A Useful probability tools

A separable process \( \{G_\phi\}_{\phi \in \Theta} \) with respect to a metric space \((\Theta, d)\) is sub-Gaussian if for any \( \lambda \in \mathbb{R} \) and \( \phi, \phi' \in \Theta \), \( \mathbb{E}[e^{\lambda(X_\phi-X_{\phi'})}] \leq e^{\lambda^2 d^2(\phi, \phi')/2} \). Let also \( \text{diam}(\Theta) = \sup_{\phi, \phi' \in \Theta} d(\phi, \phi') \) be the diameter of the metric space \((\Theta, d)\). The following result is cited from [van Handel, 2014, Theorem 5.29].

**Lemma 10.** There exists a universal constant \( C_0 < \infty \) such that for all \( z > 0 \) and \( \phi_0 \in \Theta \),

\[
\text{Pr} \left[ \sup_{\phi \in \Theta} G_\phi - G_{\phi_0} \geq C_0 \int_0^\infty \sqrt{\ln N(\Theta; d, \epsilon)} d\epsilon + z \right] \\
\leq C_0 e^{-z^2/(C_0^2 \text{diam}(\Theta))},
\]

where \( N(\Theta; d, \epsilon) \) is the covering number of the metric space \((\Theta, d)\) up to precision \( \epsilon \).

B Omitted proofs in Section 7

**Proof of Lemma 8.** Let \( T_\zeta \) be all time periods \( t \) such that \( \zeta_t = \zeta \), and define \( T_\zeta = |T_\zeta| \). We have

\[
\sum_{t} \omega_{\zeta,t}^2 \leq \sqrt{d} \sum_{\zeta \in T_\zeta} \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2. \tag{27}
\]

First by Lemma 8, we have

\[
\sum_{t \in T_\zeta} (\omega_{\zeta,t}^2)^2 \leq \ln(\det(A_{T_\zeta})) \lesssim d \ln(T_\zeta/d), \tag{28}
\]

where the last inequality is due to

\[
\det(A_{T_\zeta}) \leq \text{tr}(A_{T_\zeta})^d \leq ((T_\zeta + 1)/d)^d. \tag{29}
\]

Let us now focus on the Right-Hand Side of Eq. (27), let

\[
T_\zeta^+ := \left\{ t \in T_\zeta : \omega_{\zeta,t}^2 \geq \sqrt{d \delta^2/(T \ln^4 T \ln^2(1/\delta))} \right\}
\]

and let

\[
T_\zeta^- := \left\{ t \in T_\zeta : \omega_{\zeta,t}^2 < \sqrt{d \delta^2/(T \ln^4 T \ln^2(1/\delta))} \right\}
= T_\zeta \setminus T_\zeta^+.
\]

We have that

\[
\sum_{t \in T_\zeta^-} \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2 = \sum_{t \in T_\zeta^+} \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2 + \sum_{t \in T_\zeta^-} \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2
\]

\[
= \sum_{t \in T_\zeta^+} \sqrt{\ln((T \ln^4 T \ln^2(1/\delta)) (\omega_{\zeta,t}^2)^2/(d \delta^2)) \omega_{\zeta,t}^2}
+ \sum_{t \in T_\zeta^-} \omega_{\zeta,t}^2
\]

\[
\leq \sum_{t \in T_\zeta^+} \sqrt{\ln((T \ln^4 T \ln^2(1/\delta)) (\omega_{\zeta,t}^2)^2/(d \delta^2)) \omega_{\zeta,t}^2} + T_\zeta \sqrt{d \delta^2/(T \ln^4 T \ln^2(1/\delta))}. \tag{30}
\]

Note that the univariate function \( f(\tau) = \sqrt{\tau \ln((T \ln^4 T \ln^2(1/\delta)) \tau/(d \delta^2))} \) is concave for \( \tau \geq d \delta^2/(T \ln^4 T \ln^2(1/\delta)) \). Applying Jensen’s inequality to \( f(\tau) \) with \( \tau = (\omega_{\zeta,t}^2)^2 \) \((t \in T_\zeta^+)\), we have

\[
\sum_{t \in T_\zeta^+} \sqrt{\ln((T \ln^4 T \ln^2(1/\delta)) (\omega_{\zeta,t}^2)^2/(d \delta^2)) \omega_{\zeta,t}^2}
\]

\[
\leq |T_\zeta^+| \sqrt{\frac{\sum_{t \in T_\zeta^+} (\omega_{\zeta,t}^2)^2}{|T_\zeta^+|}} \sqrt{\ln \left( \frac{\ln \ln(|T_\zeta|/d)}{d \delta^2} \cdot \frac{\sum_{t \in T_\zeta^-} (\omega_{\zeta,t}^2)^2}{|T_\zeta^-|} \right)}
\]

\[
\leq \sqrt{dT_\zeta \ln(T_\zeta/d)} \ln \left( \frac{T \ln^4 T \ln^2(1/\delta)}{d \delta^2} \cdot \frac{\ln(|T_\zeta|/d)}{|T_\zeta|} \right)
\]

\[
\leq \sqrt{dT_\zeta \ln(T_\zeta/d)} \ln \left( \frac{T \ln^5 T}{d \delta^2} \cdot \frac{\ln(T_\zeta/d)}{T_\zeta} \right), \tag{31}
\]

where the second inequality is due to Lemma 8 and Eq. (28), and the third inequality is due to the monotonicity of the function \( g(x) = \frac{x \ln(T_\zeta/d)}{d \delta^2} \ln((T \ln^4 T \ln^2(1/\delta))/(d \delta^2)) \cdot (\ln(T_\zeta/d)/d)) \) for large enough \( x \). Combining Eq. (30), and Eq. (31), we have

\[
\sum_{t \in T_\zeta^-} \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2 \lesssim \sqrt{dT_\zeta \ln(T_\zeta/d)} \ln(T \ln^5 T/(T_\zeta \delta^3))
+ T_\zeta \delta \sqrt{d/(T \ln^4 T \ln^2(1/\delta))}. \tag{32}
\]

By Algorithm I, we know that \( \omega_{\zeta,t}^2 = \sqrt{d} \cdot \alpha_{\zeta,t}^2 \omega_{\zeta,t}^2 \geq 2^{1-\zeta} \) for all \( t \in T_\zeta \). Subsequently,

\[
(2^{-\zeta-1})^2 \cdot T_\zeta \leq \sum_{t \in T_\zeta^-} (\omega_{\zeta,t}^2)^2 \leq \sqrt{d} \cdot \max_{t \in T_\zeta^-} (\omega_{\zeta,t}^2)^2 \cdot \sum_{t \in T_\zeta^-} (\omega_{\zeta,t}^2)^2
\]

\[
\lesssim \sqrt{d} \cdot \log(T \ln^4 T \ln^2(1/\delta))/(d \delta^2) \cdot d \log T,
\]

---

1See Definition 5.22 in [van Handel, 2014] for a technical definition of separable stochastic processes.
where the last inequality holds by applying Lemma 8. Therefore,

\[ T_\zeta \lesssim 4^\zeta \cdot d^{3/2} \log T \log(T/\delta). \quad (33) \]

We first divide the resolution levels \( \zeta \in \{0, 1, \cdots, \zeta_0\} \) into two different sets: \( Z_1 := \{0, 1, \cdots, \zeta^*\} \) and \( Z_2 := \{\zeta^* < \zeta \leq \zeta_0\} \), where \( \zeta^* \) is an integer to be defined later. Clearly \( Z_1 \) and \( Z_2 \) partition \( \{0, \cdots, \zeta_0\} \). Note that \( \sqrt{d} \cdot \sum_{t \in T_\zeta} \alpha^{x_t}_{\zeta,t} \omega^{x_t}_{\zeta,t} \lesssim 2^{-\zeta} T_\zeta \) because \( \omega^{x_t}_{\zeta,t} \leq 2^{1-\zeta} \) for all \( t \in T_\zeta \).

\[
\begin{align*}
\sqrt{d} \sum_{\zeta \in Z_1} \sum_{t \in T_\zeta} \alpha^{x_t}_{\zeta,t} \omega^{x_t}_{\zeta,t} & \lesssim \sum_{\zeta=0}^{\zeta^*} 2^{-\zeta} \cdot 4^\zeta \cdot d^{3/2} \log T \log(T/\delta) \\
& \leq 2^{\zeta^*+1} \cdot d^{3/2} \log T \log(T/\delta); \quad (34)
\end{align*}
\]

\[
\begin{align*}
\sqrt{d} \sum_{\zeta \in Z_2} \sum_{t \in T_\zeta} \alpha^{x_t}_{\zeta,t} \omega^{x_t}_{\zeta,t} & \lesssim d \sum_{\zeta \in Z_2} \sqrt{T_\zeta \log(T) \log(T/\delta)} + \delta d \sqrt{T} / \log^2 T \\
& \leq d \sqrt{|Z_2| \left( \sum_{\zeta \in Z_2} T_\zeta \right) \log(T) \log \left( T \log T \cdot \frac{|Z_2|}{\delta^3 \sum_{\zeta \in Z_2} T_\zeta} \right) + \delta d \sqrt{T} / \log^2 T} \\
& \lesssim d \sqrt{|Z_2| T \log(T) \log(\log T / \delta^3) + \delta d \sqrt{T} / \log^2 T}, \quad (35)
\end{align*}
\]

where the inequality above Eq. (35) is because of the concavity of the function \( \sqrt{x \ln(T \log^3 T |Z_2| / (x \delta^3))} \) and Jensen’s inequality, and Eq. (35) is due to \( \sum_{\zeta \in Z_2} T_\zeta \leq T \) and the monotonicity of the function \( \sqrt{x \ln(T \log^3 T |Z_2| / (x \delta^3))} \).

Recall that \( \sqrt{T/d/\delta} \leq 2^\zeta_0 \leq 2 \sqrt{T/d/\delta} \). Select \( \zeta^* = \zeta_0 - \lfloor \log_2(\ln(T) \ln(T/\delta) / \delta) \rfloor \); we have that \( |Z_2| = O(\log(T \log(T/\delta) + \log(1/\delta)) \) and \( 2^{\zeta^*} \leq 2 \sqrt{T/(\sqrt{d} \ln(T) \ln(T/\delta)))} \).

Finally, we combine Eq. (27), Eq. (34), and Eq. (35), and have that

\[
\sum_{t} \omega^{x_t}_{\zeta,t} \lesssim \delta d \sqrt{T} + d \sqrt{T \log(T) \log(1/\delta) \cdot \log(T/\delta)} \lesssim d \sqrt{T \log(T) \log(1/\delta) \cdot \log(T/\delta)},
\]

which is to be demonstrated.