A Useful probability tools

A separable process [1] $\{G_\phi\}_{\phi\in\Theta}$ with respect to a metric space (Θ, d) is *sub-Gaussian* if for any $\lambda \in \mathbb{R}$ and $\phi, \phi' \in \Theta$, $\mathbb{E}[e^{\lambda(X_\phi - X_{\phi'})}] \leq e^{\lambda^2 d^2(\phi,\phi')/2}$. Let also $\mathrm{diam}(\Theta) = \sup_{\phi,\phi'\in\Theta} d(\phi, \phi')$ be the *diameter* of the metric space (Θ, d). The following result is cited from [van Handel, 2014, Theorem 5.29].

Lemma 10. *There exists a universal constant $C_0 < \infty$ such that for all $z > 0$ and $\phi_0 \in \Theta$,*

$$\Pr\left[\sup_{\phi\in\Theta} G_\phi - G_{\phi_0} \geq C_0 \int_0^\infty \sqrt{\ln N(\Theta; d, \epsilon)}\mathrm{d}\epsilon + z\right] \leq C_0 e^{-z^2/(C_0\cdot\mathrm{diam}(\Theta))},$$

where $N(\Theta; d, \epsilon)$ is the covering number of the metric space (Θ, d) up to precision ϵ.

B Omitted proofs in Section 7

Proof of Lemma 9. Let $\mathcal{T}_\zeta$ be all time periods t such that $\zeta_t = \zeta$, and define $T_\zeta = |\mathcal{T}_\zeta|$. We have

$$\sum_t \varpi^{x_t}_{\zeta_t,t} \lesssim \sqrt{d} \cdot \sum_\zeta \sum_{t\in\mathcal{T}_\zeta} \alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t}. \tag{27}$$

First by Lemma 8, we have

$$\sum_{t\in\mathcal{T}_\zeta} (\omega^{x_t}_{\zeta,t})^2 \leq \ln(\det(\Lambda_{T_\zeta})) \lesssim d\ln(T_\zeta/d), \tag{28}$$

where the last inequality is due to

$$\det(\Lambda_{T_\zeta}) \leq \mathrm{tr}(\Lambda_{T_\zeta}/d)^d \leq ((T_\zeta + 1)/d)^d. \tag{29}$$

Let us now focus on the Right-Hand Side of Eq. (27), let

$$\mathcal{T}^+_\zeta := \left\{t \in \mathcal{T}_\zeta : \omega^{x_t}_{\zeta,t} \geq \sqrt{d\delta^2/(T\ln^4 T \ln^2(1/\delta))}\right\}$$

and let

$$\mathcal{T}^-_\zeta := \left\{t \in \mathcal{T}_\zeta : \omega^{x_t}_{\zeta,t} < \sqrt{d\delta^2/(T\ln^4 T \ln^2(1/\delta))}\right\} = \mathcal{T}_\zeta \setminus \mathcal{T}^+_\zeta.$$

[1] See Definition 5.22 in [van Handel, 2014] for a technical definition of separable stochastic processes.

We have that

$$\begin{aligned}\sum_{t\in\mathcal{T}_\zeta} \alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t} &= \sum_{t\in\mathcal{T}^+_\zeta} \alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t} + \sum_{t\in\mathcal{T}^-_\zeta} \alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t}\\ &= \sum_{t\in\mathcal{T}^+_\zeta} \sqrt{\ln((T\ln^4 T\ln^2(1/\delta))(\omega^{x_t}_{\zeta,t})^2/(d\delta^2))}\omega^{x_t}_{\zeta,t} + \sum_{t\in\mathcal{T}^-_\zeta} \omega^{x_t}_{\zeta,t}\\ &\leq \sum_{t\in\mathcal{T}^+_\zeta} \sqrt{\ln((T\ln^4 T\ln^2(1/\delta))(\omega^{x_t}_{\zeta,t})^2/(d\delta^2))}\omega^{x_t}_{\zeta,t}] + T_\zeta\sqrt{d\delta^2/(T\ln^4 T\ln^2(1/\delta))}. \end{aligned} \tag{30}$$

Note that the univariate function $f(\tau) = \sqrt{\tau\ln((T\ln^4 T\ln^2(1/\delta))\tau/(d\delta^2)}$ is concave for $\tau \geq d\delta^2/(T\ln^4 T\ln^2(1/\delta))$. Applying Jensen's inequality to $f(\tau)$ with $\tau = (\omega^{x_t}_{\zeta,t})^2$ ($t \in \mathcal{T}^+_\zeta$), we have

$$\begin{aligned}&\sum_{t\in\mathcal{T}^+_\zeta} \sqrt{\ln((T\ln^4 T\ln^2(1/\delta))(\omega^{x_t}_{\zeta,t})^2/(d\delta^2))}\omega^{x_t}_{\zeta,t}\\ &\leq |\mathcal{T}^+_\zeta| \cdot \sqrt{\frac{\sum_{t\in\mathcal{T}^+_\zeta}(\omega^{x_t}_{\zeta,t})^2}{|\mathcal{T}^+_\zeta|}} \times \sqrt{\ln\left(\frac{T\ln^4 T\ln^2(1/\delta)}{d\delta^2}\cdot\frac{\sum_{t\in\mathcal{T}^+_\zeta}(\omega^{x_t}_{\zeta,t})^2}{|\mathcal{T}^+_\zeta|}\right)}\\ &\lesssim \sqrt{|\mathcal{T}^+_\zeta| d\ln(|\mathcal{T}_\zeta|/d)\ln\left(\frac{T\ln^4 T\ln^2(1/\delta)}{d\delta^2}\cdot\frac{d\ln(|\mathcal{T}_\zeta|/d)}{|\mathcal{T}^+_\zeta|}\right)}\\ &\lesssim \sqrt{dT_\zeta\ln(T_\zeta/d)\ln\left(\frac{T\ln^4 T\ln^2(1/\delta)}{d\delta^2}\cdot\frac{d\ln(T_\zeta/d)}{T_\zeta}\right)}\\ &\lesssim \sqrt{dT_\zeta\ln(T_\zeta/d)\ln(T\ln^5 T/(T_\zeta\delta^3))},\end{aligned} \tag{31}$$

where the second inequality is due to Lemma 8 and Eq. (28), and the third inequality is due to the monotonicity of the function $g(x) = \sqrt{xd\ln(T_\zeta/d)\ln((T\ln^4 T\ln^2(1/\delta))/(d\delta^2)\cdot(d\ln(T_\zeta/d)/x))}$ for large enough x. Combining Eq. (30), and Eq. (31), we have

$$\sum_{t\in\mathcal{T}_\zeta} \alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t} \lesssim \sqrt{dT_\zeta\ln(T_\zeta/d)\ln(T\ln^5 T/(T_\zeta\delta^3))} + T_\zeta\delta\sqrt{d/(T\ln^4 T\ln^2(1/\delta))}. \tag{32}$$

By Algorithm 1, we know that $\varpi^{x_t}_{\zeta,t} = \sqrt{d}\cdot\alpha^{x_t}_{\zeta,t}\omega^{x_t}_{\zeta,t} \geq 2^{1-\zeta}$ for all $t \in \mathcal{T}_\zeta$. Subsequently,

$$\begin{aligned}(2^{-\zeta-1})^2 \cdot T_\zeta &\leq \sum_{t\in\mathcal{T}_\zeta}(\varpi^{x_t}_{\zeta,t})^2 \leq \sqrt{d}\cdot\max_{t\in\mathcal{T}_\zeta}(\alpha^{x_t}_{\zeta,t})^2 \cdot \sum_{t\in\mathcal{T}_\zeta}(\omega^{x_t}_{\zeta,t})^2\\ &\lesssim \sqrt{d}\cdot\log(T\ln^4 T\ln^2(1/\delta)/(d\delta^2))\cdot d\log T,\end{aligned}$$

where the last inequality holds by applying Lemma 8. Therefore,

$$T_\zeta \lesssim 4^\zeta \cdot d^{3/2} \log T \log(T/\delta). \tag{33}$$

We first divide the resolution levels $\zeta \in \{0, 1, \cdots, \zeta_0\}$ into two different sets: $\mathcal{Z}_1 := \{0, 1, \cdots, \zeta^*\}$ and $\mathcal{Z}_2 := \{\zeta^* < \zeta \le \zeta_0\}$, where ζ^* is an integer to be defined later. Clearly $\mathcal{Z}_1$ and $\mathcal{Z}_2$ partition $\{0, \cdots, \zeta_0\}$. Note that $\sqrt{d} \cdot \sum_{t\in\mathcal{T}_\zeta} \alpha_{\zeta,t}^{x_t} \omega_{\zeta,t}^{x_t} \lesssim 2^{-\zeta} T_\zeta$ because $\varpi_{\zeta,t}^{x_t} \le 2^{1-\zeta}$ for all $t \in \mathcal{T}_\zeta$.

$$\begin{aligned}\sqrt{d} \sum_{\zeta\in\mathcal{Z}_1} \sum_{t\in\mathcal{T}_\zeta} \alpha_{\zeta,t}^{x_t} \omega_{\zeta,t}^{x_t} &\lesssim \sum_{\zeta=0}^{\zeta^*} 2^{-\zeta} \cdot 4^\zeta \cdot d^{3/2} \log T \log(T/\delta) \\ &\le 2^{\zeta^*+1} \cdot d^{3/2} \log T \log(T/\delta); \end{aligned} \tag{34}$$

$$\begin{aligned}&\sqrt{d} \sum_{\zeta\in\mathcal{Z}_2} \sum_{t\in\mathcal{T}_\zeta} \alpha_{\zeta,t}^{x_t} \omega_{\zeta,t}^{x_t} \\ &\lesssim d \sum_{\zeta\in\mathcal{Z}_2} \sqrt{T_\zeta \log(T) \log(T \log^5 T/(T_\zeta \delta^3))} + \delta d \sqrt{T}/\log^2 T \\ &\le d \sqrt{|\mathcal{Z}_2| \left(\sum_{\zeta\in\mathcal{Z}_2} T_\zeta\right) \log(T) \log\left(T \log^5 T \cdot \frac{|\mathcal{Z}_2|}{\delta^3 \sum_{\zeta\in\mathcal{Z}_2} T_\zeta}\right)} \\ &\quad + \delta d \sqrt{T}/\log^2 T \\ &\lesssim d \sqrt{|\mathcal{Z}_2| T \log(T) \log\left(\log^5 T \,|\mathcal{Z}_2|/\delta^3\right)} + \delta d \sqrt{T}/\log^2 T, \end{aligned} \tag{35}$$

where the inequality above Eq. (35) is because of the concavity of the function $\sqrt{x \ln(T \log^5 T |\mathcal{Z}_2|/(x\delta^3))}$ and Jensen's inequality, and Eq. (35) is due to $\sum_{\zeta\in\mathcal{Z}_2} T_\zeta \le T$ and the monotonicity of the function $\sqrt{x \ln(T \log^5 T |\mathcal{Z}_2|/(x\delta^3))}$.

Recall that $\sqrt{T/d}/\delta \le 2^{\zeta_0} \le 2\sqrt{T/d}/\delta$. Select $\zeta^* = \zeta_0 - \lfloor \log_2(\ln(T)\ln(T/\delta)/\delta) \rfloor$; we have that $|\mathcal{Z}_2| = O(\log\log(T/\delta) + \log(1/\delta))$ and $2^{\zeta^*} \le 2\sqrt{T}/(\sqrt{d}\ln(T)\ln(T/\delta))$.

Finally, we combine Eq. (27), Eq. (34), and Eq. (35), and have that

$$\begin{aligned}\sum_t \varpi_{\zeta_t,t}^{x_t} &\lesssim \delta d \sqrt{T} + d\sqrt{T \log T \log(1/\delta)} \cdot \log\log(T/\delta) \\ &\lesssim d\sqrt{T \log T \log(1/\delta)} \cdot \log\log(T/\delta),\end{aligned}$$

which is to be demonstrated. □