8 SUPPLEMENTARY MATERIAL

8.1 Proof of Lemma 1

The proof of Lemma 1 is an adaptation from the proof of Theorem 1 in Li et al. (2017).

Proof. Define $G(\theta) := \sum_{s=1}^{t}(\mu(X_s^T\theta) - \mu(X_s^T\theta^*))X_s$. We have $G(\theta^*) = 0$ and $G(\hat{\theta}_t) = \sum_{s=1}^{t}\epsilon_s X_s$, where ϵ_s is the sub-Gaussian noise at round s. For convenience, define $Z := G(\hat{\theta}_t)$. From mean value theorem, for any θ_1, θ_2, there exists $v \in (0,1)$ and $\bar{\theta} = v\theta_1 + (1-v)\theta_2$ such that

$$G(\theta_1) - G(\theta_2) = \left[\sum_{s=1}^{t}\mu'(X_s^T\bar{\theta})X_s X_s^T\right](\theta_1 - \theta_2) := F(\bar{\theta})(\theta_1 - \theta_2), \tag{11}$$

where $F(\bar{\theta}) = \sum_{s=1}^{t}\mu'(X_s^T\bar{\theta})X_s X_s^T$. Therefore, for any $\theta_1 \neq \theta_2$, we have

$$(\theta_1 - \theta_2)^T(G(\theta_1) - G(\theta_2)) = (\theta_1 - \theta_2)^T F(\bar{\theta})(\theta_1 - \theta_2) > 0,$$

since $\mu' > 0$ and $\lambda_{\min}(V_{t+1}) > 0$. So $G(\theta)$ is an injection from $\mathbb{R}^d$ to $\mathbb{R}^d$. Consider an η-neighborhood of θ^*, $\mathbb{B}_\eta := \{\theta : \|\theta - \theta^*\| \leq \eta\}$, where η is a constant that will be specified later such that we have $c_\eta = \inf_{\theta\in\mathbb{B}_\eta}\mu'(x^T\theta) > 0$. When $\theta_1, \theta_2 \in \mathbb{B}_\eta$, from the property of convex set, we have $\bar{\theta} \in \mathbb{B}_\eta$. From Equation 11, we have when $\theta \in \mathbb{B}_\eta$,

$$\begin{aligned}\|G(\theta)\|_{V_{t+1}^{-1}} = \|G(\theta) - G(\theta^*)\|_{V_{t+1}^{-1}} &= \sqrt{(\theta - \theta^*)^T F(\bar{\theta})V_{t+1}^{-1}F(\bar{\theta})(\theta - \theta^*)} \\ &\geq c_\eta\sqrt{\lambda_{\min}(V_{t+1})}\|\theta - \theta^*\|\end{aligned}$$

The last inequality is due to

$$F(\bar{\theta}) \succeq c_\eta \sum_{s=1}^{t} X_s X_s^T = c_\eta V_{t+1}.$$

From Lemma A in Chen et al. (1999), we have that

$$\left\{\theta : \|G(\theta) - G(\theta^*)\|_{V_{t+1}^{-1}} \leq c_\eta \eta\sqrt{\lambda_{\min}(V_{t+1})}\right\} \subset \mathbb{B}_\eta.$$

Now from Lemma 7 in Li et al. (2017), we have with probability at least $1 - \delta$,

$$\|G(\hat{\theta}_t) - G(\theta^*)\|_{V_{t+1}^{-1}} = \|Z\|_{V_{t+1}^{-1}} \leq 4R\sqrt{d + \log\frac{1}{\delta}}.$$

Therefore, when

$$\eta \geq \frac{4R}{c_\eta}\sqrt{\frac{d + \log\frac{1}{\delta}}{\lambda_{\min}(V_{t+1})}},$$

we have $\hat{\theta}_t \in \mathbb{B}_\eta$. Since $c_\eta \geq c_1 \geq c_3 > 0$ when $\eta \leq 1$, we have

$$\|\hat{\theta}_t - \theta^*\| \leq \frac{4R}{c_\eta}\sqrt{\frac{d + \log\frac{1}{\delta}}{\lambda_{\min}(V_{t+1})}} \leq 1,$$

when $\lambda_{\min}(V_{t+1}) \geq \frac{16R^2[d + \log(\frac{1}{\delta})]}{c_1^2}$. □

8.2 Proof of Lemma 2

Note that the condition of Lemma 1 holds with high probability when τ is chosen as Equation 8. This is a consequence of Proposition 1 in Li et al. (2017), which is presented below for reader's convenience.

Proposition 1 (Proposition 1 in Li et al. (2017)). *Define $V_{n+1} = \sum_{t=1}^{n} X_t X_t^T$, where X_t is drawn IID from some distribution in unit ball $\mathbb{B}^d$. Furthermore, let $\Sigma := E[X_t X_t^T]$ be the second moment matrix, let $B, \delta_2 > 0$ be two positive constants. Then there exists positive, universal constants C_1 and C_2 such that $\lambda_{\min}(V_{n+1}) \geq B$ with probability at least $1 - \delta_2$, as long as*

$$n \geq \left(\frac{C_1 \sqrt{d} + C_2 \sqrt{\log(1/\delta_2)}}{\lambda_{\min}(\Sigma)} \right)^2 + \frac{2B}{\lambda_{\min}(\Sigma)}.$$

Now we formally prove Lemma 2.

Proof. Note that from the definition of $\tilde{\theta}_0$ in the algorithm, when $j = 1$, the conclusion holds trivially. When τ is chosen as in Equation 8, we have from Lemma 1 and Proposition 1 that $\|\hat{\theta}_t - \theta^*\| \leq 1$ for all $t \geq \tau$ with probability at least $1 - \frac{2}{T^2}$. Therefore, $\hat{\theta}_{j\tau} \in \mathcal{C}$ for all $j \geq 1$ with probability at least $1 - \frac{2}{T^2}$. For the analysis below, we assume $\hat{\theta}_{j\tau} \in \mathcal{C}$ for all $j \geq 1$.

Since $\tilde{\theta}_j \in \mathcal{C}$, we have $\|\tilde{\theta}_j - \theta^*\| \leq 3$. Denote $\mathbb{B}_\eta := \{\theta : \|\theta - \theta^*\| \leq \eta\}$, we have $\tilde{\theta}_j, \hat{\theta}_{j\tau} \in \mathbb{B}_3$. For any $v > 0$, define $\bar{\theta} = v\tilde{\theta}_j + (1 - v)\hat{\theta}_{j\tau}$, since $\mathbb{B}_3$ is convex, we have $\bar{\theta} \in \mathbb{B}_3$. Therefore, we have from Assumption 2

$$\nabla^2 l_{j,\tau}(\bar{\theta}) = \sum_{s=(j-1)\tau+1}^{j\tau} \mu'(X_s^T \bar{\theta}) X_s X_s^T \succeq c_3 \sum_{s=(j-1)\tau+1}^{j\tau} X_s X_s^T.$$

Since we update $\tilde{\theta}_j$ every τ rounds and θ_j^{TS} only depends on $\tilde{\theta}_j$. For the next τ rounds, the pulled arms are only dependent on θ_j^{TS}. Therefore, the feature vectors of pulled arms among the next τ rounds are IID. According to Proposition 1 and Equation 8, and by applying a union bound, we have $\lambda_{\min}\left(\sum_{s=(j-1)\tau+1}^{j\tau} X_s X_s^T\right) \geq \frac{\alpha}{c_3}$ holds for all $j \geq 1$ with probability at least $1 - \frac{1}{T^2}$. This tells us that for all j, $l_{j,\tau}(\theta)$ is a α-strongly convex function when $\theta \in \mathbb{B}_3$. Therefore, we can apply (Theorem 3.3 of Section 3.3.1 in Hazan et al. (2016)) to get for all $j \geq 1$

$$\sum_{q=1}^{j} \left(l_{q,\tau}(\tilde{\theta}_q) - l_{q,\tau}(\hat{\theta}_{j\tau}) \right) \leq \frac{G^2}{2\alpha}(1 + \log j)$$

where G satisfies $G^2 \geq E\|\nabla l_{q,\tau}\|^2$. Note that $G \leq \tau$ since $\mu(x) \in [0, 1], Y_s \in [0, 1]$ and $\|X_s\| \leq 1$. From Jensen's Inequality, we have

$$\sum_{q=1}^{j} \left(l_{q,\tau}(\bar{\theta}_j) - l_{q,\tau}(\hat{\theta}_{j\tau}) \right) \leq \frac{G^2}{2\alpha}(1 + \log j).$$

Since $\bar{\theta}_j, \hat{\theta}_{j\tau} \in \mathbb{B}_3$, we have for any $v > 0$, if $\theta = v\bar{\theta}_j + (1 - v)\hat{\theta}_{j\tau}$, then $\nabla^2 l_{q,\tau}(\theta) \succeq \alpha I_d$ for all $1 \leq q \leq j$. Since $\sum_{q=1}^{j} \nabla l_{q,\tau}(\hat{\theta}_{j\tau}) = 0$, we have

$$\|\bar{\theta}_j - \hat{\theta}_{j\tau}\| \leq \frac{G}{\alpha} \sqrt{\frac{1 + \log j}{j}}.$$

By applying a union bound, we get the conclusion. □

8.3 Proof of Lemma 3

We utilize the concentration property of MLE. Here, we present the analysis of MLE in Li et al. (2017).

Lemma 7 (Lemma 3 in Li et al. (2017)). *Suppose $\lambda_{\min}(V_{\tau+1}) \geq 1$. For any $\delta_3 \in (0, 1)$, the following event*

$$\mathcal{E} := \left\{ \|\hat{\theta}_t - \theta^*\|_{V_{t+1}} \leq \frac{R}{c_1} \sqrt{\frac{d}{2} \log(1 + \frac{2t}{d}) + \log \frac{1}{\delta_3}} \right\}$$

holds for all $t \geq \tau$ with probability at least $1 - \delta_3$.

Proof. Note that from Proposition 1, when $\alpha \geq c_3$, $\lambda_{\min}(V_{\tau+1}) \geq 1$ holds with probability at least $1 - \frac{1}{T^2}$. The proof of Lemma 3 is simply a combination of Lemma 2 and Lemma 7 by applying a union bound. □

8.4 Proof of Lemma 4

We use formula 7.1.13 in Abramowitz and Stegun (1948) to help derive the concentration and anti-concentration inequalities for Gaussian distributed random variables. Details are shown in Lemma 8.

Lemma 8 (Formula 7.1.13 in Abramowitz and Stegun (1948))**.** *For a Gaussian distributed random variable with mean m and variance σ^2, we have for $z \geq 1$ that*

$$\mathbb{P}(|Z-m| \geq z\sigma) \leq \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{2}}.$$

For $0 < z \leq 1$, we have

$$\mathbb{P}(|Z-m| \geq z\sigma) \geq \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{2}}.$$

Now we prove Lemma 4.

Proof. Since $\theta_j^{\mathrm{TS}} | \mathcal{F}_{j\tau} \sim \mathcal{N}\left(\bar{\theta}_j, \left(2g_1(j)^2 \frac{c_3}{\alpha j} + \frac{2g_2(j)^2}{j}\right) I_d\right)$, and θ_j^{TS} is independent of $\left\{\cup_{t=j\tau+1}^{(j+1)\tau} \mathcal{A}_t\right\} = \{x_{t,a}, a \in [K], j\tau < t \leq (j+1)\tau\}$, we have for $x \in \left\{\cup_{t=j\tau+1}^{(j+1)\tau} \mathcal{A}_t\right\}$,

$$x^T(\bar{\theta}_j - \theta_j^{\mathrm{TS}}) | \mathcal{F}_{j\tau}, x \sim \mathcal{N}\left(0, \left(2g_1(j)^2 \frac{c_3}{\alpha j} + \frac{2g_2(j)^2}{j}\right) \|x\|^2\right).$$

From the property of Gaussian random variable in Lemma 8, when $u = \sqrt{2\log(T^2 K\tau)}$, we have

$$\mathbb{P}\left(|x^T(\bar{\theta}_j - \theta_j^{\mathrm{TS}})| \geq u\sqrt{2g_1(j)^2 \frac{c_3}{\alpha j}\|x\|^2 + \frac{2g_2(j)^2}{j}\|x\|^2} \middle| \mathcal{F}_{j\tau}, x\right) \leq \frac{1}{\sqrt{\pi}} e^{-\frac{u^2}{2}} \leq \frac{1}{K\tau T^2}. \tag{12}$$

We use the following property of conditional probability

$$\int_x \mathbb{P}(E|X=x,\mathcal{F}) f(X=x|\mathcal{F}) dx = \mathbb{P}(E|\mathcal{F}), \tag{13}$$

where $f(X=x|\mathcal{F})$ is the conditional $p.d.f$ of a random variable X and E is an event. Combine Equation 12 and Equation 13, we have for every $a \in [K]$ and $j\tau < t \leq (j+1)\tau$,

$$\begin{aligned}
&\mathbb{P}\left(|x_{t,a}^T(\bar{\theta}_j - \theta_j^{\mathrm{TS}})| \geq u\sqrt{2g_1(j)^2 \frac{c_3}{\alpha j} + 2g_2(j)^2/j\|x_{t,a}\|^2} \middle| \mathcal{F}_{j\tau}\right) \\
&= \int_x \mathbb{P}\left(|x_{t,a}^T(\bar{\theta}_j - \theta_j^{\mathrm{TS}})| \geq u\sqrt{2g_1(j)^2 \frac{c_3}{\alpha j} + 2g_2(j)^2/j\|x_{t,a}\|^2} \middle| \mathcal{F}_{j\tau}, x_{t,a} = x\right) f(x_{t,a} = x|\mathcal{F}_{j\tau}) dx \\
&\leq \frac{1}{K\tau T^2} \int_x f(x_{t,a} = x|\mathcal{F}_{j\tau}) dx = \frac{1}{K\tau T^2}
\end{aligned}$$

Applying a union bound, we get the conclusion. □

8.5 Proof of Lemma 5

Proof. We still use Lemma 8 to show the result. For convenience, denote $x := x_{t,*}$, $\gamma_1 := \sqrt{\frac{c_3}{\alpha j_t}}\|x\|$ and $\gamma_2 := \frac{\|x\|}{\sqrt{j_t}}$. Note that x is independent of $\theta_{j_t}^{\mathrm{TS}}$, so

$$x^T(\bar{\theta}_{j_t} - \theta_{j_t}^{\mathrm{TS}}) | \mathcal{F}_{j_t\tau}, x \sim \mathcal{N}\left(0, (2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2)\right). \tag{14}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(x^T\theta_{j_t}^{\mathrm{TS}} > x^T\theta^* \middle| \mathcal{F}_{j_t\tau}, x\right) = \mathbb{P}\left(\frac{x^T\theta_{j_t}^{\mathrm{TS}} - x^T\bar{\theta}_{j_t}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} > \frac{x^T\theta^* - x^T\bar{\theta}_{j_t}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} \middle| \mathcal{F}_{j_t\tau}, x\right) \\
&\geq \mathbb{P}\left(\frac{x^T\theta_{j_t}^{\mathrm{TS}} - x^T\bar{\theta}_{j_t}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} > \frac{g_1(j_t)\|x\|_{V_{j_t\tau+1}^{-1}} + g_2(j_t)\frac{\|x\|}{\sqrt{j_t}}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} \middle| \mathcal{F}_{j_t\tau}, x\right) \\
&\geq \mathbb{P}\left(\frac{x^T\theta_{j_t}^{\mathrm{TS}} - x^T\bar{\theta}_{j_t}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} > \frac{g_1(j_t)\sqrt{\frac{c_3}{\alpha j_t}}\|x\| + g_2(j)\frac{\|x\|}{\sqrt{j_t}}}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}} \middle| \mathcal{F}_{j_t\tau}, x\right) \\
&\geq \frac{1}{4\sqrt{\pi}}e^{-\frac{z^2}{2}},
\end{aligned}$$

where $z := \frac{g_1(j_t)\gamma_1 + g_2(j_t)\gamma_2}{\sqrt{2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2}}$. The first and second inequalities hold since $\mathcal{F}_t$ is a filtration such that $E_1(j_t)$ and $\lambda_{\min}(V_{j_t\tau+1}) \geq \frac{\alpha j_t}{c_3}$ are true. Notice that we have $0 < z \leq 1$ since

$$2g_1(j_t)^2\gamma_1^2 + 2g_2(j_t)^2\gamma_2^2 - (g_1(j_t)\gamma_1 + g_2(j_t)\gamma_2)^2 = (g_1(j_t)\gamma_1 - g_2(j_t)\gamma_2)^2 \geq 0.$$

Therefore, we get

$$\mathbb{P}\left(x^T\theta_{j_t}^{\mathrm{TS}} > x^T\theta^* \middle| \mathcal{F}_{j_t\tau}, x\right) \geq \frac{1}{4\sqrt{\pi}}e^{-\frac{z^2}{2}} \geq \frac{1}{4\sqrt{\pi e}}.$$

Similarly, using Equation 13, we get

$$\mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} > x_{t,*}^T\theta^* \middle| \mathcal{F}_{j_t\tau}\right) = \int_x \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} > x_{t,*}^T\theta^* \middle| \mathcal{F}_{j_t\tau}, x_{t,*} = x\right) f(x_{t,*} = x|\mathcal{F}_{j_t\tau})dx \geq \frac{1}{4\sqrt{\pi e}}.$$

□

8.6 Proof of Lemma 6

The technique used in this proof is extracted from Agrawal and Goyal (2013); Kveton et al. (2019).

Proof. Denote $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$. To prove the lemma, we prove the following Equation 15 holds for any possible filtration $\mathcal{F}_t$:

$$\mathbb{E}_{j_t\tau}[\Delta_{a_t}(t)\mathbb{1}(E_1(j_t) \cap E_2(j_t) \cap E_3(j_t))] \leq \left(1 + \frac{2}{\frac{1}{4\sqrt{\pi e}} - \frac{1}{T^2}}\right)\mathbb{E}_{j_t\tau}\left[H_{a_t}(t)\mathbb{1}(E_3(j_t))\right] \tag{15}$$

Denote the following set as the underesampled arms at round t,

$$S_t^C = \{i \in [K] : H_i(t) \geq \Delta_i(t)\}$$

Note that $a_t^* \in S_t^C$ for all t. The set of sufficiently sampled arms is $S_t = [K] \setminus S_t^C$. Let $J_t = \mathrm{argmin}_{i \in S_t^C} H_i(t)$ be the least uncertain undersampled arm at round t. At round t, denote $j_t = \lfloor \frac{t-1}{\tau} \rfloor$. In the steps below, we assume that event $E_1(j_t) \cap E_2(j_t)$ occurs, then

$$\begin{aligned}
\Delta_{a_t}(t) &= \Delta_{J_t}(t) + (x_{t,J_t} - X_t)^T\theta^* \\
&= \Delta_{J_t}(t) + x_{t,J_t}^T(\theta^* - \theta_{j_t}^{TS}) + (x_{t,J_t} - X_t)^T\theta_{j_t}^{TS} + X_t^T(\theta_{j_t}^{TS} - \theta^*) \\
&\leq \Delta_{J_t}(t) + H_{J_t}(t) + H_{a_t}(t) \quad \text{ since } (x_{t,J_t} - X_t)^T\theta_{j_t}^{TS} \leq 0 \\
&\leq 2H_{J_t}(t) + H_{a_t}(t) \quad \text{ since } J_t \in S_t^C.
\end{aligned}$$

The left to do is to bound $H_{J_t}(t)$ by $H_{a_t}(t)$. Since $J_t = \mathrm{argmin}_{i \in S_t^C} H_i(t)$, we have

$$\mathbb{E}_{j_t\tau}\left[H_{a_t}(t)\right] \geq \mathbb{E}_{j_t\tau}\left[H_{a_t}(t)|a_t \in S_t^C\right]\mathbb{P}\left(a_t \in S_t^C|\mathcal{F}_{j_t\tau}\right) \geq \mathbb{E}_{j_t\tau}[H_{J_t}(t)]\mathbb{P}\left(a_t \in S_t^C|\mathcal{F}_{j_t\tau}\right). \tag{16}$$

Therefore, we have

$$\mathbb{E}_{j_t\tau}\left[\Delta_{a_t}(t)\mathbb{1}(E_1(j_t)\cap E_2(j_t))\right] \le \left(1+\frac{2}{P\left(a_t\in S_t^C|\mathcal{F}_{j_t\tau}\right)}\right)\mathbb{E}_{j_t\tau}\left[H_{a_t}(t)\right] \tag{17}$$

Next, we bound $P\left(a_t\in S_t^C|\mathcal{F}_{j_t\tau}\right)$.

$$\begin{aligned}
\mathbb{P}\left(a_t\in S_t^C|\mathcal{F}_{j_t\tau}\right) &\ge \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} \ge \max_{i\in S_t} x_{t,i}^T\theta_{j_t}^{\mathrm{TS}}\middle|\mathcal{F}_{j_t\tau}\right) \qquad \text{since } a_t^*\in S_t^C\\
&\ge \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} \ge \max_{i\in S_t} x_{t,i}^T\theta_{j_t}^{\mathrm{TS}}, E_1(j_t)\cap E_2(j_t)\middle|\mathcal{F}_{j_t\tau}\right)\\
&\ge \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} \ge x_{t,*}^T\theta^*, E_1(j_t)\cap E_2(j_t)|\mathcal{F}_{j_t\tau}\right) &(18)\\
&\ge \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} \ge x_{t,*}^T\theta^*, E_1(j_t)|\mathcal{F}_{j_t\tau}\right) - \mathbb{P}\left(E_2^C(j_t)|\mathcal{F}_{j_t\tau}\right)\\
&\ge \mathbb{P}\left(x_{t,*}^T\theta_{j_t}^{\mathrm{TS}} \ge x_{t,*}^T\theta^*, E_1(j_t)|\mathcal{F}_{j_t\tau}\right) - \frac{1}{T^2}. &(19)
\end{aligned}$$

Inequality 18 holds because for all $i\in S_t$, on event $E_1(j_t)\cap E_2(j_t)$,

$$x_{t,i}^T\theta_{j_t}^{\mathrm{TS}} \le x_{t,i}^T\theta^* + H_i(t) < x_{t,i}^T\theta^* + \Delta_i(t) = x_{t,*}^T\theta^*.$$

Inequality 19 holds because of Lemma 4. When $\mathcal{F}_t$ is a filtration such that $E_1(j_t)$ and $E_3(j_t)$ are true, we have from Lemma 5 that

$$\mathbb{P}\left(a_t\in S_t^C|\mathcal{F}_{j_t\tau}\right) \ge \frac{1}{4\sqrt{\pi e}} - \frac{1}{T^2}.$$

So under such filtration, from Equation 17, we have

$$\mathbb{E}_{j_t\tau}\left[\Delta_{a_t}(t)\mathbb{1}(E_1(j_t)\cap E_2(j_t))\right] \le \left(1+\frac{2}{\frac{1}{4\sqrt{\pi e}}-\frac{1}{T^2}}\right)\mathbb{E}_{j_t\tau}\left[H_{a_t}(t)\right].$$

Since $E_3(j_t)$ is $\mathcal{F}_{j_t\tau}$-measurable, we have under such filtration,

$$\mathbb{E}_{j_t\tau}[\Delta_{a_t}(t)\mathbb{1}(E_1(j_t)\cap E_2(j_t)\cap E_3(j_t))] \le \left(1+\frac{2}{\frac{1}{4\sqrt{\pi e}}-\frac{1}{T^2}}\right)\mathbb{E}_{j_t\tau}\left[H_{a_t}(t)\mathbb{1}(E_3(j_t))\right].$$

When $\mathcal{F}_t$ is a filtration such that $E_1(j_t)\cap E_3(j_t)$ is not true, the conclusion holds trivially. This finishes our proof. □

8.7 Proof of Theorem 1

Before proving the theorem, we show a lemma below.

Lemma 9. *Let $J=\lfloor\frac{T}{\tau}\rfloor$, then*

$$\mathbb{E}\left[\sum_{t=\tau+1}^{T} H_{a_t}(t)\mathbb{1}(E_3(j_t))\right] \le \sqrt{\tau T}\left(2g_1(J)\sqrt{\frac{c_3}{\alpha}} + 2g_2(J) + u\sqrt{2g_1(J)^2\frac{c_3}{\alpha}+2g_2(J)^2}\sqrt{1+\log J}\right).$$

Proof. We know $H_{a_t}(t) = H_{a_t,1}(t) + H_{a_t,2}(t) + H_{a_t,3}(t)$ from definition, where

$$H_{i,1}(t) = g_1(j_t)\|x_{t,i}\|_{V_{j_t\tau+1}^{-1}}, \quad H_{i,2}(t) = g_2(j_t)\frac{\|x_{t,i}\|}{\sqrt{j_t}},$$
$$H_{i,3}(t) = u\sqrt{2g_1(j_t)^2\frac{c_3}{\alpha j_t}\|x_{t,i}\|^2 + 2g_2(j_t)^2\frac{\|x_{t,i}\|^2}{j_t}}$$

For all t, we have $j_t \le \lfloor \frac{T}{\tau} \rfloor$ and so $g_1(j_t) \le g_1(J)$, and $g_2(j_t) \le g_2(J)$. Since $\|X_t\|^2_{V^{-1}_{j\tau+1}} \le \lambda_{\max}(V^{-1}_{j\tau+1})\|X_t\|^2 \le \frac{c_3}{\alpha j}$ when $E_3(j)$ holds, we have

$$\mathbb{E}\left[\sum_{t=\tau+1}^{T} H_{a_t,1}(t)\mathbb{1}(E_3(j_t))\right] \le 2\tau g_1(J)\sqrt{\frac{c_3}{\alpha}J} \le 2g_1(J)\sqrt{\frac{c_3\tau}{\alpha}}\sqrt{T}. \tag{20}$$

We also have

$$\sum_{t=\tau+1}^{T} H_{a_t,2}(t) \le g_2(J)\sum_{t=\tau+1}^{T}\frac{\|X_t\|}{\sqrt{j_t}} \le 2g_2(J)\sqrt{\tau T}. \tag{21}$$

From Cauchy-Schwarz, we have

$$\begin{aligned}\sum_{t=\tau+1}^{T} H_{a_t,3}(t) &\le u\sqrt{T}\sqrt{\sum_{t=\tau+1}^{T} 2g_1(j_t)^2\frac{c_3}{\alpha j_t}\|X_t\|^2 + 2g_2(j_t)^2\frac{\|X_t\|^2}{j_t}} \\ &\le u\sqrt{T}\sqrt{2g_1(J)^2\frac{c_3\tau}{\alpha}(1+\log J) + 2g_2(J)^2\tau(1+\log J)}. \end{aligned}\tag{22}$$

Combine Equation 20, 21, 22, we get the conclusion. □

Now we formally prove Theorem 1.

Proof. Since

$$\begin{aligned}\mathbb{E}_{j_t\tau}\left[\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right] &\le \mathbb{E}_{j_t\tau}\left[\left(\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right)\mathbb{1}(E_2(j_t))\right] + \mathbb{P}(E_2^C(j_t)|\mathcal{F}_{j_t\tau}) \\ &\le \mathbb{E}_{j_t\tau}\left[\left(\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right)\mathbb{1}(E_2(j_t))\right] + \frac{1}{T^2},\end{aligned}$$

we have

$$\mathbb{E}\left[\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right] \le \mathbb{E}\left[\left(\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right)\mathbb{1}(E_2(j_t))\right] + \frac{1}{T^2}$$

From Proposition 1, when τ is chosen as in Equation 8, $E_3(j_t)$ holds with probability with at least $1 - \frac{1}{T^2}$ for every t. From the above,

$$\begin{aligned}\mathbb{E}[R(T)] &= \sum_{t=1}^{T}\mathbb{E}\left[\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right] \le \sum_{t=1}^{T}\mathbb{E}\left[\left(\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right)\mathbb{1}(E_2(j_t))\right] + \frac{1}{T} \\ &\le \mathbb{E}\left[\sum_{t=1}^{T}\left(\mu(x_{t,*}^T\theta^*) - \mu(X_t^T\theta^*)\right)\mathbb{1}(E_1(j_t)\cap E_2(j_t)\cap E_3(j_t))\right] + \sum_{t=1}^{T}\mathbb{P}(E_1^C(j_t)\cup E_3^C(j_t)) + \frac{1}{T} \\ &\le \tau + L_\mu\sum_{t=\tau+1}^{T}\mathbb{E}[\Delta_{a_t}(t)\mathbb{1}(E_1(j_t)\cap E_2(j_t)\cap E_3(j_t))] + \frac{7}{T} \\ &\le \tau + pL_\mu\sum_{t=\tau+1}^{T}\mathbb{E}\left[H_{a_t}(t)\mathbb{1}(E_3(j_t))\right] + \frac{7}{T} \qquad \text{from Lemma 6.}\end{aligned}$$

From Lemma 9, we have

$$\mathbb{E}[R(T)] \le \tau + L_\mu p\sqrt{\tau T}\left[2\sqrt{\frac{c_3}{\alpha}}g_1(J) + 2g_2(J) + u\sqrt{\frac{2c_3g_1(J)^2}{\alpha} + 2g_2(J)^2}\sqrt{1+\log\lfloor\frac{T}{\tau}\rfloor}\right] + \frac{7}{T}.$$

This ends our proof. □

8.8 Discussion

As pointed out by the reader, since $\|x_{t,a}\| \le 1$, so $\sigma_0^2, \lambda_f \le O(\frac{1}{d})$. So a more realistic assumption should be $\sigma_0^2, \lambda_f \sim O(\frac{1}{d})$. However, we found that $\sigma_0^2 \sim O(1)$ is an assumption that is widely used in literature (see Li et al. (2017)). If we assume $\sigma_0^2, \lambda_f \sim O(1/d)$, then the regret upper bound of our algorithm is $\mathbb{E}[R(T)] \le \tilde{O}(d^{\frac{5}{2}}\sqrt{T})$ and the regret upper bound of UCB-GLM (Li et al., 2017) is $\tilde{O}(d^3 + d\sqrt{T})$.