Supplementary Material:
Tight Differential Privacy for Discrete-Valued Mechanisms and for the Subsampled Gaussian Mechanism Using FFT

1 Proofs for the Results of Section 3

1.1 Integral Representation for Exact DP-Guarantees

Throughout this section we denote for neighbouring datasets X and Y the density function of $\mathcal{M}(X)$ with $f_X(t)$ and the density function of $\mathcal{M}(Y)$ with $f_Y(t)$. The definition of approximate differential privacy is equivalently given as follows.

Definition 1. *A randomised algorithm $\mathcal{M}$ with an output of one dimensional distributions satisfies (ε, δ)-DP if for every set $S \subset \mathbb{R}$ and every neighbouring datasets X and Y*

$$\int\limits_S f_X(t) \, \mathrm{d}t \leq \mathrm{e}^{\varepsilon} \int\limits_S f_Y(t) \, \mathrm{d}t + \delta \quad \textit{and} \quad \int\limits_S f_Y(t) \, \mathrm{d}t \leq \mathrm{e}^{\varepsilon} \int\limits_S f_X(t) \, \mathrm{d}t + \delta.$$

We call $\mathcal{M}$ tightly (ε, δ)-DP, if there does not exist $\delta' < \delta$ such that $\mathcal{M}$ is (ε, δ')-DP.

The auxiliary lemma 2 is needed for Lemma 3. For discrete valued distributions, it is given in [2, Lemma 1] and another version of this result using so called f-divergences is given in [1]. We prove it here for for completeness, using our formalism. In the proof, if f_X and f_Y are discrete valued distributions and if

$$f_X(t) - \mathrm{e}^{\varepsilon} f_Y(t) = \sum_i c_i \cdot \delta_{t_i}(t)$$

for some coefficients $c_i, t_i \in \mathbb{R}$, then $\max\{f_X(t) - \mathrm{e}^{\varepsilon} f_Y(t), 0\}$ denotes

$$\max\{f_X(t) - \mathrm{e}^{\varepsilon} f_Y(t), 0\} = \sum_i \max\{c_i, 0\} \cdot \delta_{t_i}(t),$$

and the set S denotes

$$S = \{t \in \mathbb{R} \, : \, f_Y(t) \geq \mathrm{e}^{\varepsilon} f_X(t)\} = \mathbb{R} \setminus \{t_i \, : \, c_i < 0\}.$$

Lemma 2. *$\mathcal{M}$ is tightly (ε, δ)-DP with*

$$\delta(\varepsilon) = \max_{X \sim Y} \left\{ \int\limits_{\mathbb{R}} \max\{f_X(t) - \mathrm{e}^{\varepsilon} f_Y(t), 0\} \, \mathrm{d}t, \int\limits_{\mathbb{R}} \max\{f_Y(t) - \mathrm{e}^{\varepsilon} f_X(t), 0\} \, \mathrm{d}t \right\}. \tag{1.1}$$

Proof. Assume $\mathcal{M}$ is tightly (ε,δ)-DP. Then, for every set $S \subset \mathbb{R}$ and for all $X \sim Y$:

$$\int\limits_S f_X(t) - \mathrm{e}^\varepsilon f_Y(t) \, \mathrm{d}t \leq \int\limits_S \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t \leq \int\limits_{\mathbb{R}} \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t.$$

We get an analogous bound for $\int_S f_Y(t) - \mathrm{e}^\varepsilon f_X(t) \, \mathrm{d}t$. Since $\mathcal{M}$ is tightly (ε,δ)-DP, by Definition 1,

$$\delta \leq \max\left\{ \int\limits_{\mathbb{R}} \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t, \int\limits_{\mathbb{R}} \max\{f_Y(t) - \mathrm{e}^\varepsilon f_X(t), 0\} \, \mathrm{d}t \right\}.$$

To show that the above inequality is tight, consider the set

$$S = \{t \in \mathbb{R} \, : \, f_X(t) \geq \mathrm{e}^\varepsilon f_Y(t)\}.$$

Then,

$$\begin{aligned} \int\limits_S f_X(t) - \mathrm{e}^\varepsilon f_Y(t) \, \mathrm{d}t &= \int\limits_S \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t \\ &= \int\limits_{\mathbb{R}} \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t. \end{aligned} \tag{1.2}$$

Next, consider the set $S = \{t \in \mathbb{R} \, : \, f_Y(t) \geq \mathrm{e}^\varepsilon f_X(t)\}$. Similarly,

$$\int\limits_S f_Y(t) - \mathrm{e}^\varepsilon f_X(t) \, \mathrm{d}t = \int\limits_{\mathbb{R}} \max\{f_Y(t) - \mathrm{e}^\varepsilon f_X(t), 0\} \, \mathrm{d}t. \tag{1.3}$$

From (1.2) and (1.3) it follows that there exists a set $S \subset \mathbb{R}$ such that either

$$\int_S f_X(t) \, \mathrm{d}t = \mathrm{e}^\varepsilon \int_S f_Y(t) \, \mathrm{d}t + \delta \quad \text{or} \quad \int_S f_Y(t) \, \mathrm{d}t = \mathrm{e}^\varepsilon \int_S f_X(t) \, \mathrm{d}t + \delta$$

for δ given by (1.1). This shows that δ given by (1.1) is tight. □

Recall from the main text that if f_X and f_Y are of the form (3.1), then the PLD distribution function is given by

$$\omega_{X/Y}(s) = \sum_{t_{X,i} = t_{Y,j}} a_{X,i} \cdot \delta_{s_i}(s), \quad s_i = \log\left(\tfrac{a_{X,i}}{a_{Y,j}}\right). \tag{1.4}$$

The following lemma gives an integral representation for the tight $\delta(\varepsilon)$-bound involving the distribution function of the PLD. For discrete valued distributions, it is originally given in [3, Lemma 5].

Lemma 3. *Let $\mathcal{M}$ be defined as above. $\mathcal{M}$ is tightly (ε,δ)-DP for*

$$\delta(\varepsilon) = \max_{X \sim Y} \max\{\delta_{X/Y}(\varepsilon), \delta_{Y/X}(\varepsilon)\},$$

where

$$\begin{aligned} \delta_{X/Y}(\varepsilon) &= \delta_{X/Y}(\infty) + \int\limits_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s}) \, \omega_{X/Y}(s) \, \mathrm{d}s, \\ \delta_{Y/X}(\varepsilon) &= \delta_{Y/X}(\infty) + \int\limits_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s}) \, \omega_{Y/X}(s) \, \mathrm{d}s, \\ \delta_{X/Y}(\infty) &= \sum_{\{t_i \, : \, \mathbb{P}(\mathcal{M}(X) = t_i) > 0, \, \mathbb{P}(\mathcal{M}(Y) = t_i) = 0\}} a_{X,i}, \\ \delta_{Y/X}(\infty) &= \sum_{\{t_i \, : \, \mathbb{P}(\mathcal{M}(Y) = t_i) > 0, \, \mathbb{P}(\mathcal{M}(X) = t_i) = 0\}} a_{Y,i}. \end{aligned}$$

Proof. We directly find from the definition of f_X and f_Y and from the definition (1.4) that

$$\begin{aligned}
\max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} &= \sum_{\{t_i \, : \, \mathbb{P}(\mathcal{M}(X)=t_i)>0, \, \mathbb{P}(\mathcal{M}(Y)=t_i)=0\}} a_{X,i} \cdot \delta_{t_{X,i}}(t) + \sum_{t_{X,i}=t_{Y,j}} \max\{a_{X,i} - \mathrm{e}^\varepsilon a_{Y,j}, 0\} \cdot \delta_{t_{X,i}}(t) \\
&= \sum_{\{t_i \, : \, \mathbb{P}(\mathcal{M}(X)=t_i)>0, \, \mathbb{P}(\mathcal{M}(Y)=t_i)=0\}} a_{X,i} \cdot \delta_{t_{X,i}}(t) + \sum_{t_{X,i}=t_{Y,j}} a_{X,i} \max\{(1 - \mathrm{e}^{\varepsilon - s_i}), 0\} \cdot \delta_{t_{X,i}}(t),
\end{aligned}$$

and therefore

$$\begin{aligned}
\int\limits_{\mathbb{R}} \max\{f_X(t) - \mathrm{e}^\varepsilon f_Y(t), 0\} \, \mathrm{d}t &= \delta_{X/Y}(\infty) + \sum_{t_{X,i}=t_{Y,j}} a_{X,i} \max\{(1 - \mathrm{e}^{\varepsilon - s_i}), 0\} \\
&= \delta_{X/Y}(\infty) + \int\limits_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, \omega_{X/Y}(s) \, \mathrm{d}s.
\end{aligned}$$

Analogously, we see that

$$\int\limits_{\mathbb{R}} \max\{f_Y(t) - \mathrm{e}^\varepsilon f_X(t), 0\} \, \mathrm{d}t = \delta_{Y/X}(\infty) + \int\limits_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, \omega_{Y/X}(s) \, \mathrm{d}s.$$

The claim follows then from Lemma 2. □

1.2 Privacy Loss Distribution of Compositions

The following theorem shows that the PLD distribution of discrete non-adaptive compositions is obtain using a discrete convolution. We first recall the definition of convolution of two generalised functions as defined in the main text. Suppose the distributions f_X and f_Y are of the form

$$\begin{aligned}
f_X(t) &= \textstyle\sum_i a_{X,i} \cdot \delta_{t_{X,i}}(t), \\
f_Y(t) &= \textstyle\sum_i a_{Y,i} \cdot \delta_{t_{Y,i}}(t),
\end{aligned}$$

where $t_{X,i}, t_{Y,i} \in \mathbb{R}$ and $a_{X,i}, a_{Y,i} \geq 0$. We define the convolution $f_X * f_Y$ as

$$(f_X * f_Y)(t) = \sum_{i,j} a_{X,i} \, a_{Y,j} \cdot \delta_{t_{X,i} + t_{Y,j}}(t). \tag{1.5}$$

The result of the following theorem is originally given in [3, Thm. 1]. For completeness we give a proof using our notation with generalised probability density functions.

Theorem 4. *Let $f_X(t)$, $f_Y(t)$, $f_{X'}(t)$ and $f_{Y'}(t)$ denote the density functions of $\mathcal{M}(X)$, $\mathcal{M}(Y)$, $\mathcal{M}'(X)$ and $\mathcal{M}'(Y)$, respectively. Denote by $\omega_{X/Y}$ the PLD distribution of $\mathcal{M}(X)$ over $\mathcal{M}(Y)$ and by $\omega_{X'/Y'}$ the PLD distribution of $\mathcal{M}'(X)$ over $\mathcal{M}'(Y)$. Denote by $\widetilde{\omega}_{X/Y}$ the PLD of the non-adaptive composition $\mathcal{M} \circ \mathcal{M}' = (\mathcal{M}, \mathcal{M}')$. The density function of $\widetilde{\omega}_{X/Y}$ is given by*

$$\widetilde{\omega}_{X/Y} = \omega_{X/Y} * \omega_{X'/Y'}.$$

Moreover,

$$\begin{aligned}
\widetilde{\delta}_{X/Y}(\infty) &:= \mathbb{P}((\mathcal{M} \circ \mathcal{M}')(X) > 0, (\mathcal{M} \circ \mathcal{M}')(Y) = 0) \\
&= 1 - \big(1 - \delta_{X/Y}(\infty)\big)\big(1 - \delta'_{X/Y}(\infty)\big),
\end{aligned}$$

where

$$\delta_{X/Y}(\infty) = \mathbb{P}(\mathcal{M}(X) > 0, \mathcal{M}(Y) = 0), \quad \delta'_{X/Y}(\infty) = \mathbb{P}(\mathcal{M}'(X) > 0, \mathcal{M}'(Y) = 0).$$

Proof. By definition of the privacy loss distribution,

$$\begin{aligned}
\widetilde{\omega}_{X/Y}(s) &= \sum_{(t_i, t_i') = (t_j, t_j')} \mathbb{P}\big((\mathcal{M} \circ \mathcal{M}')(X) = (t_i, t_i')\big) \cdot \delta_{\widetilde{s}_i}(s), \\
\widetilde{s}_i &= \log \left(\frac{(\mathcal{M} \circ \mathcal{M}')(X) = (t_i, t_i')}{(\mathcal{M} \circ \mathcal{M}')(Y) = (t_j, t_j')} \right).
\end{aligned}$$

Due to the independence of $\mathcal{M}$ and $\mathcal{M}'$,

$$\begin{aligned}\mathbb{P}\big(\mathcal{M}(X)=t_i,\, \mathcal{M}'(X)=t_i'\big) &= \mathbb{P}\big(\mathcal{M}(X)=t_i\big)\,\mathbb{P}\big(\mathcal{M}'(X)=t_i'\big),\\ \mathbb{P}\big(\mathcal{M}(Y)=t_j,\, \mathcal{M}'(Y)=t_j'\big) &= \mathbb{P}\big(\mathcal{M}(Y)=t_j\big)\,\mathbb{P}\big(\mathcal{M}'(Y)=t_j'\big).\end{aligned} \tag{1.6}$$

Therefore,

$$\log\left(\frac{\mathbb{P}\big(\mathcal{M}(X)=t_i,\, \mathcal{M}'(X)=t_i'\big)}{\mathbb{P}\big(\mathcal{M}(Y)=t_j,\, \mathcal{M}'(Y)=t_j'\big)}\right) = \log\left(\frac{\mathbb{P}\big(\mathcal{M}(X)=t_i\big)}{\mathbb{P}\big(\mathcal{M}(Y)=t_j\big)}\right) + \log\left(\frac{\mathbb{P}\big(\mathcal{M}'(X)=t_i'\big)}{\mathbb{P}\big(\mathcal{M}'(Y)=t_j'\big)}\right).$$

and

$$\widetilde{\omega}_{X/Y}(s) = \sum_{(t_i,t_i')=(t_j,t_j')} \mathbb{P}\big(\mathcal{M}(X)=t_i\big)\,\mathbb{P}\big(\mathcal{M}'(X)=t_i'\big)\cdot \delta_{s_i+s_i'}(s), \tag{1.7}$$

where

$$s_i = \log\left(\frac{\mathbb{P}\big(\mathcal{M}(X)=t_i\big)}{\mathbb{P}\big(\mathcal{M}(Y)=t_j\big)}\right), \quad s_i' = \log\left(\frac{\mathbb{P}\big(\mathcal{M}'(X)=t_i'\big)}{\mathbb{P}\big(\mathcal{M}'(Y)=t_j'\big)}\right).$$

We see from (1.7) that $\widetilde{\omega}_{X/Y} = \omega_{X/Y} * \omega_{X'/Y'}$ with convolution defined in (1.5). The expression for $\widetilde{\delta}_{X/Y}(\infty)$ follows directly from its definition and from the independence of the mechanisms (1.6). □

Theorem 4 directly gives the following representation for tight $\delta(\varepsilon)$ of compositions.

Corollary 5. *Consider k consecutive applications of a mechanism $\mathcal{M}$. Let $\varepsilon > 0$. The composition is tightly (ε,δ)-DP for δ given by*

$$\delta(\varepsilon) = \max_{X\sim Y} \max\{\delta_{X/Y}(\varepsilon), \delta_{Y/X}(\varepsilon)\},$$

where

$$\delta_{X/Y}(\varepsilon) = 1 - (1-\delta_{X/Y}(\infty))^k + \int_{\varepsilon}^{\infty} (1-\mathrm{e}^{\varepsilon - s})\left(\omega_{X/Y} *^k \omega_{X/Y}\right)(s)\,\mathrm{d}s,$$

where $(\omega_{X/Y} *^k \omega_{X/Y})(s)$ denotes the density function obtained by convolving $\omega_{X/Y}$ by itself k times (an analogous formula holds for $\delta_{Y/X}(\varepsilon)$).

2 Proofs for the Results of Section 4

2.1 Grid Approximation

Recall from Section 4 of the main text: we place the PLD distribution on a grid $X_n = \{x_0,\ldots,x_{n-1}\}$, $n \in \mathbb{Z}^+$, where

$$x_i = -L + i\Delta x, \quad \Delta x = 2L/n. \tag{2.1}$$

Suppose the distribution ω of the PLD is of the form

$$\omega(s) = \sum_{i=0}^{n-1} a_i \cdot \delta_{s_i}(s), \tag{2.2}$$

where $a_i \geq 0$ and $-L \leq s_i \leq L - \Delta x$, $0 \leq i \leq n-1$. We define the grid approximations

$$\begin{aligned}\omega^{\mathrm{L}}(s) &= \sum_{i=0}^{n-1} a_i \cdot \delta_{s_i^{\mathrm{L}}}(s), \quad s_i^{\mathrm{L}} = \sup\{x \in X_n \,:\, s_i \geq x\},\\ \omega^{\mathrm{R}}(s) &= \sum_{i=0}^{n-1} a_i \cdot \delta_{s_i^{\mathrm{R}}}(s), \quad s_i^{\mathrm{R}} = \inf\{x \in X_n \,:\, s_i \leq x\}.\end{aligned} \tag{2.3}$$

Lemma 6. *Let $\delta(\varepsilon)$ be given by the integral formula of Lemma 3 and let $\delta^{\mathrm{L}}(\varepsilon)$ and $\delta^{\mathrm{R}}(\varepsilon)$ be defined analogously by ω^{L} and ω^{R}. Then for all $\varepsilon > 0$ we have*

$$\delta^{\mathrm{L}}(\varepsilon) \leq \delta(\varepsilon) \leq \delta^{\mathrm{R}}(\varepsilon). \tag{2.4}$$

Proof. The claim follows from the definition (2.3) and from the fact that $(1-\mathrm{e}^{\varepsilon-s})$ is a monotonously increasing function of s. $\square$

Corollary 7. *Lemma 6 directly generalises to convolutions. Namely, if*

$$(\omega *^k \omega)(s) = \sum_i a_i \cdot \delta_{s_i}(s)$$

for some coefficients $a_i \geq 0$, $s_i \in \mathbb{R}$, then from the definition (1.5) *it follows that*

$$(\omega^{\mathrm{L}} *^k \omega^{\mathrm{L}})(s) = \sum_i a_i \cdot \delta_{s_i^{\mathrm{L}}}(s)$$

for some s_i^{L} such that $s_i^{\mathrm{L}} \leq s_i$ for all i. And similarly, then

$$(\omega^{\mathrm{R}} *^k \omega^{\mathrm{R}})(s) = \sum_i a_i \cdot \delta_{s_i^{\mathrm{R}}}(s)$$

for some s_i^{R} such that $s_i^{\mathrm{R}} \geq s_i$ for all i. And since $(1-\mathrm{e}^{\varepsilon-s})$ is a monotonously increasing function of s for $s \geq \varepsilon$, the inequality (2.4) *holds also in case $\delta(\varepsilon)$, $\delta^{\mathrm{L}}(\varepsilon)$ and $\delta^{\mathrm{R}}(\varepsilon)$ is determined by $\omega *^k \omega$, $\omega^{\mathrm{L}} *^k \omega^{\mathrm{L}}$ and $\omega^{\mathrm{R}} *^k \omega^{\mathrm{R}}$, respectively.*

The following bounds for the moment generating functions will be used in the error analysis.

Lemma 8. *Let ω, ω^{R} and ω^{L} be defined as in* (2.2) *and* (2.3) *and let $0 < \lambda < (\Delta x)^{-1}$. Then*

$$\mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{L}}}] \leq \mathbb{E}[\mathrm{e}^{\lambda \omega}], \qquad \mathbb{E}[\mathrm{e}^{-\lambda \omega^{\mathrm{L}}}] \leq \tfrac{1}{1-\lambda \Delta x} \mathbb{E}[\mathrm{e}^{-\lambda \omega}] \tag{2.5}$$

and

$$\mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{R}}}] \leq \tfrac{1}{1-\lambda \Delta x} \mathbb{E}[\mathrm{e}^{\lambda \omega}], \qquad \mathbb{E}[\mathrm{e}^{-\lambda \omega^{\mathrm{R}}}] \leq \mathbb{E}[\mathrm{e}^{-\lambda \omega}]. \tag{2.6}$$

Proof. The condition $\mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{L}}}] \leq \mathbb{E}[\mathrm{e}^{\lambda \omega}]$ follows directly from the definition (2.3):

$$\mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{L}}}] = \sum_{i=0}^{n-1} a_i \mathrm{e}^{\lambda s_i^{\mathrm{L}}} \leq \sum_{i=0}^{n-1} a_i \mathrm{e}^{\lambda s_i} = \mathbb{E}[\mathrm{e}^{\lambda \omega}],$$

since $s_i^{\mathrm{L}} \leq s_i$ for all $0 \leq i \leq n-1$. The proof for the condition $\mathbb{E}[\mathrm{e}^{-\lambda \omega^{\mathrm{R}}}] \leq \mathbb{E}[\mathrm{e}^{-\lambda \omega}]$ goes similarly.

Using the Lipschitz continuity of the exponential function, we see that

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{R}}}] - \mathbb{E}[\mathrm{e}^{\lambda \omega}] &= \sum_{i=0}^{n-1} a_i \big(\mathrm{e}^{\lambda s_i^{\mathrm{R}}} - \mathrm{e}^{\lambda s_i}\big) \\ &\leq \sum_{i=0}^{n-1} a_i \lambda \left| s_i^{\mathrm{R}} - s_i \right| \mathrm{e}^{\lambda s_i^{\mathrm{R}}} \\ &\leq \lambda \Delta x \sum_{i=0}^{n-1} a_i \mathrm{e}^{\lambda s_i^{\mathrm{R}}} = \lambda \Delta x \, \mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{R}}}]. \end{aligned}$$

Thus

$$(1-\lambda \Delta x) \, \mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{R}}}] \leq \mathbb{E}[\mathrm{e}^{\lambda \omega}]$$

from which the condition $\mathbb{E}[\mathrm{e}^{\lambda \omega^{\mathrm{R}}}] \leq \frac{1}{1-\lambda \Delta x} \, \mathbb{E}[\mathrm{e}^{\lambda \omega}]$ follows. The proof for the condition $\mathbb{E}[\mathrm{e}^{-\lambda \omega^{\mathrm{L}}}] \leq \frac{1}{1-\lambda \Delta x} \, \mathbb{E}[\mathrm{e}^{-\lambda \omega}]$ goes similarly. $\square$

2.2 FFT Evaluation for Truncated Convolutions of Periodic Distributions

We next prove the lemma showing that the truncated convolutions of periodic distributions can be evaluated using FFT. Suppose ω is defined on X_n such that

$$\omega(s) = \sum_{i=0}^{n-1} a_i \cdot \delta_{s_i}(s), \tag{2.7}$$

where $a_i \geq 0$ and $s_i = i\Delta x$. The convolutions can then be written as

$$(\omega * \omega)(s) = \sum_{i,j} a_i a_j \cdot \delta_{s_i+s_j}(s) = \sum_i \Big(\sum_j a_j a_{i-j} \Big) \cdot \delta_{s_i}(s).$$

We define $\widetilde{\omega}$ to be a $2L$-periodic extension of ω such that

$$\widetilde{\omega}(s) = \sum_{m \in \mathbb{Z}} \sum_i a_i \cdot \delta_{s_i + m \cdot 2L}(s).$$

In case the distribution ω is defined on an equidistant grid, FFT can be used to evaluate the approximation $\widetilde{\omega} \circledast \widetilde{\omega}$:

Lemma 9. *Let ω be of the form (2.7), such that n is even and $s_i = -L + i\Delta x$, $0 \leq i \leq n-1$. Define*

$$\boldsymbol{a} = \begin{bmatrix} a_0 & \dots & a_{n-1} \end{bmatrix}^{\mathrm{T}} \quad \textit{and} \quad D = \begin{bmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Then,

$$(\widetilde{\omega} \circledast^k \widetilde{\omega})(s) = \sum_{i=0}^{n-1} b_i^k \cdot \delta_{s_i}(s),$$

where

$$b_i^k = \big[D\, \mathcal{F}^{-1} \big(\mathcal{F}(D\boldsymbol{a})^{\odot k} \big) \big]_i ,$$

and $^{\odot k}$ denotes the elementwise power of vectors.

Proof. Assume n is even and $s_i = -L + i\Delta x$, $0 \leq i \leq n-1$. From the the truncation and periodisation it follows that $\widetilde{\omega} \circledast \widetilde{\omega}$ is of the form

$$(\widetilde{\omega} \circledast \widetilde{\omega})(s) = \sum_{i=0}^{n-1} b_i \cdot \delta_{s_i}(s), \qquad b_i = \sum_{j=n/2}^{3n/2-1} a_j\, a_{i-j} \ \text{(indices modulo } n). \tag{2.8}$$

Denoting $\widetilde{\boldsymbol{a}} = D\boldsymbol{a}$, we see that the coefficients b_i in (2.8) are given by the expression

$$b_{i+n/2} = \sum_{j=0}^{n-1} \widetilde{a}_j\, \widetilde{a}_{i-j} \ \text{(indices modulo } n),$$

to which we can apply DFT and the convolution theorem [4]. I.e., when $0 \leq i \leq n-1$,

$$b_{i+n/2} = \big[\mathcal{F}^{-1} \big(\mathcal{F}(\widetilde{\boldsymbol{a}}) \odot \mathcal{F}(\widetilde{\boldsymbol{a}}) \big) \big]_i = \big[\mathcal{F}^{-1} \big(\mathcal{F}(D\boldsymbol{a}) \odot \mathcal{F}(D\boldsymbol{a}) \big) \big]_i , \ \text{(indices modulo } n) \tag{2.9}$$

where $\odot$ denotes the elementwise product of vectors. From (2.9) we find that

$$b_i = \big[D\mathcal{F}^{-1} \big(\mathcal{F}(D\boldsymbol{a}) \odot \mathcal{F}(D\boldsymbol{a}) \big) \big]_i , \ \text{(indices modulo } n).$$

By induction this generalises to k-fold compositions and we arrive at the claim. □

3 Proof of Theorem 10

We next prove step by step the main theorem, i.e., Theorem 10 of the main text. We start by splitting the error induced by Algorithm 1 into three terms.

Lemma 10. *Let ω be a generalised distribution and denote by $\widetilde{\delta}(\varepsilon)$ the result of Algorithm 1. Total error of the approximation can be split as follows:*

$$\left| \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \widetilde{\delta}(\varepsilon) \right| \leq I_1(L) + I_2(L) + I_3(L),$$

where

$$\begin{aligned} I_1(L) &= \int_{L}^{\infty} (\omega *^k \omega)(s)\, \mathrm{d}s, \\ I_2(L) &= \int_{\varepsilon}^{L} (\omega *^k \omega - \omega \circledast^k \omega)(s)\, \mathrm{d}s, \\ I_3(L) &= \int_{\varepsilon}^{L} \left| (\omega \circledast^k \omega - \widetilde{\omega} \circledast^k \widetilde{\omega})(s) \right|\, \mathrm{d}s, \end{aligned}$$

where, for a generalised density function of the form $\sum_i a_i \cdot \delta_{s_i}(s)$, the absolute value denotes

$$\left| \sum_i a_i \cdot \delta_{s_i}(s) \right| = \sum_i |a_i| \cdot \delta_{s_i}(s).$$

Proof. By adding and subtracting terms and using the triangle inequality, we get

$$\begin{aligned} \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \widetilde{\delta}(\varepsilon) &= \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s \\ &\quad + \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\widetilde{\omega} \circledast^k \widetilde{\omega})(s)\, \mathrm{d}s. \end{aligned} \tag{3.1}$$

Since $0 \leq (1 - \mathrm{e}^{\varepsilon - s}) < 1$ for all $s \geq \varepsilon$, we have for the first term on the right hand side of (3.1):

$$0 \leq \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s \leq \int_{L}^{\infty} (\omega *^k \omega)(s)\, \mathrm{d}s. \tag{3.2}$$

Similarly, adding and subtracting $\int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\omega \circledast^k \omega)(s)\, \mathrm{d}s$ the second term on the right hand side of (3.1), we find that

$$\left| \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\omega *^k \omega)(s)\, \mathrm{d}s - \int_{\varepsilon}^{L} (1 - \mathrm{e}^{\varepsilon - s})(\widetilde{\omega} \circledast^k \widetilde{\omega})(s)\, \mathrm{d}s \right| \leq I_2(L) + I_3(L)$$

which shows the claim. □

We next consider separately each of the three terms stated in Theorem 10. Each of them are bounded using the Chernoff bound [5]

$$\mathbb{P}[X \geq t] = \mathbb{P}[\mathrm{e}^{\lambda X} \geq \mathrm{e}^{\lambda t}] \leq \frac{\mathbb{E}[\mathrm{e}^{\lambda X}]}{\mathrm{e}^{\lambda t}}$$

which holds for any random variable X and for all $\lambda > 0$. If ω is of the form

$$\omega(s) = \sum_{i=0}^{n-1} a_i \cdot \delta_{s_i}(s), \quad s_i = \log\left(\frac{a_{X,i}}{a_{Y,i}}\right),$$

where $a_{X,i}, a_{Y,i} \geq 0$, $s_i \in \mathbb{R}$, $0 \leq i \leq n-1$, the moment generating function is given by

$$\mathbb{E}[\mathrm{e}^{\lambda \omega_{X/Y}}] = \int\limits_{-\infty}^{\infty} \mathrm{e}^{\lambda s} \omega(s)\, \mathrm{d}s = \sum_{i=1}^{n} \mathrm{e}^{\lambda s_i} \cdot a_{X,i} = \sum_{i=1}^{n} \left(\frac{a_{X,i}}{a_{Y,i}}\right)^{\lambda} a_{X,i}. \tag{3.3}$$

3.1 Tail Bound for the Convolved PLDs

Denote $S_k := \sum_{i=1}^{k} \omega_i$, where ω_i denotes the PLD random variable of the ith mechanism. Since ω_i's are independent, $\mathbb{E}[\mathrm{e}^{\lambda S_k}] = \prod_{i=1}^{k} \mathbb{E}[\mathrm{e}^{\lambda \omega_i}]$ and the Chernoff bound shows that for any $\lambda > 0$

$$\int_L^{\infty} (\omega *^k \omega)(s)\, \mathrm{d}s = \mathbb{P}[S_k \geq L] \leq \prod_{i=1}^{k} \mathbb{E}[\mathrm{e}^{\lambda \omega_i}]\, \mathrm{e}^{-\lambda L}.$$

If ω_i's are i.i.d. and distributed as ω, and if $\alpha(\lambda) = \log(\mathbb{E}[\mathrm{e}^{\lambda \omega}])$, then

$$I_1(L) = \int\limits_{L}^{\infty} (\omega *^k \omega)(s)\, \mathrm{d}s \leq \mathrm{e}^{k\alpha(\lambda)} \mathrm{e}^{-\lambda L}. \tag{3.4}$$

3.2 Error Arising from the Periodisation

We define $\alpha^+(\lambda)$ and $\alpha^-(\lambda)$ via the moment generating function of the PLD as

$$\alpha^+(\lambda) = \log(\mathbb{E}[\mathrm{e}^{\lambda \omega}]) \quad \text{and} \quad \alpha^-(\lambda) = \log(\mathbb{E}[\mathrm{e}^{-\lambda \omega}]). \tag{3.5}$$

Using the Chernoff bound, the required error bounds can be obtained using $\alpha^+(\lambda)$ and $\alpha^-(\lambda)$.

Lemma 11. *Let ω be defined as above and suppose $s_i \in [-L, L]$ for all $0 \leq i \leq n-1$. Then,*

$$I_3(L) = \int\limits_{\varepsilon}^{L} \left|(\omega \circledast^k \omega - \widetilde{\omega} \circledast^k \widetilde{\omega})(s)\right|\, \mathrm{d}s \leq \left(\mathrm{e}^{k\alpha^+(\lambda)} + \mathrm{e}^{k\alpha^-(\lambda)}\right) \frac{\mathrm{e}^{-L\lambda}}{1 - \mathrm{e}^{-L\lambda}}.$$

Proof. Let ω and its $2L$-periodic continuation $\widetilde{\omega}(s)$ be of the form

$$\omega(s) = \sum_i a_i \cdot \delta_{s_i}(s) \quad \text{and} \quad \widetilde{\omega}(s) = \sum_i \widetilde{a}_i \cdot \delta_{s_i}(s)$$

for some $a_i, \widetilde{a}_i \geq 0$, $s_i = i\Delta x$. By definition of the truncated convolution $\circledast$ (see the main text),

$$\begin{aligned}
(\widetilde{\omega} \circledast^k \widetilde{\omega})(s) &= \sum_{-L \leq s_{j_1} < L} \widetilde{a}_{j_1} \sum_{-L \leq s_{j_2} < L} \widetilde{a}_{j_2} \ldots \sum_{-L \leq s_{j_{k-1}} < L} \widetilde{a}_{j_{k-1}} \sum_i \widetilde{a}_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s) \\
&= \sum_{-L \leq s_{j_1} < L} a_{j_1} \sum_{-L \leq s_{j_2} < L} a_{j_2} \ldots \sum_{-L \leq s_{j_{k-1}} < L} a_{j_{k-1}} \sum_i \widetilde{a}_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s) \\
&= \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \ldots \sum_{j_{k-1}} a_{j_{k-1}} \sum_i \widetilde{a}_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s),
\end{aligned}$$

since $\widetilde{a}_i = a_i$ for all i such that $-L \leq s_i < L$. Furthermore,

$$\begin{aligned}
(\omega \circledast^k \omega)(s) &= \sum_{-L \leq s_{j_1} < L} a_{j_1} \sum_{-L \leq s_{j_2} < L} a_{j_2} \ldots \sum_{-L \leq s_{j_{k-1}} < L} a_{j_{k-1}} \sum_i a_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s) \\
&= \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \ldots \sum_{j_{k-1}} a_{j_{k-1}} \sum_i a_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s).
\end{aligned}$$

Thus

$$(\widetilde{\omega} \circledast^k \widetilde{\omega} - \omega \circledast^k \omega)(s) = \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \dots \sum_{j_{k-1}} a_{j_{k-1}} \sum_{i} \widehat{a}_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s), \tag{3.6}$$

where

$$\widehat{a}_i = \widetilde{a}_i - a_i = \begin{cases} 0, & \text{if } -L \leq s_i < L, \\ a_{i \bmod n}, & \text{else.} \end{cases} \tag{3.7}$$

From (3.6) we see that

$$\begin{aligned} \int\limits_{\varepsilon}^{L} \left| (\omega \circledast^k \omega - \widetilde{\omega} \circledast^k \widetilde{\omega})(s) \right| \, \mathrm{d}s &\leq \int\limits_{\mathbb{R}} \left| (\omega \circledast^k \omega - \widetilde{\omega} \circledast^k \widetilde{\omega})(s) \right| \, \mathrm{d}s \\ &= \int\limits_{\mathbb{R}} \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \dots \sum_{j_{k-1}} a_{j_{k-1}} \sum_{i} \widehat{a}_{i-j_1-\ldots-j_{k-1}} \cdot \delta_{s_i}(s) \, \mathrm{d}s \\ &= \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \dots \sum_{j_{k-1}} a_{j_{k-1}} \sum_{i} \widehat{a}_{i-j_1-\ldots-j_{k-1}}. \end{aligned} \tag{3.8}$$

From (3.7) we see that

$$\begin{aligned} \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_2} \dots \sum_{j_{k-1}} a_{j_{k-1}} \sum_{i} \widehat{a}_{i-j_1-\ldots-j_{k-1}} &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \mathbb{P}\big((2n-1)L \leq \omega *^k \omega < (2n+1)L \big) \\ &= \sum_{n \in \mathbb{Z}^-} \mathbb{P}\big((2n-1)L \leq \omega *^k \omega < (2n+1)L \big) \\ &\quad + \sum_{n \in \mathbb{Z}^+} \mathbb{P}\big((2n-1)L \leq \omega *^k \omega < (2n+1)L \big) \\ &\leq \sum_{n \in \mathbb{Z}^-} \mathbb{P}\big(\omega *^k \omega \leq (2n+1)L \big) + \sum_{n \in \mathbb{Z}^+} \mathbb{P}\big(\omega *^k \omega \geq (2n-1)L \big). \end{aligned} \tag{3.9}$$

We also see that

$$\sum_{n \in \mathbb{Z}^-} \mathbb{P}\big(\omega *^k \omega \leq (2n+1)L \big) = \sum_{n \in \mathbb{Z}^+} \mathbb{P}\big((-\omega) *^k (-\omega) \geq (2n-1)L \big).$$

Using the bounds (3.8), (3.9) and the Chernoff bound (3.4), we find that for all $\lambda > 0$

$$\int\limits_{\varepsilon}^{L} \left| (\omega *^k \omega - \widetilde{\omega} \circledast^k \widetilde{\omega})(s) \right| \, \mathrm{d}s \leq \sum_{\ell=1}^{\infty} \mathrm{e}^{k\alpha^+(\lambda)} \mathrm{e}^{-\ell L \lambda} + \mathrm{e}^{k\alpha^-(\lambda)} \mathrm{e}^{-\ell L \lambda} = \big(\mathrm{e}^{k\alpha^+(\lambda)} + \mathrm{e}^{k\alpha^-(\lambda)} \big) \frac{\mathrm{e}^{-L\lambda}}{1 - \mathrm{e}^{-L\lambda}}.$$

□

3.3 Error Arising from the Truncation of the Convolution Integrals

Next, assume that the generalised distribution ω of the PLD is of the form

$$\omega(s) = \sum_{i} a_i \cdot \delta_{s_i}(s),$$

where $a_i \geq 0$ and $s_i = i \Delta x$.

The following lemma gives a bound for the truncation error $\int\limits_{\varepsilon}^{L} \left| (\omega *^k \omega - \omega \circledast^k \omega)(s) \right| \, \mathrm{d}s$ in terms of the moment generating function of ω. Notice that this result applies also for the case where the support of the PLD distribution are outside of the interval $[-L, L]$.

Lemma 12. *Let ω be defined as above. For all $\lambda > 0$,*

$$I_2(L) = \int\limits_{\varepsilon}^{L} (\omega *^k \omega - \omega \circledast^k \omega)(s) \, \mathrm{d}s \leq \left(\frac{\mathrm{e}^{k\alpha^+(\lambda)} - \mathrm{e}^{\alpha^+(\lambda)}}{\mathrm{e}^{\alpha^+(\lambda)} - 1} + \frac{\mathrm{e}^{\alpha^-(\lambda)} - \mathrm{e}^{k\alpha^-(\lambda)}}{1 - \mathrm{e}^{\alpha^-(\lambda)}} \right) \mathrm{e}^{-L\lambda}.$$

Proof. By adding and subtracting $(\omega *^k \omega) \circledast \omega$, we may write

$$\omega *^k \omega - \omega \circledast^k \omega = (\omega *^{k-1} \omega) * \omega - (\omega *^{k-1} \omega) \circledast \omega + (\omega *^{k-1} \omega - \omega \circledast^{k-1} \omega) \circledast \omega. \tag{3.10}$$

Let $\ell \in \mathbb{Z}^+$. Let ω be of the form $\omega(s) = \sum_i a_i \cdot \delta_{s_i}(s)$ and let the convolution $\omega *^\ell \omega$ be of the form $(\omega *^\ell \omega)(s) = \sum_i c_i \cdot \delta_{s_i}(s)$ for some $a_i, c_i \geq 0$, $s_i = i\Delta x$. From the definition of the operators $*$ and $\circledast$ it follows that

$$\begin{aligned} \big((\omega *^\ell \omega) * \omega - (\omega *^\ell \omega) \circledast \omega\big)(s) &= \sum_i \Big(\sum_j c_j a_{i-j} \Big) \cdot \delta_{s_i}(s) - \sum_i \Big(\sum_{-L \leq s_j < L} c_j a_{i-j} \Big) \cdot \delta_{s_i}(s) \\ &= \sum_i \Big(\sum_{s_j < -L,\, s_j \geq L} c_j a_{i-j} \Big) \cdot \delta_{s_i}(s). \end{aligned}$$

Therefore

$$\begin{aligned} \int\limits_{\mathbb{R}} \big((\omega *^\ell \omega) * \omega - (\omega *^\ell \omega) \circledast \omega\big)(s) \, \mathrm{d}s &= \int\limits_{\mathbb{R}} \sum_i \Big(\sum_{s_j < -L,\, s_j \geq L} c_j a_{i-j} \Big) \cdot \delta_{s_i}(s) \, \mathrm{d}s \\ &= \sum_{s_j < -L,\, s_j \geq L} c_j \int\limits_{\mathbb{R}} \sum_i a_{i-j} \cdot \delta_{s_i}(s) \, \mathrm{d}s \\ &= \sum_{s_j < -L,\, s_j \geq L} c_j \\ &= \mathbb{P}\Big(\omega *^\ell \omega < -L\Big) + \mathbb{P}\Big(\omega *^\ell \omega \geq L\Big) \\ &\leq \mathrm{e}^{\ell \alpha^+(\lambda)} \mathrm{e}^{-L\lambda} + \mathrm{e}^{\ell \alpha^-(\lambda)} \mathrm{e}^{-L\lambda} \end{aligned} \tag{3.11}$$

for all $\lambda > 0$. The last inequality follows from the Chernoff bound. Similarly, let $\omega *^\ell \omega - \omega \circledast^\ell \omega$ be of the form

$$(\omega *^\ell \omega - \omega \circledast^\ell \omega)(s) = \sum_i \widetilde{c}_i \cdot \delta_{s_i}(s)$$

for some $\widetilde{c}_i \geq 0$, $s_i = i\Delta x$. Then

$$\begin{aligned} \int\limits_{\mathbb{R}} \big((\omega *^\ell \omega - \omega \circledast^\ell \omega) \circledast \omega\big)(s) \, \mathrm{d}s &= \int\limits_{\mathbb{R}} \sum_i \Big(\sum_{-L \leq s_j < L} \widetilde{c}_j a_{i-j} \Big) \cdot \delta_{s_i}(s) \, \mathrm{d}s \\ &= \sum_{-L \leq s_j < L} \widetilde{c}_j \int\limits_{\mathbb{R}} \sum_i a_{i-j} \cdot \delta_{s_i}(s) \, \mathrm{d}s \\ &\leq \sum_{-L \leq s_j < L} \widetilde{c}_j \\ &\leq \int\limits_{\mathbb{R}} (\omega *^\ell \omega - \omega \circledast^\ell \omega)(s) \, \mathrm{d}s. \end{aligned} \tag{3.12}$$

Using (3.10), (3.11) and (3.12), we see that for all $\lambda > 0$,

$$\begin{aligned} \int\limits_{\varepsilon}^{L} (\omega *^k \omega - \omega \circledast^k \omega)(s) \, \mathrm{d}s &\leq \int\limits_{\mathbb{R}} (\omega *^k \omega - \omega \circledast^k \omega)(s) \, \mathrm{d}s \\ &\leq \mathrm{e}^{(k-1)\alpha^+(\lambda)} \mathrm{e}^{-L\lambda} + \mathrm{e}^{(k-1)\alpha^-(\lambda)} \mathrm{e}^{-L\lambda} + \int\limits_{\mathbb{R}} (\omega *^{k-1} \omega - \omega \circledast^{k-1} \omega)(s) \, \mathrm{d}s. \end{aligned} \tag{3.13}$$

Using (3.13) recursively, we see that for all $\lambda > 0$,

$$\begin{aligned} \int\limits_{\varepsilon}^{L} (\omega *^k \omega - \omega \circledast^k \omega)(s) \, \mathrm{d}s &\leq \sum_{\ell=1}^{k-1} \mathrm{e}^{\ell \alpha^+(\lambda)} \mathrm{e}^{-L\lambda} + \sum_{\ell=1}^{k-1} \mathrm{e}^{\ell \alpha^-(\lambda)} \mathrm{e}^{-L\lambda} \\ &= \left(\frac{\mathrm{e}^{k\alpha^+(\lambda)} - \mathrm{e}^{\alpha^+(\lambda)}}{\mathrm{e}^{\alpha^+(\lambda)} - 1} + \frac{\mathrm{e}^{k\alpha^-(\lambda)} - \mathrm{e}^{\alpha^-(\lambda)}}{\mathrm{e}^{\alpha^-(\lambda)} - 1} \right) \mathrm{e}^{-L\lambda}. \end{aligned}$$

□

3.4 Proof of Theorem 10 (Total Error)

Proof of Theorem 10. Let $\alpha^+(\lambda)$ and $\alpha^-(\lambda)$ be defined as in (3.5). Combining the bound (3.4) and the bounds given by Lemmas 11 and 12, we find that

$$\begin{aligned}\left|\delta(\varepsilon)-\widetilde{\delta}(\varepsilon)\right| \leq & \,\mathrm{e}^{k\alpha^+(\lambda)}\mathrm{e}^{-\lambda L} + \left(\mathrm{e}^{k\alpha^+(\lambda)}+\mathrm{e}^{k\alpha^-(\lambda)}\right)\frac{\mathrm{e}^{-L\lambda}}{1-\mathrm{e}^{-L\lambda}} + \left(\frac{\mathrm{e}^{k\alpha^+(\lambda)}-\mathrm{e}^{\alpha^+(\lambda)}}{\mathrm{e}^{\alpha^+(\lambda)}-1}+\frac{\mathrm{e}^{k\alpha^-(\lambda)}-\mathrm{e}^{\alpha^-(\lambda)}}{\mathrm{e}^{\alpha^-(\lambda)}-1}\right)\mathrm{e}^{-L\lambda} \\ \leq & \,\mathrm{e}^{k\alpha^+(\lambda)}\frac{\mathrm{e}^{-L\lambda}}{1-\mathrm{e}^{-L\lambda}} + \left(\mathrm{e}^{k\alpha^+(\lambda)}+\mathrm{e}^{k\alpha^-(\lambda)}\right)\frac{\mathrm{e}^{-L\lambda}}{1-\mathrm{e}^{-L\lambda}} \\ & + \left(\frac{\mathrm{e}^{k\alpha^+(\lambda)}-\mathrm{e}^{\alpha^+(\lambda)}}{\mathrm{e}^{\alpha^+(\lambda)}-1}+\frac{\mathrm{e}^{k\alpha^-(\lambda)}-\mathrm{e}^{\alpha^-(\lambda)}}{\mathrm{e}^{\alpha^-(\lambda)}-1}\right)\frac{\mathrm{e}^{-L\lambda}}{1-\mathrm{e}^{-L\lambda}} \\ = & \left(\frac{2\mathrm{e}^{(k+1)\alpha^+(\lambda)}-\mathrm{e}^{k\alpha^+(\lambda)}-\mathrm{e}^{\alpha^+(\lambda)}}{\mathrm{e}^{\alpha^+(\lambda)}-1}+\frac{\mathrm{e}^{(k+1)\alpha^-(\lambda)}-\mathrm{e}^{\alpha^-(\lambda)}}{\mathrm{e}^{\alpha^-(\lambda)}-1}\right)\frac{\mathrm{e}^{-L\lambda}}{1-\mathrm{e}^{-L\lambda}}.\end{aligned}$$

□

4 Theorem 11: Tight Bound for Multidimensional Mechanisms via One Dimensional Distributions

The following results shows that the tight (ε,δ)-bound for a multidimensional mechanism $\mathcal{M}$ can be obtained by analysis of one dimensional distributions, in case the neighbouring datasets X and Y leading to the maximal $\delta(\varepsilon)$ are known.

Theorem 13. *Consider a function $f : \mathcal{X}^N \rightarrow \mathbb{R}^d$ and a randomised mechanism $\mathcal{M}$ of the form $\mathcal{M}(X) = f(X)+Z$, where Z_i's are independent random variables. Suppose the data sets X and Y lead to the $\delta(\varepsilon)$-upper bound, and denote $\Delta = f(X)-f(Y)$. Then, the tight (ε,δ)-bound for $\mathcal{M}$ is given by the tight (ε,δ)-bound for the non-adaptive compositions of one-dimensional random variables*

$$\Delta_i + Z_i \quad \textit{and} \quad Z_i, \quad 1 \leq i \leq d.$$

Proof. The claim can be shown simply by observing that the privacy loss distribution generated by $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ and the privacy loss distribution generated by compositions $(\Delta_1+Z_1,\ldots,\Delta_d+Z_d)$ and $(Z_1,\ldots,Z_d)$ are the same. □

5 Experiments of Section 6.2

We next show how to use the Fourier accountant for obtaining the (ε,δ)-bound of Figure 4. Essentially, we show how to obtain the PLD for a subsampled multivariate mechanism, where the neighbouring distributions are known and fixed (i.e., $\Delta = f(X)-f(Y)$ is fixed and $f(X)$ is sampled with probability q and $f(Y)$ with probability $1-q$).

Now denote the density functions for one-dimensional mechanisms $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ by

$$f_X(t) := \sum_i a_{X,i}\cdot\delta_{t_{X,i}}(t) \quad \text{and} \quad f_Y(t) := \sum_i a_{Y,i}\cdot\delta_{t_{Y,i}}(t),$$

respectively.

Then, for the d-fold compositions

$$\big(\mathcal{M}(X),\ldots,\mathcal{M}(X)\big) \quad \text{and} \quad \big(\mathcal{M}(Y),\ldots,\mathcal{M}(Y)\big),$$

the density functions are given by the convolutions

$$\widetilde{f}_X(t) = \sum_{(i_1,\ldots,i_d)} a_{X,i_1}\cdots a_{X,i_d}\cdot\delta_{t_{X,i_1}+\ldots+t_{X,i_d}}(t) \quad \text{and} \quad \widetilde{f}_Y(t) = \sum_{(i_1,\ldots,i_d)} a_{Y,i_1}\cdots a_{Y,i_d}\cdot\delta_{t_{Y,i_1}+\ldots+t_{Y,i_d}}(t),$$

respectively.

By definition, the PLD generated by the distributions

$$q \cdot \widetilde{f}_X + (1-q) \cdot \widetilde{f}_Y \quad \text{and} \quad \widetilde{f}_Y,$$

is of the form

$$\widetilde{\omega}(s) = \sum_{(i_1,\ldots,i_d)} \Big(q \cdot a_{X,i_1} \cdots a_{X,i_d} + (1-q) \cdot a_{Y,i_1} \cdots a_{Y,i_d} \Big) \cdot \delta_{\widetilde{s}_i}(s), \tag{5.1}$$

where

$$\begin{aligned} \widetilde{s}_i &= \log \left(\frac{q \cdot a_{X,i_1} \cdots a_{X,i_d} + (1-q) \cdot a_{Y,i_1} \cdots a_{Y,i_d}}{a_{Y,i_1} \cdots a_{Y,i_d}} \right) \\ &= \log \left(q \cdot \frac{a_{X,i_1} \cdots a_{X,i_d}}{a_{Y,i_1} \cdots a_{Y,i_d}} + (1-q) \right) \\ &= \log \Big(q \cdot \exp\big(s_{i_1} + \ldots s_{i_d}\big) + (1-q) \Big), \end{aligned}$$

where

$$s_i = \log \left(\frac{a_{X,i}}{a_{Y,i}} \right)$$

for all i. Thus, if we have the distributions

$$\omega_1(s) = \sum_{(i_1,\ldots,i_d)} a_{X,i_1} \cdots a_{X,i_d} \cdot \delta_{s_{i_1} + \ldots s_{i_d}}(s) \tag{5.2}$$

and

$$\omega_2(s) = \sum_{(i_1,\ldots,i_d)} a_{Y,i_1} \cdots a_{Y,i_d} \cdot \delta_{s_{i_1} + \ldots s_{i_d}}(s), \tag{5.3}$$

we can form the PLD $\widetilde{\omega}$ by the change of variable

$$s \to \log\big(q \cdot s + (1-q)\big)$$

and summing the coefficients as in (5.1). On the other hand, we can obtain ω_1 and ω_2 by using the Fourier accountant to the d-fold convolutions of the distributions

$$\sum_i a_{X,i} \cdot \delta_{s_i}(s) \quad \text{and} \quad \sum_i a_{Y,i} \cdot \delta_{s_i}(s).$$

Also, the $\delta(\infty)$-probabilities can be evaluated straightforwardly for $q \cdot \widetilde{f}_X + (1-q) \cdot \widetilde{f}_Y$ and $\widetilde{f}_Y$.

6 Section 6.3: The Subsampled Gaussian Mechanism

In this Section we give an error analysis for the approximations given in Section 6.3. Recall first the form of the PLD for the subsampled Gaussian mechanism. For a subsampling ratio $0 < q < 1$ and noise level $\sigma > 0$, the continuous PLD distribution is given by

$$\omega(s) = \begin{cases} f(g(s))g'(s), & \text{if } s > \log(1-q), \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

where

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} [q \mathrm{e}^{\frac{-(t-1)^2}{2\sigma^2}} + (1-q)\mathrm{e}^{-\frac{t^2}{2\sigma^2}}], \qquad g(s) = \sigma^2 \log \left(\frac{\mathrm{e}^s - (1-q)}{q} \right) + \frac{1}{2}. \tag{6.2}$$

In order to carry out an error analysis for the approximations given in Section 6.3, we define the infinite extending grid approximations of $\omega_{\min}$ and $\omega_{\max}$. Let $L > 0$, $n \in \mathbb{Z}^+$, $\Delta x = 2L/n$ and let the grid X_n be defined as in (2.1). Define

$$\omega_{\min}(s) = \sum_{i=0}^{n-1} c_i^- \cdot \delta_{s_i}(s), \qquad \omega_{\max}(s) = \sum_{i=0}^{n-1} c_i^+ \cdot \delta_{s_i}(s),$$

where $s_i = i\Delta x$ and

$$c_i^- = \Delta x \cdot \min_{s \in [s_i, s_{i+1}]} \omega(s), \qquad c_i^+ = \Delta x \cdot \max_{s \in [s_{i-1}, s_i]} \omega(s). \tag{6.3}$$

Define

$$\omega_{\min}^\infty(s) = \sum_{i \in \mathbb{Z}} c_i^- \cdot \delta_{s_i}(s), \qquad \omega_{\max}^\infty(s) = \sum_{i \in \mathbb{Z}} c_i^+ \cdot \delta_{s_i}(s), \tag{6.4}$$

where c_i^- and c_i^+ are as defined in (6.3). We find that ω as defined in (6.1) has one stationary point which we determine numerically. Using this, the numerical values of c_i^- and c_i^+ are obtained.

We obtain approximations for the lower and upper bounds $\delta_{\min}(\varepsilon)$ and $\delta_{\max}(\varepsilon)$ of Section 6.3 by running Algorithm 1 for $\omega_{\min}^\infty$ and $\omega_{\max}^\infty$ using some prescribed parameter values n and L. This is equivalent to running Algorithm 1 for the truncated distributions $\omega_{\min}$ and $\omega_{\max}$. However, to obtain the bounds of Theorem 10 (and subsequently strict lower and upper bounds for $\delta(\varepsilon)$), the error analysis has to be carried out for the distributions $\omega_{\min}^\infty$ and $\omega_{\max}^\infty$. To this end, we need bounds for the moment generating functions of $-\omega_{\min}^\infty$, $\omega_{\min}^\infty - \omega_{\max}^\infty$ and $\omega_{\max}^\infty$.

However, we first show that $\omega_{\min}^\infty$ and $\omega_{\max}^\infty$ indeed give lower and upper bounds for $\delta(\varepsilon)$.

Lemma 14. *Let $\delta(\varepsilon)$ be given by the integral formula of Lemma 3 for some privacy loss distribution ω and for some $\delta(\infty) \geq 0$. Let $\delta_{\min}^\infty(\varepsilon)$ and $\delta_{\max}^\infty(\varepsilon)$ be defined analogously by $\omega_{\min}^\infty$ and $\omega_{\max}^\infty$. Then for all $\varepsilon > 0$ we have*

$$\delta_{\min}^\infty(\varepsilon) \leq \delta(\varepsilon) \leq \delta_{\max}^\infty(\varepsilon).$$

Proof. From the definition (6.4) and from the fact that $(1 - \mathrm{e}^{\varepsilon - s})$ is a monotonously increasing function of s it follows that the discrete sums $\delta_{\min}^\infty(\varepsilon)$ and $\delta_{\max}^\infty(\varepsilon)$ are the lower and upper Riemann sums for the continuous integral $\delta(\varepsilon)$ on the partition $\{i\Delta x : i \in \mathbb{Z}\}$. This shows the claim. □

Lemma 14 directly generalises to convolutions:

Corollary 15. *Consider a single composition, i.e., suppose the PLD is given by $\omega * \omega$ for a distribution ω of the form (6.1). Let $\omega_{\max}^\infty$ be defined as in (6.4). We have that*

$$\begin{aligned}
\int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s})\, (\omega * \omega)(s) \, \mathrm{d}s &= \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s}) \int_{-\infty}^\infty \omega(t)\, \omega(s-t) \, \mathrm{d}t \, \mathrm{d}s \\
&= \int_{-\infty}^\infty \omega(t) \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s})\, \omega(s-t) \, \mathrm{d}s \, \mathrm{d}t \\
&\leq \int_{-\infty}^\infty \omega(t) \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s})\, \omega_{\max}^\infty(s-t) \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_{-\infty}^\infty \omega(t) \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s}) \sum_{i \in \mathbb{Z}} c_i^+ \cdot \delta_{s_i + t}(s) \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_{-\infty}^\infty \omega(t) \sum_{s_i + t > \varepsilon} (1 - \mathrm{e}^{\varepsilon - (s_i + t)})\, c_i^+ \, \mathrm{d}t \\
&\leq \int_{-\infty}^\infty \omega_{\max}^\infty(t) \sum_{s_i + t > \varepsilon} (1 - \mathrm{e}^{\varepsilon - (s_i + t)})\, c_i^+ \, \mathrm{d}t \\
&= \sum_{s_i + s_j > \varepsilon} (1 - \mathrm{e}^{\varepsilon - (s_i + s_j)})\, c_i^+ c_j^+ \\
&= \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s}) \sum_{i,j} c_i^+ c_j^+ \delta_{s_i + s_j}(s) \, \mathrm{d}s \\
&= \int_\varepsilon^\infty (1 - \mathrm{e}^{\varepsilon - s})\, (\omega_{\max}^\infty * \omega_{\max}^\infty)(s) \, \mathrm{d}s.
\end{aligned} \tag{6.5}$$

Showing that

$$\int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, (\omega * \omega)(s) \geq \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, (\omega_{\mathrm{min}}^{\infty} * \omega_{\mathrm{min}}^{\infty})(s) \, \mathrm{d}s$$

goes analogously. Inductively, bounding as in (6.5)*, we also see that*

$$\begin{aligned} \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, (\omega *^k \omega)(s) \, \mathrm{d}s &\leq \sum_{s_{i_1} + \ldots + s_{i_k} > \varepsilon} (1 - \mathrm{e}^{\varepsilon - (s_{i_1} + \ldots + s_{i_k})}) \, a_{i_1} \cdot \ldots \cdot a_{i_k} \\ &= \int_{\varepsilon}^{\infty} (1 - \mathrm{e}^{\varepsilon - s}) \, (\omega_{\mathrm{max}}^{\infty} *^k \omega_{\mathrm{max}}^{\infty})(s) \, \mathrm{d}s \end{aligned}$$

and similarly for the lower bound determined by the convolutions of $\omega_{\mathrm{min}}^{\infty}$*.*

To evaluate $\alpha^{+}(\lambda)$ and $\alpha^{-}(\lambda)$ in the upper bound of Theorem 10 of the main text, we need the moment generating functions of $-\omega_{\mathrm{min}}^{\infty}$, $\omega_{\mathrm{min}}^{\infty}$, $-\omega_{\mathrm{max}}^{\infty}$ and $\omega_{\mathrm{max}}^{\infty}$. We first state the following auxiliary lemma needed to bound these moment generating functions.

Lemma 16. *For all* $s \geq 1$ *and* $0 < q \leq \frac{1}{2}$*:*

$$\omega(s) \leq \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{(\sigma^2 s + C)^2}{2\sigma^2}},$$

where $C = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$*.*

Proof. When $s \geq 1$,

$$\mathrm{e}^s - (1 - q) \geq \frac{1}{2} \mathrm{e}^s \tag{6.6}$$

and subsequently

$$g(s) = \sigma^2 \log \left(\frac{\mathrm{e}^s - (1-q)}{q} \right) + \frac{1}{2} \geq \sigma^2 s + \widetilde{C},$$

where $\widetilde{C} = \sigma^2 \log(\frac{1}{2q}) + \frac{1}{2}$. We see that when $0 < q \leq \frac{1}{2}$, we have $g(s) \geq \frac{1}{2}$. From (6.2) we see that

$$f(g(s)) \leq \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(\sigma^2 s + \widetilde{C} - 1)^2}{2\sigma^2}},$$

Furthermore, when $s \geq 1$, from (6.6) it follows that

$$g'(s) = \frac{\sigma^2 \mathrm{e}^s}{\mathrm{e}^s - (1-q)} \leq 2\sigma^2.$$

Thus, when $s \geq 1$,

$$\omega(s) \leq \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{(\sigma^2 s + C)^2}{2\sigma^2}},$$

where $C = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$. □

Using Lemma 16, we can bound the moment generating function of $\omega_{\mathrm{max}}^{\infty}$ as follows. We note that $\mathbb{E}[\mathrm{e}^{\lambda \omega_{\mathrm{max}}}]$ can be evaluated numerically.

Lemma 17. *Let* $0 < \lambda \leq L$ *and assume* $\sigma \geq 1$ *and* $\Delta x \leq c \cdot L$, $0 < c < 1$*. The moment generating function of* $\omega_{\mathrm{max}}^{\infty}$ *can be bounded as*

$$\mathbb{E}[\mathrm{e}^{\lambda \omega_{\mathrm{max}}^{\infty}}] \leq \mathbb{E}[\mathrm{e}^{\lambda \omega_{\mathrm{max}}}] + \mathrm{err}(\lambda, L, \sigma),$$

where

$$\mathrm{err}(\lambda, L, \sigma) = \mathrm{e}^{c\lambda L} \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\frac{\lambda(2C - \lambda)}{2\sigma^2}} \mathrm{erfc} \left(\frac{(1-c)\sigma^2 L + C - \lambda}{\sqrt{2}\sigma} \right). \tag{6.7}$$

Here ω_{max} *is the restriction of* $\omega_{\mathrm{max}}^{\infty}$ *to the interval* $[-L, L]$ *(i.e., as defined in equation (14) of the main text) and the constant* C *is as defined in Lemma 16.*

Proof. Assuming $L > |\log 1 - q|$ (i.e., $\omega(s) = 0$ for all $s < -L$), the moment generating function of $\omega^{\infty}_{\max}$ is given by

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{\lambda \omega^{\infty}_{\max}}] &= \int_{-\infty}^{L} \mathrm{e}^{\lambda s} \omega^{\infty}_{\max}(s)\,\mathrm{d}s + \int_{L}^{\infty} \mathrm{e}^{\lambda s} \omega^{\infty}_{\max}(s)\,\mathrm{d}s \\
&= \int_{-L}^{L} \mathrm{e}^{\lambda s} \omega^{\infty}_{\max}(s)\,\mathrm{d}s + \int_{L}^{\infty} \mathrm{e}^{\lambda s} \omega^{\infty}_{\max}(s)\,\mathrm{d}s \qquad (6.8)\\
&= \mathbb{E}[\mathrm{e}^{\lambda \omega_{\max}}] + \sum_{i \geq n} \Delta x \cdot \mathrm{e}^{\lambda i \Delta x} \cdot c_i^{+}.
\end{aligned}$$

From Lemma 16 it follows that

$$c_i^{+} = \max_{s \in [s_{i-1}, s_i]} \omega(s) \leq \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{(\sigma^2 s_{i-1} + C)^2}{2\sigma^2}},$$

where $C = \sigma^2 \log(\frac{1}{2q}) - \frac{1}{2}$, $s_i = i\Delta x$. Thus

$$\begin{aligned}
\sum_{i \geq n} \mathrm{e}^{\lambda i \Delta x} \cdot c_i^{+} &= \mathrm{e}^{\lambda \Delta x} \sum_{i \geq n} \mathrm{e}^{\lambda s_{i-1}} \cdot c_i^{+} \\
&\leq \mathrm{e}^{\lambda \Delta x} \sigma \sqrt{\frac{2}{\pi}} \sum_{i \geq n} \Delta x \cdot \mathrm{e}^{\lambda s_{i-1}} \mathrm{e}^{-\frac{(\sigma^2 s_{i-1} + C)^2}{2\sigma^2}} \\
&= \mathrm{e}^{\lambda \Delta x} \sigma \sqrt{\frac{2}{\pi}} \sum_{i \geq n} \Delta x \cdot \mathrm{e}^{\frac{-(\sigma^2 s_{i-1} + C - \lambda)^2 - \lambda(2C - \lambda)}{2\sigma^2}} \qquad (6.9)\\
&= \mathrm{e}^{\lambda \Delta x} \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \sum_{i \geq n} \Delta x \cdot \mathrm{e}^{-\frac{(\sigma^2 s_{i-1} + C - \lambda)^2}{2\sigma^2}}.
\end{aligned}$$

Assuming $\sigma \geq 1$ and $\lambda \leq L$, $\Delta x \leq c \cdot L$, we further see that

$$\begin{aligned}
\mathrm{e}^{\lambda \Delta x} \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \sum_{i \geq n} \Delta x \cdot \mathrm{e}^{-\frac{(\sigma^2 s_{i-1} + C - \lambda)^2}{2\sigma^2}} &\leq \mathrm{e}^{\lambda \Delta x} \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \int_{L-\Delta x}^{\infty} \mathrm{e}^{-\frac{(\sigma^2 s + C - \lambda)^2}{2\sigma^2}}\,\mathrm{d}s \\
&\leq \mathrm{e}^{c\lambda L} \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \int_{(1-c)L}^{\infty} \mathrm{e}^{-\frac{(\sigma^2 s + C - \lambda)^2}{2\sigma^2}}\,\mathrm{d}s \qquad (6.10)\\
&= \mathrm{e}^{c\lambda L} \sigma \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \frac{\sqrt{2}}{\sigma} \mathrm{erfc}\left(\frac{(1-c)\sigma^2 L + C - \lambda}{\sqrt{2}\sigma}\right) \\
&= \mathrm{e}^{c\lambda L} \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\frac{\lambda(2C-\lambda)}{2\sigma^2}} \mathrm{erfc}\left(\frac{(1-c)\sigma^2 L + C - \lambda}{\sqrt{2}\sigma}\right).
\end{aligned}$$

□

Using a reasoning similar to the proof of Lemma 17, we get the following. We note that $\mathbb{E}[\mathrm{e}^{-\lambda \omega_{\max}}]$, $\mathbb{E}[\mathrm{e}^{\lambda \omega_{\min}}]$ and $\mathbb{E}[\mathrm{e}^{-\lambda \omega_{\min}}]$ can be evaluated numerically.

Corollary 18. *The moment generating functions of $-\omega^{\infty}_{\max}$, $\omega^{\infty}_{\min}$ and $-\omega^{\infty}_{\min}$ can be bounded as*

$$\mathbb{E}[\mathrm{e}^{-\lambda \omega^{\infty}_{\max}}] \leq \mathbb{E}[\mathrm{e}^{-\lambda \omega_{\max}}] + \mathrm{err}(\lambda, L, \sigma),$$

$$\mathbb{E}[\mathrm{e}^{\lambda \omega^{\infty}_{\min}}] \leq \mathbb{E}[\mathrm{e}^{\lambda \omega_{\min}}] + \mathrm{err}(\lambda, L, \sigma),$$

$$\mathbb{E}[\mathrm{e}^{-\lambda \omega^{\infty}_{\min}}] \leq \mathbb{E}[\mathrm{e}^{-\lambda \omega_{\min}}] + \mathrm{err}(\lambda, L, \sigma),$$

where $\mathrm{err}(\lambda, L, \sigma)$ *is defined as in* (6.7).

Proof. Assuming $L > |\log 1 - q|$ (i.e., $\omega(s) = 0$ for all $s < -L$), the moment generating function of $-\omega_{\max}^{\infty}$ is given by

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\lambda\omega_{\max}^{\infty}}] &= \int_{-\infty}^{L} \mathrm{e}^{-\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s + \int_{L}^{\infty} \mathrm{e}^{-\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s \\ &= \int_{-L}^{L} \mathrm{e}^{-\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s + \int_{L}^{\infty} \mathrm{e}^{-\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s \qquad (6.11) \\ &\leq \int_{-L}^{L} \mathrm{e}^{-\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s + \int_{L}^{\infty} \mathrm{e}^{\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s.\end{aligned}$$

After bounding the term $\int_L^\infty \mathrm{e}^{\lambda s}\omega_{\max}^{\infty}(s)\,\mathrm{d}s$ as in the proof of Lemma 17, the first claim follows. Bounding $\mathbb{E}[\mathrm{e}^{\lambda\omega_{\min}^{\infty}}]$ and $\mathbb{E}[\mathrm{e}^{-\lambda\omega_{\min}^{\infty}}]$ can be carried out analogously to (6.11). □

***Remark* 19.** *In the experiments, the effect of the error term* $\mathrm{err}(\lambda, L, \sigma)$ *was found to be negligible (less than* 10^{-90} *in the experiments of Figure 3).*

7 Description of Learning Rate Cooling Used for Experiments of Figure 2b.

When running the feedforward network experiment of Figure 2b, we set the initial learning rate $\eta = 0.02$. When $n = 2400$ and $|B| = 500$, starting from epoch 13, and when $n = 3000$ and $|B| = 300$, starting from epoch 5, the learning rate η is linearly decreased after each epoch such that it is zero at the end of the training.

References

[1] Gilles Barthe and Federico Olmedo. Beyond differential privacy: composition theorems and relational logic for f-divergences between probabilistic programs. In *Proceedings of the 40th international conference on Automata, Languages, and Programming-Volume Part II*, pages 49–60, 2013.

[2] Sebastian Meiser and Esfandiar Mohammadi. Tight on budget?: Tight bounds for r-fold approximate differential privacy. In *Proceedings of the 2018 ACM SIGSAC Conference on Computer and Communications Security*, pages 247–264. ACM, 2018.

[3] David M Sommer, Sebastian Meiser, and Esfandiar Mohammadi. Privacy loss classes: The central limit theorem in differential privacy. *Proceedings on Privacy Enhancing Technologies*, 2019(2):245–269, 2019.

[4] Thomas G Stockham Jr. High-speed convolution and correlation. In *Proceedings of the April 26-28, 1966, Spring joint computer conference*, pages 229–233. ACM, 1966.

[5] Martin J Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press, 2019.