Error-Bounded Correction of Noisy Labels
— Supplementary Material —

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1. Additional (Synthetic) Experiment for Validation of the Bound

In Section 2.3 of the submitted manuscript, we used the output of deep neural networks f as an approximation of η on the CIFAR10 dataset. We provided empirical estimates of the constants C and λ in the Tsybakov condition for η, as well as estimates of the probability Pr[ŷ = h*(x), f̂(x) < ∆].

In this section, we provide additional experiments on a synthetic data set generated using a mixture-of-Gaussians distribution. In this ideal setting, we know η, τ01, τ10, η exactly. We can a) use η̂ as the classifier and b) evaluate the constants in Tsybakov condition for η in order to evaluate the upper bound in Theorem 1.

**Estimation of Tsybakov condition constants.** We let Pr(x) be a mixture of Gaussian distribution in a 10 dimensional feature space, x ∼ 1/2N(0, I_{10×10}) + 1/2N(1, I_{10×10}). We sample from the two components with equal probability. If x comes from component N(0, I_{10×10}), it is given label 0. Otherwise, if x comes from component N(1, I_{10×10}), it is given label 1. The true conditional distribution is η(x) = \exp\{-\frac{1}{2}||x||^2\} / \exp\{-\frac{1}{2}||x||^2\} + \exp\{-\frac{1}{2}||x-1||^2\}.

Following the idea of our experiment on CIFAR10 in the manuscript (Section 2.4), we estimate Pr [η(x) − 1/2 ≤ t] for values of t sampled between 0 and 0.9 using the empirical frequency \( p_t = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{|η(x)−1/2|≤t\}}(x) \). Note that if the Tsybakov condition is tight, log(p_t) approximates log(Ct^λ). The samples for log(t) and correspondingly, log(Ct^λ) ≈ log(p_t) are drawn as blue dots in Figure 1(a). The ordinary least square (OLS) linear regression results is drawn as a red line. We found the estimated values of C and λ to be 0.58 and 1.27 respectively. The estimation is high is confidence: the determinant coefficient \( R^2 \) equals 0.904, and we have a p-value which is less than 10⁻⁴.

**Estimation of the error bound, and its tightness.** We also introduce label noise using predefined transition probability τ01 and τ10. We can estimate C and λ as mentioned above, and know τ01, τ10, η(x), and thus, η̂(x). Therefore we can evaluate the error bound in Theorem 1. We plot the error bound as a function of ε in Figures 1(b) and (c) (drawn green curves).

Finally, we assume a perfect noisy classifier f = η̂. In other words, ε = 0. We empirically show that when f(x) < ∆, the probability of η̂ being correct (i.e., ŷ = h*(x)) is zero (blue lines in Figures 1(b) and (c)).

**Validation of the label-correction algorithm.** To the same synthetic dataset, we also apply our LRT-Correction algorithm and validate the bound in Corollary 1. Since we know η̂(x), τ01 and τ10, we calculate the correction error bound of Corollary 1 in closed form. We draw the bound w.r.t. the error ε in orange curves in Figure 2. Finally, we run our label correction algorithm using the perfect noisy classifier f = η̂ and validate that the corrected labels are very close to clean (the success rate is limited by the asymmetry level of the noise pattern). See blue lines in Figure 2.
Figure 1. Synthetic experiment using Mixture of Gaussian at noise level 20%. (a): Check of Tsybakov condition using linear regression, where y-axis is the proportion of data points at distance t from decision boundary. (b): Proportion of labels that are not correct (not consistent with Bayes optimal decision rule) and the proposed upper bound. (c): Same as (b) but labels are corrupted with asymmetric noise. (d): t-SNE of the clean data. (e): t-SNE of the data with symmetric noise. (f): t-SNE of the data with asymmetric noise.

Figure 2. Performance of LRT algorithm given $\tilde{\eta}(x)$ v.s the proposed upper bound. (a): Symmetric noise ($\tau_{10} = \tau_{01} = 0.3$). (b): Asymmetric noise ($\tau_{10} = 0.2, \tau_{01} = 0.3$). (c): Asymmetric noise ($\tau_{10} = 0.1, \tau_{01} = 0.3$). (d): Asymmetric noise ($\tau_{10} = 0.3, \tau_{01} = 0$)
2. Proof of Theorem 2

Define $m_x := \arg \max_i f_i(x)$, $u_x := \arg \max_i \eta_i(x)$ and $s_x := \arg \max_{i \neq u_x} \eta_i(x)$. Let $[N_c] := \{1, 2, \cdots, N_c\}$. Finally, define $\epsilon_i(x) := |f_i(x) - \hat{\eta}_i(x)|$ and $\epsilon := \max_{x, i} \epsilon_i(x)$.

For multi-class scenario, we know $\forall i \in [N_c], \hat{\eta}_i(x) = \sum_j \tau_{ji} \eta_j(x)$. We also restate the multi-class Tsybakov condition here:

**Assumption 1** (Multi-class Tsybakov Condition). $\exists C, \lambda > 0$ and $t_0 \in (0, 1]$ such that for all $t \leq t_0$,

$$\Pr [ ||\eta_{u_x}(x) - \eta_{s_x}(x) || \leq t] \leq Ct^\lambda$$

**Theorem 2.** Assume $\eta(x)$ fulfills multi-class Tsybakov condition for constant $C, \lambda > 0$ and $t_0 \in (0, 1]$. Assume that $\epsilon \leq t_0 \min_i \tau_{i,i}$. For $\Delta = \min \left[1, \min_x \left[ \tau_{\hat{y}, \hat{y}} \eta_{s_x}(x) + \sum_{j \neq \hat{y}} \tau_{j, \hat{y}} \eta_j(x) \right] \right]$:

$$\Pr_{(x, \hat{y}) \sim D} \left[ \hat{y} = h^*(x), f_{\hat{y}}(x) < \Delta \right] \leq C [O(\epsilon)]^\lambda$$

**Proof.**

$$\Pr_{(x, \hat{y}) \sim D} \left[ \hat{y} = h^*(x), f_{\hat{y}}(x) < \Delta \right] = \Pr [ \hat{\eta}_{\hat{y}}(x) \geq \eta_{s_x}(x), f_{\hat{y}}(x) < \Delta ]$$

$$\leq \Pr [ \hat{\eta}_{\hat{y}}(x) \geq \eta_{s_x}(x), \hat{\eta}_{\hat{y}}(x) < \Delta + \epsilon ]$$

$$\leq \Pr [ \hat{\eta}_{\hat{y}}(x) \geq \eta_{s_x}(x), \hat{\eta}_{\hat{y}}(x) < \Delta + \epsilon ]$$

$$= \Pr \left[ \hat{\eta}_{\hat{y}}(x) \geq \eta_{s_x}(x), \sum_j \tau_{j, \hat{y}} \eta_j(x) < \Delta + \epsilon \right]$$

$$= \Pr \left[ \hat{\eta}_{\hat{y}}(x) \geq \eta_{s_x}(x), \eta_{\hat{y}}(x) < \frac{\Delta - \sum_j \tau_{j, \hat{y}} \eta_j(x)}{\tau_{\hat{y}, \hat{y}}} + \frac{\epsilon}{\tau_{\hat{y}, \hat{y}}} \right]$$

$$= \Pr \left[ \eta_{s_x}(x) \leq \eta_{\hat{y}}(x) < \frac{\Delta - \sum_{j \neq \hat{y}} \tau_{j, \hat{y}} \eta_j(x)}{\tau_{\hat{y}, \hat{y}}} + \frac{\epsilon}{\tau_{\hat{y}, \hat{y}}} \right]$$

(1)

Remember that $\Delta = \min \left[1, \min_x \left[ \tau_{\hat{y}, \hat{y}} \eta_{s_x}(x) + \sum_{j \neq \hat{y}} \tau_{j, \hat{y}} \eta_j(x) \right] \right] \leq \tau_{\hat{y}, \hat{y}} \eta_{s_x}(x) + \sum_{j \neq \hat{y}} \tau_{j, \hat{y}} \eta_j(x)$. Then if we substitute $\Delta$ in (1) with $\tau_{\hat{y}, \hat{y}} \eta_{s_x}(x) + \sum_{j \neq \hat{y}} \tau_{j, \hat{y}} \eta_j(x)$, continuing the derivation of (1), we will end up with:
Lemma 1. (Algorithm Multiclass-Theorem Guarantee). Assume \( \eta(x) \) fulfills multi-class Tsybakov condition for constant \( C > 0, \lambda > 0 \) and \( t_0 \in (0,1) \). Assume that \( \epsilon \leq t_0 \min_i \tau_{i,i} \), which implies that \( \frac{\epsilon}{\tau_{y,y}} \leq t_0 \). This complete the proof for this case.

3. Proof of Theorem 3

Lemma 1. (Algorithm Multiclass-Theorem Guarantee). Assume \( \eta(x) \) fulfills multi-class Tsybakov condition for constant \( C > 0, \lambda > 0 \) and \( t_0 \in (0,1) \). Assume that \( \epsilon \leq t_0 \min_i \tau_{i,i} \). Let \( \tilde{y}_{\text{new}} \) denote the output of the LRT-Correction with \( x, \tilde{y}_x, f \), and the given \( \delta \), then:

1. Sensitivity Optimized Critical Value. Let \( \delta = \min_x \left[ \frac{\tau_{y,y} \eta_{x}(x) + \sum_{j \neq y} \tau_{j,y} \eta_j(x)}{f_{m_x}(x)} \right] \) then:

\[
\Pr_{(x,y) \sim \mathcal{D}} [\tilde{y}_{\text{new}} \neq h^*(x), \tilde{y} \text{ is rejected}] \leq C [O(\epsilon)]^\lambda + \Pr_{(x,y) \sim \mathcal{D}} [u_x \neq m_x, u_x \neq \tilde{y}]
\]

2. Specificity Optimized Critical Value. Let \( \delta = \max_x \left[ \frac{f_{\tilde{y}}(x)}{\tau_{m_x,m_x} \eta_{x}(x) + \sum_{j \neq m_x} \tau_{j,m_x} \eta_j(x)} \right] \) then:

\[
\Pr_{(x,y) \sim \mathcal{D}} [\tilde{y}_{\text{new}} \neq h^*(x), \tilde{y} \text{ is accepted}] \leq C [O(\epsilon)]^\lambda + \Pr_{(x,y) \sim \mathcal{D}} [u_x \neq m_x, u_x \neq \tilde{y}]
\]

Proof. First look at cases where \( \tilde{y} \) is rejected.

\[
\Pr [\tilde{y}_{\text{new}} \neq h^*(x), \tilde{y} \text{ is rejected}]
= \Pr [\tilde{y}_{\text{new}} \neq h^*(x), \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta]
= \Pr [\tilde{y}_{\text{new}} = m_x \neq h^*(x) = \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta] + \Pr [\tilde{y}_{\text{new}} = m_x \neq h^*(x) = u_x, u_x \neq \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta]
\leq \Pr [h^*(x) = \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta] + \Pr [\tilde{y}_{\text{new}} = m_x \neq h^*(x) = u_x, u_x \neq \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta]
\]

For the first term in (2), we have:

\[
\Pr [h^*(x) = \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m_x}(x)} < \delta] = \Pr [h^*(x) = \tilde{y}, f_{\tilde{y}}(x) < \delta f_{m_x}(x)]
\leq \Pr [\eta_{\tilde{y}}(x) \geq \eta_{m_x}(x), \eta_{\tilde{y}}(x) - \epsilon < \delta f_{m_x}(x)]
\leq \Pr \left[ \eta_{m_x}(x) \leq \eta_{\tilde{y}}(x) + \frac{\delta f_{m_x}(x) - \sum_{j \neq \tilde{y}} \tau_{j,y} \eta_j(x)}{\tau_{y,y}} + \frac{\epsilon}{\tau_{y,y}} \right]
\]
We substitute $\delta$ in (3) with $\frac{\tau_{y,y} \eta_\text{new}(x) + \sum_{j \neq y} \tau_{j,y} \eta_j(x)}{f_{\text{max}}(x)}$ and continue the calculation:

$$
\Pr \left[ h^*(x) = \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} < \delta \right] \\
\leq \Pr \left[ \eta_{\text{new}}(x) \leq \eta_{\tilde{y}}(x) < \frac{\delta f_{\text{max}}(x) - \sum_{j \neq \tilde{y}} \tau_{j,\tilde{y}} \eta_j(x)}{\tau_{y,\tilde{y}}} + \frac{\epsilon}{\tau_{y,\tilde{y}}} \right] \\
\leq \Pr \left[ \eta_{\text{new}}(x) \leq \eta_{\tilde{y}}(x) \leq \eta_{\text{new}}(x) + \frac{\epsilon}{\tau_{y,\tilde{y}}} \right] \\
\leq C \left( \frac{\epsilon}{\tau_{y,\tilde{y}}} \right)^{\lambda} 
$$

(4)

In (4), the Tsybakov condition holds here because $\epsilon \leq t_0 \min_i \tau_{ii}$, which implies $\frac{\epsilon}{\tau_{y,\tilde{y}}} \leq t_0$.

For the second term in (2), we have:

$$
\Pr \left[ \tilde{y}_{\text{new}} = m_x \neq h^*(x) = u_x, u_x \neq \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} < \delta \right] \leq \Pr \left[ u_x \neq m_x, u_x \neq \tilde{y} \right] 
$$

(5)

for which our algorithm currently doesn’t have a good way to deal with and we will leave it as future research problem.

Finally, summarize every piece and we finished the proof for cases where $\tilde{y}$ is rejected:

$$
\Pr \left[ \tilde{y}_{\text{new}} \neq h^*(x), \tilde{y} \text{ is rejected} \right] \leq (4) + (5) \\
\leq C \left( \frac{\epsilon}{\tau_{u_x,\text{new}}} \right)^{\lambda} + \Pr \left[ u_x \neq m_x, u_x \neq \tilde{y} \right] \\
= C [O(\epsilon)]^{\lambda} + \Pr \left[ u_x \neq m_x, u_x \neq \tilde{y} \right]
$$

For cases where $\tilde{y}$ is accepted:

$$
\Pr \left[ \tilde{y}_{\text{new}} \neq h^*(x), \tilde{y} \text{ is accepted} \right] = \Pr \left[ \tilde{y}_{\text{new}} \neq h^*(x), \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} \geq \delta \right] \\
= \Pr \left[ \tilde{y}_{\text{new}} = \tilde{y} \neq h^*(x) = m_x, \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} \geq \delta \right] + \Pr \left[ \tilde{y}_{\text{new}} = \tilde{y} \neq h^*(x), m_x \neq h^*(x), \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} \geq \delta \right] \\
= \Pr \left[ \eta_{\text{new}}(x) \geq \eta_{\text{new}}(x), f_{\text{max}}(x) \leq \frac{f_{\tilde{y}}(x)}{\delta} \right] + \Pr \left[ u_x \neq m_x, u_x \neq \tilde{y} \right] 
$$

(6)

For the first term in (6), we have:

$$
\Pr \left[ \eta_{\text{new}}(x) \geq \eta_{\text{new}}(x), f_{\text{max}}(x) \leq \frac{f_{\tilde{y}}(x)}{\delta} \right] \leq \Pr \left[ \eta_{\text{new}}(x) \geq \eta_{\text{new}}(x), \eta_{\text{new}}(x) - \epsilon \leq f_{\tilde{y}}(x)/\delta \right] \\
= \Pr \left[ \eta_{\text{new}}(x) \leq \eta_{\text{new}}(x) \leq \frac{f_{\tilde{y}}(x)/\delta - \sum_{j \neq \text{new}} \tau_{j,\text{new}} \eta_j(x)}{\tau_{\text{max},\text{new}}} + \frac{\epsilon}{\tau_{\text{max},\text{max}}} \right] 
$$

(7)

Firstly, observe that if $\delta > 1$, then $\Pr \left[ \tilde{y}_{\text{new}} = \tilde{y} \neq h^*(x), \frac{f_{\tilde{y}}(x)}{f_{\text{max}}(x)} \geq \delta \right] = 0$ due to the definition of $m_x$.

Then notice that $\delta = \max_x \frac{f_{\tilde{y}}(x)}{\tau_{\text{max},\text{new}} \eta_{\text{new}}(x) + \sum_{j \neq \text{new}} \tau_{j,\text{new}} \eta_j(x) \geq \frac{f_{\tilde{y}}(x)}{\tau_{\text{max},\text{max}} \eta_{\text{max}}(x) + \sum_{j \neq \text{max}} \tau_{j,\text{max}} \eta_j(x)}$. If we substitute $\delta$ in (7) with
We give following several facts based on our theorem: which compete the proof for cases that are accepted.

Now we summarize all pieces and we get:

\[
\Pr[\eta_{m_2}(x) \geq \eta_{s_2}(x), f_{m_2}(x) \leq f_{\bar{y}}(x)/\delta] = \left(\frac{\epsilon}{\tau_{m_2, m_2}}\right)^\lambda \leq C \left(\frac{\epsilon}{\tau_{m_2, m_2}}\right)^\lambda + \Pr[u_x \neq m_2, u_x \neq \bar{y}]
\]

For the second term in (6), our algorithm cannot deal with it properly. We will leave it as the future research problem.

Now we summarize all pieces and we get:

\[
\Pr[\eta_{new} \neq h^*(x), \bar{y} \text{ is accepted}] = (6) \leq (8) + \Pr[u_x \neq m_2, u_x \neq \bar{y}]
\]

which compete the proof for cases that are accepted.

We give following several facts based on our theorem:

1. For binary case, if we set \( \delta = \frac{1 - |\tau_0 - \tau_1|}{1 + |\tau_0 - \tau_1|} \) and further assume \( \epsilon \leq t_0(1 - \tau_{10} - \tau_{01}) - \frac{|\tau_0 - \tau_1|}{2} \), we have:

\[
\Pr_{(x,y) \sim D}[\eta_{new} \neq h^*(x)] = C \left[\frac{\tau_{10} - \tau_{01}}{2(1 - \tau_{10} - \tau_{01})} + \frac{\epsilon}{1 - \tau_{10} - \tau_{01}}\right]^{\lambda}
\]

**Proof.** For binary case, we have:

\[
\begin{align*}
\Pr_{(x,y) \sim D} & \left[\eta_{new} \neq h^*(x)\right] = \Pr_{(x,y) \sim D} \left[\eta_{new} \neq h^*(x), \bar{y} \text{ is rejected}\right] + \Pr_{(x,y) \sim D} \left[\eta_{new} \neq h^*(x), \bar{y} \text{ is accepted}\right] \\
& = \Pr[\eta_{\bar{y}}(x) > \frac{1}{2}, f_{\bar{y}}(x) < \delta] + \Pr[\eta_{\bar{y}}(x) \leq \frac{1}{2}, f_{\bar{y}}(x) \geq \delta] \\
& \leq \Pr[\eta_{\bar{y}}(x) > \frac{1}{2}, f_{\bar{y}}(x) < \delta] + \Pr[\eta_{\bar{y}}(x) \leq \frac{1}{2}, f_{\bar{y}}(x) \geq \delta] \\
& \leq \Pr[\eta_{\bar{y}}(x) > \frac{1}{2}, \eta_{\bar{y}}(x) < \frac{\delta}{1 + \delta} + \epsilon] + \Pr[\eta_{\bar{y}}(x) \leq \frac{1}{2}, \eta_{\bar{y}}(x) \geq \frac{\delta}{1 + \delta} - \epsilon] \\
& = \Pr \left[ \frac{1}{2} < \eta_{\bar{y}}(x) < \frac{\delta}{1 + \delta} - \tau_1 - \tau_0 + \frac{\epsilon}{1 - \tau_{10} - \tau_{01}} \right] + \Pr \left[ \frac{\delta}{1 + \delta} - \tau_1 - \tau_0 - \frac{\epsilon}{1 - \tau_{10} - \tau_{01}} \leq \eta_{\bar{y}}(x) \leq \frac{1}{2} \right]
\end{align*}
\]

Observe that \( \delta = \frac{1 - |\tau_0 - \tau_1|}{1 + |\tau_0 - \tau_1|} \leq \frac{1 - \tau_0 - \tau_1}{1 + \tau_0 - \tau_1} \). We also have \( \frac{\delta}{1 + \delta} = \frac{1 - |\tau_0 - \tau_1|}{2} \leq \frac{1}{2} \). Now we substitute \( \delta = \frac{1 - \tau_0 - \tau_1}{1 + \tau_0 - \tau_1} \) in the first term of (9) and substitute \( \frac{\delta}{1 + \delta} \) with \( \frac{1}{2} \) in the second term of (9), by algebra we know
Theorem 3. Assume \( \delta \) and further assume that \( \epsilon \leq \frac{1}{2} \min_{x} |\tilde{\eta}_{u_{x}}(x) - \tilde{\eta}_{x_{u}}(x)| \), we have:

(a) Sensitivity Optimized Critical Value. Let \( \delta = \min_{x} \left[ \tau_{\tilde{y}, y_{u_{x}}(x)} + \sum_{j \neq y_{u_{x}}} \tau_{y_{j}, y_{u_{x}}(x)} \right] \) then:

\[
\Pr_{(x,y) \sim D} \left[ \tilde{y}_{\text{new}} \neq h^{*}(x), \tilde{y} \text{ is rejected} \right] \leq C \left[ O(\epsilon) \right]^{\lambda}
\]

(b) Specificity Optimized Critical Value. Let \( \delta = \max_{x} \left[ \frac{f_{\tilde{y}}(x)}{\tau_{\eta_{u_{x}}, m_{x}} - \tau_{\tilde{y}, y_{u_{x}}(x)} + \tau_{y_{u_{x}}, m_{x}}} \right] \) then:

\[
\Pr_{(x,y) \sim D} \left[ \tilde{y}_{\text{new}} \neq h^{*}(x), \tilde{y} \text{ is accepted} \right] \leq C \left[ O(\epsilon) \right]^{\lambda}
\]

Proof. Observe that under symmetric noise scenario, \( \forall i \in [N_c], \eta_{u_{x}}(x) \geq \eta_{i}(x) \) implies that \( \tilde{\eta}_{u_{x}}(x) \geq \tilde{\eta}_{i}(x) \), i.e. \( h^{*}(x) = \tilde{h}^{*}(x) \). To show this:

\[
\eta_{u_{x}}(x) \geq \eta_{i}(x) \iff [1 - N_c \tau] \eta_{u_{x}}(x) \geq [1 - N_c \tau] \eta_{i}(x) \iff [1 - (N_c - 1) \tau] \eta_{u_{x}}(x) - \tau \eta_{u_{x}}(x) \geq [1 - (N_c - 1) \tau] \eta_{i}(x) - \tau \eta_{i}(x) \iff [1 - (N_c - 1) \tau] \eta_{u_{x}}(x) + \tau \eta_{x}(x) \geq [1 - (N_c - 1) \tau] \eta_{i}(x) + \tau \eta_{u_{x}}(x) \iff [1 - (N_c - 1) \tau] \eta_{u_{x}}(x) + \tau \eta_{x}(x) \geq [1 - (N_c - 1) \tau] \eta_{i}(x) + \tau \eta_{u_{x}}(x) \iff \sum_{j \neq u_{x}} \tau_{\tilde{y}_{j}, y_{u_{x}}(x)} \geq \sum_{j \neq u_{x}} \tau_{\tilde{y}_{j}, y_{u_{x}}(x)}
\]

Since \( \tilde{\eta}_{u_{x}}(x) \geq \tilde{\eta}_{x_{u}}(x) + 2 \epsilon, \) then \( \tilde{\eta}_{u_{x}}(x) - \epsilon \geq \tilde{\eta}_{i}(x) + \epsilon \) and thus \( f_{u_{x}} \geq f_{\tilde{y}}(x) \) \( \forall i \in [N_c] \), which implies \( f_{m_{x}}(x) = f_{u_{x}}(x) \). As a result, second term in (2) and second term in (6) will be 0. \( \square \)

Theorem 3. Assume \( \eta \) and \( f \) satisfy the same conditions as Lemma 1. Also assume \( \xi < \delta \) and further assume that \( \epsilon \leq \min_{i} \left( \frac{t_{0, \delta, \xi - \xi_{i}}}{\xi_{i}} \right) \). Let \( \tilde{y}_{\text{new}} \) be the output of the LRT-Correction with \( (x, \tilde{y}), f, \) and the approximate \( \tilde{\delta} \). Then:

1. Sensitivity Optimized Critical Value. Let \( \delta = \min_{x} \left[ \frac{\tau_{\tilde{y}, y_{u_{x}}(x)} + \sum_{j \neq y_{u_{x}}} \tau_{y_{j}, y_{u_{x}}(x)}}{f_{m_{x}}(x)} \right] \) then:

\[
\Pr_{(x,y) \sim D} \left[ \tilde{y}_{\text{new}} \neq h^{*}(x), \tilde{y} \text{ is rejected} \right] \leq C \left[ O(\max(\epsilon, \xi)) \right]^{\lambda} + \Pr \left[ u_{x} \neq m_{x}, u_{x} \neq \tilde{y} \right]
\]
2. Specificity Optimized Critical Value. Let \( \delta = \max_{x} \frac{f_{x}(x)}{\tau_{m, x, \eta}(x)} \) then:

\[
\Pr_{(x, y) \sim D} \left[ \tilde{y}_{new} \neq h^{*}(x), \tilde{y} \text{ is accepted} \right] \leq C \left[ O(\max(\epsilon, \xi)) \right]^{\lambda} + \Pr \left[ u_{x} \neq m_{x}, u_{x} \neq \tilde{y} \right]
\]

Proof. The proof will be similar to the proof of Lemma 1, but we need to adjust the error introduced by picking \( \hat{\delta} \). Recall that \( \xi \) and \( \epsilon \) are both less than one.

If we pick \( \hat{\delta} \) instead of \( \delta \), then for (3) in Lemma 1, we have:

\[
\Pr \left[ h^{*}(x) = \tilde{y}, \frac{f_{\tilde{y}}(x)}{f_{m}(x)} < \hat{\delta} \right] = \Pr \left[ h^{*}(x) = \tilde{y}, f_{\tilde{y}}(x) < \hat{\delta} f_{m}(x) \right] 
\]

\[
\leq \Pr \left[ \eta_{\tilde{y}}(x) \leq \eta_{m}(x), \tilde{y}_{\tilde{y}}(x) - \epsilon < \hat{\delta} f_{m}(x) \right] 
\]

\[
\leq \Pr \left[ \eta_{m}(x) \leq \eta_{\tilde{y}}(x) < \frac{\delta f_{m}(x) - \sum_{j \neq \tilde{y}} \tau_{j, \tilde{y}} \eta_{j}(x)}{\tau_{\tilde{y}, \tilde{y}}} + \frac{\epsilon}{\tau_{\tilde{y}, \tilde{y}}} \right] 
\]

\[
\leq \Pr \left[ \eta_{m}(x) \leq \eta_{\tilde{y}}(x) < \frac{(\delta + \xi) f_{m}(x) - \sum_{j \neq \tilde{y}} \tau_{j, \tilde{y}} \eta_{j}(x)}{\tau_{\tilde{y}, \tilde{y}}} + \frac{\epsilon}{\tau_{\tilde{y}, \tilde{y}}} \right] 
\]

\[
\leq \Pr \left[ \eta_{m}(x) \leq \eta_{\tilde{y}}(x) < \frac{\delta f_{m}(x) - \sum_{j \neq \tilde{y}} \tau_{j, \tilde{y}} \eta_{j}(x)}{\tau_{\tilde{y}, \tilde{y}}} + \frac{\epsilon + \xi}{\tau_{\tilde{y}, \tilde{y}}} \right] 
\]

\[
\leq C \left[ \frac{\epsilon + \xi}{\tau_{\tilde{y}, \tilde{y}}} \right]^{\lambda} \tag{10}
\]

The same upper bound holds for (5) with the same reason. Then:

\[
\Pr \left[ \tilde{y}_{new} \neq h^{*}(x), \tilde{y} \text{ is rejected} \right] \leq (10) + \Pr \left[ u_{x} \neq m_{x}, u_{x} \neq \tilde{y} \right] 
\]

\[
= C \left[ O(\max(\epsilon, \xi)) \right]^{\lambda} + \Pr \left[ u_{x} \neq m_{x}, u_{x} \neq \tilde{y} \right]
\]

We next analyze (7) in Lemma 1:

\[
\Pr \left[ \eta_{m}(x) \geq \eta_{m}(x), f_{m}(x) \geq f_{\tilde{y}}(x)/\hat{\delta} \right] \leq \Pr \left[ \eta_{m}(x) \geq \eta_{m}(x), \eta_{m}(x) - \epsilon \geq f_{\tilde{y}}(x)/\hat{\delta} \right]
\]

\[
= \Pr \left[ \eta_{m}(x) \leq \eta_{m}(x) \leq \frac{f_{\tilde{y}}(x)/\hat{\delta} - \sum_{j \neq m_{x}} \tau_{j, m_{x}} \eta_{j}(x)}{\tau_{m, m_{x}}} + \frac{\epsilon}{\tau_{m, m_{x}}} \right] 
\]

\[
\leq \Pr \left[ \eta_{m}(x) \leq \eta_{m}(x) \leq \frac{f_{\tilde{y}}(x)/(\hat{\delta} - \xi) - \sum_{j \neq m_{x}} \tau_{j, m_{x}} \eta_{j}(x)}{\tau_{m, m_{x}}} + \frac{\epsilon}{\tau_{m, m_{x}}} \right] 
\]

\[
= \Pr \left[ 0 < \eta_{m}(x) - \eta_{m}(x) < \frac{f_{\tilde{y}}(x)/(\hat{\delta} - \xi) - \sum_{j \neq m_{x}} \tau_{j, m_{x}} \eta_{j}(x)}{\tau_{m, m_{x}}} - \eta_{m}(x) + \frac{\epsilon}{\tau_{m, m_{x}}} + \frac{\xi f_{\tilde{y}}(x)}{\hat{\delta} - \xi} \right]
\]

Observe that \( \frac{\xi}{\hat{\delta} - \xi} = \frac{\delta}{\hat{\delta} - \xi} \frac{\xi}{\hat{\delta}} = [1 + O(\xi)] \frac{\xi}{\hat{\delta}}, \) where second equality comes from Taylor expansion. Then we substitute the \( \delta \) as what we did in Lemma 1 and continue the calculation:
Here Tsybakove condition holds, because $\frac{\epsilon}{\tau_{m_x,m_x}} + \frac{\xi}{\delta \tau_{m_x,m_x}} + \frac{\xi^2}{\delta^2 \tau_{m_x,m_x}} \leq t_0$. As a result:

$$\Pr \left[ y_{new} \neq h^*(x), \bar{y} \text{ is accepted} \right] \leq (11) + \Pr \left[ u_{x} \neq m_x, u_{x} \neq \bar{y} \right] \leq C \left[ O(\max(\epsilon, \xi)) \right]^\lambda + \Pr \left[ u_{x} \neq m_x, u_{x} \neq \bar{y} \right]$$

which complete the proof for cases that are accepted.

Other terms will not be affected by the choice of $\delta$. By now we completes the proof. \qed