8 SUPPLEMENTARY MATERIAL

8.1 Proof of Lemma 1

The proof of Lemma 1 is an adaptation from the proof of Theorem 1 in Li et al. (2017).

Proof. Define \( G(\theta) := \sum_{s=1}^{t} (\mu(X_s^T \theta) - \mu(X_s^T \theta^*)) X_s \). We have \( G(\theta^*) = 0 \) and \( G(\hat{\theta}_t) = \sum_{s=1}^{t} \epsilon_s X_s \), where \( \epsilon_s \) is the sub-Gaussian noise at round \( s \). For convenience, define \( Z := G(\hat{\theta}_t) \). From mean value theorem, for any \( \theta_1, \theta_2 \), there exists \( v \in (0, 1) \) and \( \hat{\theta} = v \theta_1 + (1-v) \theta_2 \) such that

\[
G(\theta_1) - G(\theta_2) = \left[ \sum_{k=1}^{t} \mu'(X_s^T \hat{\theta}) X_s X_s^T \right] (\theta_1 - \theta_2) := F(\hat{\theta})(\theta_1 - \theta_2),
\]

where \( F(\hat{\theta}) = \sum_{s=1}^{t} \mu'(X_s^T \hat{\theta}) X_s X_s^T \). Therefore, for any \( \theta_1 \neq \theta_2 \), we have

\[
(\theta_1 - \theta_2)^T (G(\theta_1) - G(\theta_2)) = (\theta_1 - \theta_2)^T F(\hat{\theta})(\theta_1 - \theta_2) > 0,
\]

since \( \mu' > 0 \) and \( \lambda_{\min}(V_{t+1}) > 0 \). So \( G(\theta) \) is an injection from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). Consider an \( \eta \)-neighborhood of \( \theta^* \), \( \mathbb{B}_\eta := \{ \theta : ||\theta - \theta^*|| \leq \eta \} \), where \( \eta \) is a constant that will be specified later such that we have \( c_\eta = \inf_{\theta \in \mathbb{B}_\eta} \mu'(x^T \theta) > 0 \).

When \( \theta_1, \theta_2 \in \mathbb{B}_\eta \), from the property of convex set, we have \( \theta \in \mathbb{B}_\eta \). From Equation 11, we have when \( \theta \in \mathbb{B}_\eta \),

\[
||G(\theta)||_{V_{t+1}^{-1}} = ||G(\theta) - G(\theta^*)||_{V_{t+1}^{-1}} = \sqrt{(\theta - \theta^*)^T F(\theta) V_{t+1}^{-1} F(\theta)(\theta - \theta^*)} \geq c_\eta \sqrt{\lambda_{\min}(V_{t+1})} ||\theta - \theta^*||
\]

The last inequality is due to

\[
F(\hat{\theta}) \geq c_\eta \sum_{s=1}^{t} X_s X_s^T = c_\eta V_{t+1}.
\]

From Lemma A in Chen et al. (1999), we have that

\[
\{ \theta : ||G(\theta) - G(\theta^*)||_{V_{t+1}^{-1}} \leq c_\eta \eta \sqrt{\lambda_{\min}(V_{t+1})} \} \subset \mathbb{B}_\eta.
\]

Now from Lemma 7 in Li et al. (2017), we have with probability at least \( 1 - \delta \),

\[
||G(\hat{\theta}_t) - G(\theta^*)||_{V_{t+1}^{-1}} = ||Z||_{V_{t+1}^{-1}} \leq 4R \sqrt{\frac{d + \log \left( \frac{1}{\delta} \right)}{\eta^2}}.
\]

Therefore, when

\[
\eta \geq \frac{4R}{c_\eta} \sqrt{\frac{d + \log \left( \frac{1}{\delta} \right)}{\lambda_{\min}(V_{t+1})}},
\]

we have \( \hat{\theta}_t \in \mathbb{B}_\eta \). Since \( c_\eta \geq c_1 \geq c_3 > 0 \) when \( \eta \leq 1 \), we have

\[
||\hat{\theta}_t - \theta^*|| \leq \frac{4R}{c_\eta} \sqrt{\frac{d + \log \left( \frac{1}{\delta} \right)}{\lambda_{\min}(V_{t+1})}} \leq 1,
\]

when \( \lambda_{\min}(V_{t+1}) \geq \frac{16R^2 [d + \log(\frac{1}{\delta})]}{c_1^2} \).

\[
\]

8.2 Proof of Lemma 2

Note that the condition of Lemma 1 holds with high probability when \( \tau \) is chosen as Equation 8. This is a consequence of Proposition 1 in Li et al. (2017), which is presented below for reader’s convenience.
Proposition 1 (Proposition 1 in Li et al. (2017)). Define $V_{n+1} = \sum_{i=1}^{n} X_i X_i^T$, where $X_i$ is drawn IID from some distribution in unit ball $\mathbb{B}^d$. Furthermore, let $\Sigma := E[X_i X_i^T]$ be the second moment matrix, let $B, \delta_2 > 0$ be two positive constants. Then there exists positive, universal constants $C_1$ and $C_2$ such that $\lambda_{\min}(V_{n+1}) \geq B$ with probability at least $1 - \delta_2$, as long as

$$n \geq \left( \frac{C_1 \sqrt{d} + C_2 \sqrt{\log(1/\delta_2)}}{\lambda_{\min}(\Sigma)} \right)^2 + \frac{2B}{\lambda_{\min}(\Sigma)}.$$

Now we formally prove Lemma 2.

Proof. Note that from the definition of $\hat{\theta}_0$ in the algorithm, when $j = 1$, the conclusion holds trivially. When $\tau$ is chosen as in Equation 8, we have from Lemma 1 and Proposition 1 that $\|\hat{\theta}_1 - \theta^*\| \leq 1$ for all $t \geq \tau$ with probability at least $1 - \frac{2}{d^2}$. Therefore, $\hat{\theta}_\tau \in C$ for all $j = 1$ with probability at least $1 - \frac{2}{d^2}$. For the analysis below, we assume $\hat{\theta}_\tau \in C$ for all $j \geq 1$.

Since $\hat{\theta}_j \in C$, we have $\|\hat{\theta}_j - \theta^*\| \leq 3$. Denote $\mathbb{B}_\eta := \{\theta : \|\theta - \theta^*\| \leq \eta\}$, we have $\hat{\theta}_j, \hat{\theta}_\tau \in \mathbb{B}_3$. For any $v > 0$, define $\bar{v} = v \hat{\theta}_j + (1 - v)\hat{\theta}_\tau$, since $\mathbb{B}_3$ is convex, we have $\bar{v} \in \mathbb{B}_3$. Therefore, we have from Assumption 2

$$\nabla^2 l_{j, \tau}(\bar{v}) = \sum_{s=(j-1)\tau+1}^{j\tau} \mu'(X_s^T \bar{v}) X_s X_s^T \geq c_3 \sum_{s=(j-1)\tau+1}^{j\tau} X_s X_s^T.$$

Since we update $\hat{\theta}_j$ every $\tau$ rounds and $\theta_j^{TS}$ only depends on $\hat{\theta}_j$, For the next $\tau$ rounds, the pulled arms are only dependent on $\theta_j^{TS}$. Therefore, the feature vectors of pulled arms among the next $\tau$ rounds are IID. According to Proposition 1 and Equation 8, and by applying a union bound, we have $\lambda_{\min}\left(\sum_{s=(j-1)\tau+1}^{j\tau} X_s X_s^T\right) \geq \frac{G^2}{2}\alpha$ holds for all $j \geq 1$ with probability at least $1 - \frac{2}{d^2}$. This tells us that for all $j$, $l_{j, \tau}(\bar{v})$ is an $\alpha$-strongly convex function when $v \in \mathbb{B}_3$. Therefore, we can apply (Theorem 3.3 of Section 3.3.1 in Hazan et al. (2016)) to get for all $j \geq 1$

$$\sum_{q=1}^{j} \left( l_{q, \tau}(\bar{v}) - l_{q, \tau}(\hat{\theta}_\tau) \right) \leq \frac{G^2}{2\alpha} (1 + \log j),$$

where $G$ satisfies $G^2 \geq E[\|\nabla l_{q, \tau}\|^2]$. Note that $G \leq \tau$ since $\mu(x) \in [0, 1], Y_s \in [0, 1]$ and $\|X_s\| \leq 1$. From Jensen’s Inequality, we have

$$\sum_{q=1}^{j} \left( l_{q, \tau}(\bar{v}) - l_{q, \tau}(\hat{\theta}_\tau) \right) \leq \frac{G^2}{2\alpha} (1 + \log j).$$

Since $\hat{\theta}_j, \hat{\theta}_\tau \in \mathbb{B}_3$, we have for any $v > 0$, if $v = v \hat{\theta}_j + (1 - v)\hat{\theta}_\tau$, then $\nabla^2 l_{q, \tau}(\bar{v}) \geq \alpha I_d$ for all $1 \leq q \leq j$. Since $\sum_{q=1}^{j} \nabla l_{q, \tau}(\hat{\theta}_\tau) = 0$, we have

$$\|\bar{v} - \hat{\theta}_\tau\| \leq \frac{G}{\alpha} \sqrt{\frac{1 + \log j}{j}}.$$

By applying a union bound, we get the conclusion. \hfill $\square$

8.3 Proof of Lemma 3

We utilize the concentration property of MLE. Here, we present the analysis of MLE in Li et al. (2017).

Lemma 7 (Lemma 3 in Li et al. (2017)). Suppose $\lambda_{\min}(V_{\tau+1}) \geq 1$. For any $\delta_3 \in (0, 1)$, the following event

$$\mathcal{E} := \left\{ \|\hat{\theta}_t - \theta^*\|_{V_{\tau+1}} \leq \frac{R}{c_1} \sqrt{\frac{d}{2} \log(1 + \frac{2t}{d}) + \log \frac{1}{\delta_3}} \right\}$$

holds for all $t \geq \tau$ with probability at least $1 - \delta_3$.

Proof. Note that from Proposition 1 when $\alpha \geq c_3$, $\lambda_{\min}(V_{\tau+1}) \geq 1$ holds with probability at least $1 - \frac{1}{d^2}$. The proof of Lemma 3 is simply a combination of Lemma 2 and Lemma 7 by applying a union bound. \hfill $\square$
### 8.4 Proof of Lemma 4

We use formula 7.1.13 in [Abramowitz and Stegun 1948](#) to help derive the concentration and anti-concentration inequalities for Gaussian distributed random variables. Details are shown in Lemma 8.

**Lemma 8.** (Formula 7.1.13 in [Abramowitz and Stegun 1948](#)). For a Gaussian distributed random variable with mean $m$ and variance $\sigma^2$, we have for $z \geq 1$ that

$$
\mathbb{P}(|Z - m| \geq z\sigma) \leq \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{2}}.
$$

For $0 < z \leq 1$, we have

$$
\mathbb{P}(|Z - m| \geq z\sigma) \geq \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{2}}.
$$

Now we prove Lemma 4.

**Proof.** Since $\theta_j^{TS}|F_j \sim \mathcal{N}\left(\tilde{\theta}_j, \left(2g_1(j)^2\frac{\gamma_3}{\alpha_j} + \frac{2g_2(j)^2}{j}\right)I_d\right)$, and $\theta_j^{TS}$ is independent of $\{\cup_{t=\tau}^{t+1+\tau} A_t\} = \{x_{t,a}, a \in [K], j\tau < t \leq (j+1)\tau\}$, we have for $x \in \{\cup_{t=\tau}^{t+1+\tau} A_t\}$,

$$
x^T(\tilde{\theta}_j - \theta_j^{TS})|F_j, x \sim \mathcal{N}\left(0, \left(2g_1(j)^2\frac{\gamma_3}{\alpha_j} + \frac{2g_2(j)^2}{j}\right)\|x\|^2\right).
$$

From the property of Gaussian random variable in Lemma 8 when $u = \sqrt{2\log(T^2K\tau)}$, we have

$$
\mathbb{P}\left(|x^T(\tilde{\theta}_j - \theta_j^{TS})| \geq u\sqrt{2g_1(j)^2\frac{\gamma_3}{\alpha_j}\|x\|^2 + \frac{2g_2(j)^2}{j}\|x\|^2}|F_j, x\right) \leq \frac{1}{\sqrt{\pi}} e^{-\frac{u^2}{2}} \leq \frac{1}{K\tau T^2}. \tag{12}
$$

We use the following property of conditional probability

$$
\int_x \mathbb{P}(E|X = x, F)f(X = x|F)dx = \mathbb{P}(E|F), \tag{13}
$$

where $f(X = x|F)$ is the conditional p.d.f of a random variable $X$ and $E$ is an event. Combine Equation 12 and Equation 13, we have for every $a \in [K]$ and $j\tau < t \leq (j+1)\tau$,

$$
\begin{align*}
\mathbb{P}\left(|x^T_{t,a}(\tilde{\theta}_j - \theta_j^{TS})| \geq u\sqrt{2g_1(j)^2\frac{\gamma_3}{\alpha_j}\|x_{t,a}\|^2}|F_j, x\right) &= \int_x \mathbb{P}\left(|x^T_{t,a}(\tilde{\theta}_j - \theta_j^{TS})| \geq u\sqrt{2g_1(j)^2\frac{\gamma_3}{\alpha_j}|x_{t,a}|^2}|F_j, x_{t,a} = x\right)f(x_{t,a} = x|F_j, x)dx \\
&\leq \frac{1}{K\tau T^2} \int_x f(x_{t,a} = x|F_j, x)dx = \frac{1}{K\tau T^2}.
\end{align*}
$$

Applying a union bound, we get the conclusion. \qed

### 8.5 Proof of Lemma 5

**Proof.** We still use Lemma 8 to show the result. For convenience, denote $x := x_{t,\tau}, \gamma_1 := \sqrt{\frac{\gamma_3}{\alpha_j}}\|x\|$ and $\gamma_2 := \frac{\|x\|}{\sqrt{\gamma_1}}$. Note that $x$ is independent of $\theta_j^{TS}$, so

$$
x^T(\tilde{\theta}_j - \theta_j^{TS})|F_j, x \sim \mathcal{N}\left(0, (2g_1(j)^2\gamma_1^2 + 2g_2(j)^2\gamma_2^2)\right). \tag{14}
$$
Therefore,

\[
\mathbb{P}(x^T \theta_{j_t}^{TS} > x^T \theta^* | F_{j_t \tau}, x) = \mathbb{P} \left( \frac{x^T \theta_{j_t}^{TS} - x^T \bar{\theta}_{j_t}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} > \frac{x^T \theta^* - x^T \bar{\theta}_{j_t}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} \middle| F_{j_t \tau}, x \right)
\]

\[
\geq \mathbb{P} \left( \frac{x^T \theta_{j_t}^{TS} - x^T \bar{\theta}_{j_t}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} > \frac{g_1(j_t) \|x\|_{V_{j_t}^{-1}} + g_2(j_t) \|x\|_{V_{j_t}^{-1}}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} \middle| F_{j_t \tau}, x \right)
\]

\[
\geq \mathbb{P} \left( \frac{x^T \theta_{j_t}^{TS} - x^T \bar{\theta}_{j_t}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} > \frac{g_1(j_t) \|x\|_{V_{j_t}^{-1}} + g_2(j_t) \|x\|_{V_{j_t}^{-1}}}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} \right)
\]

\[
\geq \frac{1}{4\sqrt{\pi}} e^{-\frac{z^2}{2}},
\]

where \( z := \frac{g_1(j_t) \gamma_1 + g_2(j_t) \gamma_2}{\sqrt{2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2}} \). The first and second inequalities hold since \( F_t \) is a filtration such that \( E_1(j_t) \) and \( \lambda_{\min}(V_{j_{t+1}}) \geq \frac{\alpha_{j_t}}{c_3} \) are true. Notice that we have \( 0 < z \leq 1 \) since

\[
2g_1(j_t) 2\gamma_1^2 + 2g_2(j_t) 2\gamma_2^2 - (g_1(j_t) \gamma_1 + g_2(j_t) \gamma_2)^2 = (g_1(j_t) \gamma_1 - g_2(j_t) \gamma_2)^2 \geq 0.
\]

Therefore, we get

\[
\mathbb{P}(x^T \theta_{j_t}^{TS} > x^T \theta^* | F_{j_t \tau}, x) \geq \frac{1}{4\sqrt{\pi}} e^{-\frac{z^2}{2}} \geq \frac{1}{4\sqrt{\pi}e}.
\]

Similarly, using Equation (13) we get

\[
\mathbb{P}(x_t^T \theta_{j_t}^{TS} > x_t^T \theta^* | F_{j_t \tau}) = \int_x \mathbb{P}(x_t^T \theta_{j_t}^{TS} > x_t^T \theta^* | F_{j_t \tau}, x_t = x) f(x_t = x | F_{j_t \tau}) dx \geq \frac{1}{4\sqrt{\pi}e}.
\]

### 8.6 Proof of Lemma (3)

The technique used in this proof is extracted from Agrawal and Goyal (2013); Kveton et al. (2019).

**Proof.** Denote \( \mathbb{E}[\cdot] := \mathbb{E}[\cdot | F_t] \). To prove the lemma, we prove the following Equation holds for any possible filtration \( F_t \):

\[
\mathbb{E}_{j_t \tau}[\Delta_{a_t}(t) \mathbb{1}(E_1(j_t) \cap E_2(j_t) \cap E_3(j_t))] \leq \left( 1 + \frac{2}{\frac{1}{4\pi e} - \frac{1}{T^2}} \right) \mathbb{E}_{j_t \tau}[H_{a_t}(t) \mathbb{1}(E_3(j_t))]
\]

(15)

Denote the following set as the undersampled arms at round \( t \),

\[
S_t^C = \{i \in [K] : H_i(t) \geq \Delta_i(t)\}
\]

Note that \( a_t^* \in S_t^C \) for all \( t \). The set of sufficiently sampled arms is \( S_t = [K] \setminus S_t^C \). Let \( J_t = \text{argmin}_{i \in S_t^C} H_i(t) \) be the least uncertain undersampled arm at round \( t \). At round \( t \), denote \( j_t = \lfloor \frac{t - 1}{\tau} \rfloor \). In the steps below, we assume that event \( E_1(j_t) \cap E_2(j_t) \) occurs, then

\[
\Delta_{a_t}(t) = \Delta_{j_t}(t) + (x_{t,j_t} - X_j)^T \theta^*
\]

\[
= \Delta_{j_t}(t) + x_{t,j_t}^T (\theta^* - \theta_{j_t}^{TS}) + (x_{t,j_t} - X_j)^T \theta_{j_t}^{TS} + X_j^T (\theta_{j_t}^{TS} - \theta^*)
\]

\[
\leq \Delta_{j_t}(t) + H_{j_t}(t) + H_{a_t}(t) \quad \text{since} \quad (x_{t,j_t} - X_j)^T \theta_{j_t}^{TS} \leq 0
\]

\[
\leq 2H_{j_t}(t) + H_{a_t}(t) \quad \text{since} \quad J_t \in S_t^C.
\]

The left to do is to bound \( H_{j_t}(t) \) by \( H_{a_t}(t) \). Since \( J_t = \text{argmin}_{i \in S_t^C} H_i(t) \), we have

\[
\mathbb{E}_{j_t \tau}[H_{a_t}(t)] \geq \mathbb{E}_{j_t \tau}[H_{a_t}(t)|a_t \in S_t^C] \mathbb{P}(a_t \in S_t^C | F_{j_t \tau}) \geq \mathbb{E}_{j_t \tau}[H_{j_t}(t)] \mathbb{P}(a_t \in S_t^C | F_{j_t \tau}).
\]

(16)
Therefore, we have
\[
\mathbb{E}_{j,t} [\Delta_3(t) \mathbb{I}(E_1(j_t) \cap E_2(j_t))] \leq \left(1 + \frac{2}{P(a_t \in S^C_t | \mathcal{F}_{j,t})}\right) \mathbb{E}_{j,t} [H_{a_t}(t)]
\] (17)

Next, we bound \(P(a_t \in S^C_t | \mathcal{F}_{j,t})\).
\[
P(a_t \in S^C_t | \mathcal{F}_{j,t}) \geq P\left(x_{i,t}^T \theta_{j_t}^{TS} \geq \max_{i \in S_t} x_{i,t}^T \theta_{j_t}^{TS} | \mathcal{F}_{j,t}\right) \quad \text{since } a_t^* \in S^C_t
\]
\[
\geq P\left(x_{i,t}^T \theta_{j_t}^{TS} \geq \max_{i \in S_t} x_{i,t}^T \theta_{j_t}^{TS}, E_1(j_t) \cap E_2(j_t) | \mathcal{F}_{j,t}\right)
\]
\[
\geq P\left(x_{i,t}^T \theta_{j_t}^{TS} \geq x_{i,t}^T \theta^*, E_1(j_t) \cap E_2(j_t) | \mathcal{F}_{j,t}\right) - P\left(E_2^C(j_t) | \mathcal{F}_{j,t}\right)
\]
\[
\geq P\left(x_{i,t}^T \theta_{j_t}^{TS} \geq x_{i,t}^T \theta^*, E_1(j_t) | \mathcal{F}_{j,t}\right) - \frac{1}{T^2}.
\]
(18)

Inequality (18) holds because for all \(i \in S_t\), on event \(E_1(j_t) \cap E_2(j_t)\),
\[
x_{i,t}^T \theta_{j_t}^{TS} \leq x_{i,t}^T \theta^* + H_1(t) < x_{i,t}^T \theta^* + \Delta_t(t) = x_{i,t}^T \theta^*.
\]

Inequality (19) holds because of Lemma 8. When \(\mathcal{F}_t\) is a filtration such that \(E_1(j_t)\) and \(E_3(j_t)\) are true, we have from Lemma 8 that
\[
P\left(a_t \in S^C_t | \mathcal{F}_{j,t}\right) \geq \frac{1}{4\sqrt{\pi e}} - \frac{1}{T^2}.
\]

So under such filtration, from Equation (17) we have
\[
\mathbb{E}_{j,t} [\Delta_3(t) \mathbb{I}(E_1(j_t) \cap E_2(j_t))] \leq \left(1 + \frac{2}{4\sqrt{\pi e}} - \frac{1}{T^2}\right) \mathbb{E}_{j,t} [H_{a_t}(t)].
\]

Since \(E_3(j_t)\) is \(\mathcal{F}_{j,t}\)-measurable, we have under such filtration,
\[
\mathbb{E}_{j,t} [\Delta_3(t) \mathbb{I}(E_1(j_t) \cap E_2(j_t) \cap E_3(j_t))] \leq \left(1 + \frac{2}{4\sqrt{\pi e}} - \frac{1}{T^2}\right) \mathbb{E}_{j,t} [H_{a_t}(t) \mathbb{I}(E_3(j_t))].
\]

When \(\mathcal{F}_t\) is a filtration such that \(E_1(j_t) \cap E_3(j_t)\) is not true, the conclusion holds trivially. This finishes our proof. \(\Box\)

8.7 Proof of Theorem

Before proving the theorem, we show a lemma below.

Lemma 9. Let \(J = \left[\frac{T}{T}\right]\), then
\[
\mathbb{E} \left[\sum_{t=T+1}^{T} H_{a_t}(t) \mathbb{I}(E_3(j_t))\right] \leq \sqrt{T} \left(2g_1(J) \sqrt{\frac{c_3}{\alpha} t} + 2g_2(J) \sqrt{\frac{2g_1(J)^2 c_3}{\alpha} t} + 2g_2(J)^2 \sqrt{1 + \log J}\right).
\]

Proof. We know \(H_{a_t}(t) = H_{a_t,1}(t) + H_{a_t,2}(t) + H_{a_t,3}(t)\) from definition, where
\[
H_{i,1}(t) = g_1(j_t) \|x_{i,t}\|_{V_{i,t+1}^{-1}} \quad \text{and} \quad H_{i,2}(t) = g_2(j_t) \|x_{i,t}\|_{\sqrt{j_t}}^2,
\]
\[
H_{i,3}(t) = u \sqrt{2g_1(j_t)^2 \frac{c_3}{\alpha j_t} \|x_{i,t}\|^2 + 2g_2(j_t)^2 \|x_{i,t}\|^2_{j_t}}.
\]
For all $t$, we have $j_t \leq \lceil \frac{T}{J} \rceil$ and so $g_1(j_t) \leq g_1(J)$, and $g_2(j_t) \leq g_2(J)$. Since $\|X_t\|_{\gamma_{j+1}}^2 \leq \lambda_{\text{max}}(V_{j+1}^{-1})\|X_t\|^2 \leq \frac{C_3}{\alpha J}$ when $E_3(j_t)$ holds, we have

$$
\mathbb{E} \left[ \sum_{t=\tau+1}^{T} H_{a_1}(t) \mathbb{I}(E_3(j_t)) \right] \leq 2\tau g_1(J) \sqrt{\frac{C_3}{\alpha J}} \leq 2 g_1(J) \sqrt{\frac{C_3}{\alpha J} T}.
$$

(20)

We also have

$$
\sum_{t=\tau+1}^{T} H_{a_1}(t) \sum_{t=\tau+1}^{T} \|X_t\|_{\gamma_t} \leq 2g_2(J) \sqrt{T}.
$$

(21)

From Cauchy-Schwarz, we have

$$
\sum_{t=\tau+1}^{T} H_{a_1}(t) \leq u \sqrt{T} \sum_{t=\tau+1}^{T} 2g_1(j_t)^2 \frac{C_3}{\alpha J_t} \|X_t\|^2 + 2g_2(j_t)^2 \frac{\|X_t\|^2}{j_t}
$$

$$
\leq u \sqrt{T} \sqrt{2g_1(J)^2 \frac{C_3T}{\alpha} (1 + \log J) + 2g_2(J)^2 (1 + \log J)}.
$$

(22)

Combine Equation 20, 21, 22, we get the conclusion.

Now we formally prove Theorem 1.

**Proof.** Since

$$
\mathbb{E}_{j_t} [\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)] \leq \mathbb{E}_{j_t} \left[ (\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)) \mathbb{I}(E_2(j_t)) \right] + \mathbb{P}(E_{2}^C(j_t) | \mathcal{F}_{j_t})
$$

$$
\leq \mathbb{E}_{j_t} \left[ (\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)) \mathbb{I}(E_2(j_t)) \right] + \frac{1}{T^2},
$$

we have

$$
\mathbb{E} [\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)] \leq \mathbb{E} \left[ (\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)) \mathbb{I}(E_2(j_t)) \right] + \frac{1}{T^2}
$$

From Proposition 2 when $\tau$ is chosen as in Equation 8, $E_3(j_t)$ holds with probability with at least $1 - \frac{1}{T^2}$ for every $t$. From the above,

$$
\mathbb{E}[R(T)] = \sum_{t=1}^{T} \mathbb{E} [\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)] \leq \sum_{t=1}^{T} \mathbb{E} \left[ (\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)) \mathbb{I}(E_2(j_t)) \right] + \frac{1}{T}
$$

$$
\leq \mathbb{E} \left[ \sum_{t=1}^{T} (\mu(x_{t,e}^T \theta^*) - \mu(X_t^T \theta^*)) \mathbb{I}(E_1(j_t) \cap E_2(j_t) \cap E_3(j_t)) \right] + \mathbb{P}(E_1^C(j_t) \cup E_2^C(j_t) \cup E_3^C(j_t)) + \frac{1}{T}
$$

$$
\leq \tau + L_{\mu} \sum_{t=\tau+1}^{T} \mathbb{E}[\Delta_{a_1}(t) \mathbb{I}(E_1(j_t) \cap E_2(j_t) \cap E_3(j_t))] + \frac{7}{T}
$$

$$
\leq \tau + pL_{\mu} \sum_{t=\tau+1}^{T} \mathbb{E}[H_{a_1}(t) \mathbb{I}(E_3(j_t))] + \frac{7}{T} \text{ from Lemma 6}
$$

From Lemma 3 we have

$$
\mathbb{E}[R(T)] \leq \tau + L_{\mu} p \sqrt{T} \left[ 2 \sqrt{\frac{C_3}{\alpha} g_1(J)} + 2 g_2(J) + u \sqrt{\frac{2C_3 g_1(J)^2}{\alpha}} + 2g_2(J)^2 \sqrt{1 + \log \frac{T}{\tau}} \right] + \frac{7}{T}.
$$

This ends our proof.

\[
8.8 \text{ Discussion}
\]

As pointed out by the reader, since $\|x_{t,e}\| \leq 1$, so $\sigma_0^2, \lambda_f \leq O(\frac{1}{d})$. So a more realistic assumption should be $\sigma_0^2, \lambda_f \sim O(d)$. However, we found that $\sigma_0^2 \sim O(1)$ is an assumption that is widely used in literature (see Li et al. 2017). If we assume $\sigma_0^2, \lambda_f \sim O(1/d)$, then the regret upper bound of our algorithm is $\mathbb{E}[R(T)] \leq \tilde{O}(d^2 \sqrt{T})$ and the regret upper bound of UCB-GLM (Li et al. 2017) is $\tilde{O}(d^3 + d \sqrt{T})$. 