^*)] \right) \right). \]
By Lemma 3.1, we have that
\[
\hat{Q}_{\text{DRO}}(\pi) = -\inf_{\alpha \geq 0} \left( \alpha \log \left( \hat{W}_n(\pi, \alpha) \right) + \alpha \delta \right),
\]
and
\[
Q_{\text{DRO}}(\pi) = -\inf_{\alpha \geq 0} \left( \alpha \log \left( \mathbb{E}[W_i(\pi, \alpha)] \right) + \alpha \delta \right) = -\mathbb{V}(\mathbb{E}[W_i(\pi, \alpha)]).
\]
We remain to show that
\[
P(\hat{Q}_{\text{DRO}}(\pi) \neq -\mathbb{V}(\hat{W}_n(\pi, \alpha))) \to 0, \text{ as } n \to \infty.
\]

Since Donsker classes are Glivenko–Cantelli classes, we have the uniform convergence
\[
\sup_{\alpha \in [\alpha/2, 2\alpha]} \left| \hat{W}_n(\pi, \alpha) - \mathbb{E}[W_i(\pi, \alpha)] \right| \to 0 \text{ a.s.}
\]
Therefore, we further have
\[
\sup_{\alpha \in [\alpha/2, 2\alpha]} \left| \left( \alpha \log \left( \hat{W}_n(\pi, \alpha) \right) + \alpha \delta \right) - \left( \alpha \log \left( \mathbb{E}[W_i(\pi, \alpha)] \right) + \alpha \delta \right) \right| \to 0 \text{ a.s.}
\]
given \(\mathbb{E}[W_i(\pi, \alpha)]\) is bounded away from zero in (A.2). Let
\[
\epsilon = \min \left\{ \alpha/2 \log \left( \mathbb{E}[W_i(\pi, \alpha/2)] \right) + \alpha \delta/2, 2\alpha \log \left( \mathbb{E}[W_i(\pi, 2\alpha)] \right) + 2\alpha \delta \right\} - \left( \alpha^* \log \left( \mathbb{E}[W_i(\pi, \alpha^*)] \right) + \alpha^* \delta \right) > 0.
\]
Then, given the event
\[
\left\{ \sup_{\alpha \in [\alpha/2, 2\alpha]} \left| \left( \alpha \log \left( \hat{W}_n(\pi, \alpha) \right) + \alpha \delta \right) - \left( \alpha \log \left( \mathbb{E}[W_i(\pi, \alpha)] \right) + \alpha \delta \right) \right| < \epsilon / 2 \right\},
\]
we have
\[
\alpha^* \log \left( \hat{W}_n(\pi, \alpha) \right) + \alpha^* \delta < \min \left\{ \alpha/2 \log \left( \hat{W}_n(\pi, \alpha/2) \right) + \alpha \delta/2, 2\alpha \log \left( \hat{W}_n(\pi, 2\alpha) \right) + 2\alpha \delta \right\},
\]
which means \(\hat{Q}_{\text{DRO}}(\pi) = -\mathbb{V}(\hat{W}_n(\pi, \alpha))\) by the convexity of \(\alpha \log \left( \hat{W}_n(\pi, \alpha) \right) + \alpha \delta\).

Finally, we complete the proof by Slutsky’s lemma:
\[
\sqrt{n} \left( \hat{Q}_{\text{DRO}}(\pi) - Q_{\text{DRO}}(\pi) \right) = \sqrt{n} \left( \hat{Q}_{\text{DRO}}(\pi) - Q_{\text{DRO}}(\pi) \right) + \sqrt{n} \left( V(\mathbb{E}[W_i(\pi, \alpha)]) - V(\hat{W}_n(\pi, \alpha)) \right) \to 0 + N \left( 0, (\alpha^*)^2 \left( \mathbb{E}[W_i(\pi, \alpha^*)] \right)^2 \mathbb{V}[W_i(\pi, \alpha^*)] \right) + N \left( 0, (\alpha^*)^2 \left( \mathbb{E}[W_i(\pi, \alpha^*)] \right)^2 \mathbb{V}[W_i(\pi, \alpha^*)] \right).
\]

\[\square\]

**Appendix A.4  Proof of lemma 3.5**

We use a similar technique as presented in [5, Lemma 5].
Consider the function \( f(\beta) = \log \mathbb{E}_{\pi_0} [\exp(-\beta Y(X))] \). The following equalities hold

\[
\begin{align*}
  f(0) = 0, \quad f'(0) &= -\mathbb{E}_{\pi_0} [Y(\pi(X))] \quad \text{and} \quad f''(0) = \text{Var} (Y(\pi(X))).
\end{align*}
\]

Therefore, by second-order Taylor expansion around 0 and \( \beta = 1/\alpha \), we have

\[
\phi(\pi, \alpha) = \mathbb{E}[Y(\pi(X))] - \frac{\text{Var}(Y(\pi(X)))}{2\alpha} - \alpha \delta + o(1/\alpha).
\]

Then, the optimal solution is

\[
\alpha^* = \sqrt{\frac{\text{Var}(Y(\pi(X)))}{2\delta}} + o(1/\delta).
\]

### Appendix A.5 Proof of Proposition 4.1

We first show the upper bound of empirical optimal dual variable for both direct and stable formulations.

**Lemma A4.** Let the empirical optimal dual variables \( \hat{\alpha}_n \) and \( \hat{\alpha}^\text{stable}_n \) be defined as \( \hat{\alpha}_n = \arg \max_{\alpha \geq 0} \left\{ \hat{\phi}_n(\pi, \alpha) \right\} \) and \( \hat{\alpha}^\text{stable}_n = \arg \max_{\alpha \geq 0} \left\{ \hat{\phi}^\text{stable}_n(\pi, \alpha) \right\} \), respectively. Suppose that Assumption 1 is imposed, we have

\[
\hat{\alpha}_n \leq \frac{M}{\log (S^{\pi}_n) + \delta} \quad \text{and} \quad \hat{\alpha}^\text{stable}_n \leq \frac{M}{\delta},
\]

if \( \log (S^{\pi}_n) + \delta > 0 \).

**Proof.** Notice that we have

\[
\lim_{\alpha \to 0} -\alpha \log \hat{W}_n(\pi, \alpha) - \alpha \delta \geq \min_{i \in \{1, 2, \ldots, n\}} (Y_i) \geq 0, \quad \text{and} \quad \lim_{\alpha \to 0} -\alpha \log \hat{W}^\text{stable}_n(\pi, \alpha) - \alpha \delta \geq 0.
\]

In fact, in view of the following inequalities

\[
-\alpha \log \hat{W}_n(\pi, \alpha) - \alpha \delta \leq M - \alpha \log (S^{\pi}_n) - \alpha \delta, \quad \text{and} \quad -\alpha \log \hat{W}^\text{stable}_n(\pi, \alpha) - \alpha \delta \leq M - \alpha \delta,
\]

we have the desired results.

\( \square \)

**Proof of Proposition 4.1.** Notice that

\[
\hat{Q}^\text{stable}(\pi) = \sup_{\alpha \geq 0} \left\{ -\alpha \log \hat{W}_n(\pi, \alpha) - \alpha \delta + \alpha \log (S^{\pi}_n) \right\}.
\]

By Lemma A4, if \( S^{\pi}_n > \exp(-\delta/2) \), we have

\[
\hat{\alpha}_n \leq \frac{2M}{\delta} \quad \text{and} \quad \hat{\alpha}^\text{stable}_n \leq \frac{M}{\delta}.
\]
and further

$$\left| \hat{Q}_{DRO}^{\text{stable}}(\pi) - \hat{Q}_{DRO}(\pi) \right| \leq \frac{2M}{\delta} \log \left( S_\pi^n \right).$$

Notice that $|\log (x)| \leq 2 |x - 1|$ when $x \geq \frac{1}{4}$. Hence, we have when $S_\pi^n \geq \frac{1}{4}$.

$$\left| \hat{Q}_{DRO}^{\text{stable}}(\pi) - \hat{Q}_{DRO}(\pi) \right| \leq \frac{4M}{\delta} \left| S_\pi^n - 1 \right|. \quad (A.3)$$

Recall that $S_\pi^n = \sum_{i=1}^n \mathbb{1}_{\{\pi(X_i) = A_i\}} \frac{\mathbb{1}_{\{\pi(X_i) = A_i\}}}{\pi_0(A_i|X_i)}$ with $\mathbb{E} \left[ \mathbb{1}_{\{\pi(X_i) = A_i\}} \frac{\mathbb{1}_{\{\pi(X_i) = A_i\}}}{\pi_0(A_i|X_i)} \right] = 1$ and $\mathbb{1}_{\{\pi(X_i) = A_i\}} \in [0, 1/\eta]$. By Hoeffding’s inequality (see, for example, Theorem 2.8 in [3]), we have

$$P \left( |S_\pi^n - 1| > t \right) \leq 2 \exp \left( -\frac{2t^2}{n\eta} \right) \quad \text{for } t > 0.$$ 

Let $t = \log(2/\epsilon) / \left( \sqrt{2n\eta} \right)$ and $t < 1 - \max \left\{ 1/4, \exp(-\delta/2) \right\}$, which is equivalent to

$$n > \frac{\log(2/\epsilon)}{(1 - \max \left\{ 1/4, \exp(-\delta/2) \right\})} / \eta^2 / 2.$$ 

Finally, when $n > \frac{1}{2} \left( \frac{\log(2/\epsilon)}{(1 - \max \left\{ 1/4, \exp(-\delta/2) \right\})} \right)^2$, with probability $1 - \epsilon$, we obtain

$$\left| \hat{Q}_{DRO}^{\text{stable}}(\pi) - \hat{Q}_{DRO}(\pi) \right| \leq \frac{2\sqrt{2}M \log(2/\epsilon)}{\delta \eta} \sqrt{n}.$$ 

References


