A Proofs

A.1 Proof of Theorem 2

**Theorem 2.** For any fixed $\gamma$, let $z \sim N(0, \gamma^2)$, define $D_s = \mathbb{E}[(\sin(Q(z)) - \sin(z))^2]$, $\zeta_s = \text{Cov}(\sin(Q(z)) - \sin(z), \sin(z))$, and $D_c$ and $\zeta_c$ analogously for cosine function. Further denote $\triangle_c = \mathbb{E}[(\cos(Q(z)) - e^{-\frac{c}{2}}]$ and $\tilde{D}_c = D_c - \triangle_c^2$. Denote $V_s^* = \frac{1}{2} \left[ 1 - e^{-2\rho^2\gamma^2} \right]$ and $V_c^* = \frac{1}{2} \left[ 1 + e^{-2\rho^2\gamma^2} \right] - e^{-\rho^2\gamma^2}$. Assume $x, y$ are two normalized samples with correlation $\rho$. Then at $\gamma$, $K_Q(x,y)$ is lower and upper bounded respectively by

$$K(x,y) - D_s - D_c + 2e^{-\gamma^2(1-\rho^2)/2} (C_1_+ + C_2_-),$$

$$K(x,y) + D_s + D_c + 2e^{-\gamma^2(1-\rho^2)/2} (C_1_- + C_2_+),$$

where $C_1\pm = (C_1 C_2 \pm \sqrt{(1-C_1^2)(1-C_2^2)}) \sqrt{D_s V_s^*}$, $C_2\pm = [(C_3 C_4 \pm \sqrt{(1-C_3^2)(1-C_4^2)}) \sqrt{D_c V_c^*} + e^{-2\gamma^2\triangle_c}],$

with

$$C_1 = \sqrt{\frac{2\zeta_s}{D_s(1-e^{-2\gamma^2})}}, \quad C_2 = \frac{e^{-\gamma^2(1-\rho^2)/2} - e^{-\gamma^2(1+\rho^2)/2}}{\sqrt{2(1-e^{-2\gamma^2})} V_s^*},$$

$$C_3 = \sqrt{\frac{\zeta_c}{D_c(1+ e^{-2\gamma^2}) - e^{-\gamma^2}}}, \quad C_4 = \frac{1}{2} \left[ e^{-\gamma^2(1-\rho^2)/2} + e^{-\gamma^2(1+\rho^2)/2} \right] - \frac{e^{-\gamma^2(1+\rho^2)/2}}{\sqrt{(1+ e^{-2\gamma^2}) - e^{-\gamma^2}}} V_c^*.$$

**Proof.** First we look at the sine function. In this proof, we will use the notation $(u,v) = (w^T x, w^T y) \sim N \left( 0, \gamma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$ to denote the projected data. $Q$ is a general quantizer applied to the linear random projections. We have

$$\mathbb{E}[(\sin(Q(u)) - \sin(u))^2] = \mathbb{E}[(\sin(Q(u)) - \sin(u) + \sin(u))$$

$$(\sin(Q(v)) - \sin(v) + \sin(v))]$$

$$= \mathbb{E}[(\sin(Q(u)) - \sin(u))(\sin(Q(v)) - \sin(v))]$$

$$+ 2\mathbb{E}[(\sin(Q(u)) - \sin(u)) \sin(v)] + \mathbb{E}[\sin(u) \sin(v)]$$

$$= T_1 + 2T_2 + E[\sin(u) \sin(v)].$$

By Young’s inequality, we have

$$-D_s \leq T_1 \leq D_s. \tag{9}$$

To bound the second term, the following identities would be useful. For $u \sim N(0, \gamma^2)$,

$$\mathbb{E}[\cos(u)] = e^{-\gamma^2/2},$$

$$\mathbb{E}[\sin(au) \sin(bu)] = \frac{1}{2} \left[ e^{-\frac{\gamma^2(a-b)^2}{2}} - e^{-\frac{\gamma^2(a+b)^2}{2}} \right],$$

$$\mathbb{E}[\cos(au) \cos(bu)] = \frac{1}{2} \left[ e^{-\frac{\gamma^2(a-b)^2}{2}} + e^{-\frac{\gamma^2(a+b)^2}{2}} \right].$$
Since we can write \( v = \rho u + \sqrt{1 - \rho^2} Z \) with \( Z \sim N(0, \gamma^2) \) independent of \( u \), we have

\[
T_2 = \mathbb{E}\left[ (\sin(Q(u)) - \sin(u)) \sin(\rho u + \sqrt{1 - \rho^2} Z) \right] \\
= \mathbb{E}\left[ (\sin(Q(u)) - \sin(u)) \sin(\rho u) \cos(\sqrt{1 - \rho^2} Z) \right] \\
+ \mathbb{E}\left[ (\sin(Q(u)) - \sin(u)) \cos(\rho u) \sin(\sqrt{1 - \rho^2} Z) \right] \\
= e^{-\frac{\gamma^2(1 - \rho^2)}{2}} \mathbb{E}\left[ (\sin(Q(u)) - \sin(u)) \sin(\rho u) \right].
\]

(10)

By assumption,

\[
\text{Cov}[\sin(Q(u)) - \sin(u), \sin(u)] = \zeta_s, \\
\text{Var}[\sin(Q(u)) - \sin(u)] = D_s,
\]

Now we can compute

\[
\text{Var}[\sin(u)] = \frac{1}{2} \left[ 1 - e^{-2\gamma^2} \right], \\
\text{Cov}[\sin(\rho u), \sin(u)] = \frac{1}{2} \left[ e^{-\frac{\gamma^2(1 - \rho^2)}{2}} - e^{-\frac{\gamma^2(1 + \rho^2)}{2}} \right], \\
\text{Var}[\sin(\rho u)] = \frac{1}{2} \left[ 1 - e^{-2\rho^2\gamma^2} \right] \equiv V_s^*.
\]

The correlation coefficients are

\[
C_1 \triangleq \text{Corr}[\sin(Q(u)) - \sin(u), \sin(u)] = \frac{\sqrt{2}\zeta_s}{\sqrt{D_s(1 - e^{-2\gamma^2})}}, \\
C_2 \triangleq \text{Corr}[\sin(\rho u), \sin(u)] = \frac{e^{-\frac{\gamma^2(1 - \rho^2)}{2}} - e^{-\frac{\gamma^2(1 + \rho^2)}{2}}}{\sqrt{(1 - e^{-2\gamma^2})(1 - e^{-2\rho^2\gamma^2})}}.
\]

By Cauchy-Schwartz inequality, we know that \( \text{Corr}[\sin(Q(u)) - \sin(u), \sin(\rho u)] \) is bounded between

\[
C_{11} \triangleq C_1 C_2 - \sqrt{(1 - C_1^2)(1 - C_2^2)}, \\
C_{u1} \triangleq C_1 C_2 + \sqrt{(1 - C_1^2)(1 - C_2^2)}.
\]

Therefore, we have that

\[
C_{11} \sqrt{D_s V_s^*} \leq \mathbb{E}\left[ (\sin(Q(u)) - \sin(u)) \sin(\rho u) \right] \\
\leq C_{u1} \sqrt{D_s V_s^*}.
\]

Combining with (8), (9) and (10) gives the expression for the sine part. For the cosine, we can use similar approach. From now on, denote \( Q = Q_{c, \gamma} \). In particular, we can have

\[
\mathbb{E}[\cos(Q(u)) \cos(Q(v))] = T_3 + 2T_4 + \mathbb{E}[\cos(u) \cos(v)],
\]

where

\[
T_3 = \mathbb{E}[(\cos(Q(u)) - \cos(u))(\cos(Q(v)) - \cos(v))], \\
T_4 = \mathbb{E}[(\cos(Q(u)) - \cos(u)) \cos(v)].
\]

Similarly, we have

\[-D_c \leq T_3 \leq D_c,\]

and

\[
T_4 = e^{-\frac{\gamma^2(1 - \rho^2)}{2}} \mathbb{E}\left[ (\cos(Q(u)) - \cos(u)) \cos(\rho u) \right].
\]
Similarly, we can obtain
\[
\text{Cov}[\cos(Q(u)) - \cos(u), \cos(u)] = \zeta_c, \\
\text{Var}[\cos(Q(u)) - \cos(u)] = \hat{D}_c, \\
\text{Var}[\cos(u)] = \frac{1}{2} \left[ 1 + e^{-2\gamma^2} \right] - e^{-\gamma^2}, \\
\text{Cov}[\cos(\rho u), \cos(u)] \\
= \frac{1}{2} \left[ e^{-\frac{x^2+\gamma^2}{2}} + e^{-\frac{y^2+\gamma^2}{2}} \right] - e^{-\frac{x^2+y^2}{2}}, \\
\text{Var}[\cos(\rho u)] = \frac{1}{2} \left[ 1 + e^{-2\gamma^2} \right] - e^{-\gamma^2} \triangleq V_c^*.
\]

The remaining part is similar, where we use Cauchy-Schwartz to bound the correlation of \(\cos(Q(u)) - \cos(u)\) and \(\cos(\rho u)\). We omit it for conciseness. The desired result is obtained by combining two parts and noticing that \(\mathbb{E}[\sin(u) \sin(v) + \cos(u) \cos(v)] = e^{-\gamma^2(1-\rho)} = K(u, v)\).

\[\square\]

### A.2 Proof of Theorem 3

**Theorem 3. (Uniform Approximation Error)** Assume the sample space \(\mathcal{S}\) is the unit sphere (normalized data). Let QRP-RFF estimators be defined as (6). Let \(\Gamma \sim N(0, \gamma^2)\) in Definition 2. Suppose a quantizer \(Q\) is mean smooth w.r.t. \(\sin\) and \(\cos\) functions with Lipschitz constant \(L_Q^\rho\) and \(L_Q^c\), respectively. Then for \(\forall \epsilon > 0\), with probability at least \(1 - 4e^{-k\epsilon^2/256}\),
\[|\bar{K}_Q(x, y) - K_Q(x, y)| \leq \epsilon, \quad \text{for } \forall x, y \in \mathcal{S},\]
when \(k \geq \frac{512d}{\epsilon^2} \log \left( \frac{64\max(L_Q^\rho, L_Q^c)\gamma}{\epsilon} + 1 \right)\).

**Proof.** We denote the sample space (unit sphere) as \(\mathcal{S} = \mathbb{S}^{d-1}\). Let \(\tilde{S}_\triangle\) be a \(\triangle\)-net placed on \(\mathcal{S}\). We then can express any \(x \in \mathcal{S}\) as \(x = \tilde{x} + r_x\), for the center \(\tilde{x} \in \tilde{S}_\triangle\) and \(\|r_x\| \leq \triangle\).

Define
\[
K_Q^\rho(x, y) = \mathbb{E}[\sin(Q(w^T x)) \sin(Q(w^T y))], \\
K_Q^c(x, y) = \mathbb{E}[\cos(Q(w^T x)) \cos(Q(w^T y))].
\]

We have
\[
|\bar{K}_Q(x, y) - K_Q(x, y)| = |\bar{K}_Q^\rho(x, y) + \bar{K}_Q^c(x, y) - K_Q^\rho(x, y) - K_Q^c(x, y)| \\
\leq |\bar{K}_Q^\rho(x, y) - K_Q^\rho(x, y)| + |\bar{K}_Q^c(x, y) - K_Q^c(x, y)|. \quad (11)
\]

As before, we mainly provide details on the sine part, and the reasoning applies to the cosine part similarly. For any \(x, y \in \mathcal{S}\), firstly we assume that the following two events hold:
\[
\Omega_1 : \sup_{\tilde{x} \in \tilde{S}_\triangle} \frac{1}{k} \sum_{i=1}^{k} \sup_{\|r\| \leq \triangle} |\sin(Q(w^T \tilde{x} + w^T r)) - \sin(Q(w^T \tilde{x}))| \leq L_Q^\rho \gamma \triangle + \epsilon_1, \\
\Omega_2 : \sup_{\tilde{x}, \tilde{y} \in \tilde{S}_\triangle} |\bar{K}_Q^\rho(\tilde{x}, \tilde{y}) - K_Q^\rho(\tilde{x}, \tilde{y})| \leq \epsilon_2.
\]

For any \(x, y \in \mathcal{S}\), we have the following bound by triangle inequality,
\[
|\bar{K}_Q^\rho(x, y) - K_Q^\rho(x, y)| \leq |\bar{K}_Q^\rho(x, y) - \bar{K}_Q^\rho(\tilde{x}, y)| + |\bar{K}_Q^\rho(\tilde{x}, y) - \bar{K}_Q^\rho(\tilde{x}, \tilde{y})| \\
+ |\bar{K}_Q^\rho(\tilde{x}, \tilde{y}) - K_Q^\rho(\tilde{x}, \tilde{y})| + |K_Q^\rho(x, y) - K_Q^\rho(\tilde{x}, \tilde{y})| \triangleq T_1 + T_2 + T_3 + T_4. \quad (12)
\]
We now bound these terms separately. We have

\[ T_1 = \frac{1}{k} \left| \sum_{i=1}^{k} \sin(Q(w_i^T \hat{x} + w_i^T r_x)) \sin(Q(w_i^T \hat{y} + w_i^T r_y)) - \sin(Q(w_i^T \bar{x})) \sin(Q(w_i^T \bar{y} + w_i^T r_y)) \right| \]

\[ = \frac{1}{k} \left| \sum_{i=1}^{k} [\sin(Q(w_i^T \hat{x} + w_i^T r_x)) - \sin(Q(w_i^T \bar{x}))] \sin(Q(w_i^T \bar{y} + w_i^T r_y)) \right| \]

\[ \leq L_Q \gamma \triangle + \epsilon_1, \]

where the last line is due to event \( \Omega_1 \) and boundedness of sine function. Similarly,

\[ T_2 = \frac{1}{k} \left| \sum_{i=1}^{k} \sin(Q(w_i^T \hat{x})) [\sin(Q(w_i^T \bar{y} + w_i^T r_y)) - \sin(Q(w_i^T \bar{y}))] \right| \]

\[ \leq L_Q \gamma \triangle + \epsilon_1. \]

The event \( \Omega_2 \) directly implies that

\[ T_3 \leq \epsilon_2. \]

For \( T_4 \), by mean smoothness assumption we have

\[ T_4 = \left| \mathbb{E} \left[ \sin(Q(w_i^T \hat{x} + w_i^T r_x)) \sin(Q(w_i^T \hat{y} + w_i^T r_y)) - \sin(Q(w_i^T \bar{x})) \sin(Q(w_i^T \bar{y})) \right] \right| \]

\[ = \left| \mathbb{E} \left[ \sin(Q(w_i^T \hat{x} + w_i^T r_x)) \sin(Q(w_i^T \hat{y} + w_i^T r_y)) - \sin(Q(w_i^T \bar{x})) \sin(Q(w_i^T \bar{y})) \right. \right. \]

\[ + \left. \left. \sin(Q(w_i^T \hat{x} + w_i^T r_x)) \sin(Q(w_i^T \bar{y} + w_i^T r_y)) - \sin(Q(w_i^T \bar{x} + w_i^T r_x)) \sin(Q(w_i^T \bar{y})) \right] \right| \]

\[ \leq \mathbb{E} \left[ \left| \sin(Q(w_i^T \hat{x} + w_i^T r_x)) - \sin(Q(w_i^T \bar{x} + w_i^T r_x)) \right| \right] \]

\[ \leq L_Q^x \mathbb{E} \left[ \|w_i^T r_x\| + \|w_i^T r_y\| \right] \]

\[ \leq 2L_Q^x \gamma \triangle. \]

Summing up ingredients together in (12) we get that in event \( \Omega_1 \) and \( \Omega_2 \), we have

\[ |\hat{K}_Q^x(x, y) - K_Q^x(x, y)| \leq 2\epsilon_1 + \epsilon_2 + 4L_Q^x \gamma \triangle. \]

To derive a high probability bound, we now investigate the two events. First, we have the complement

\[ P \left[ \Omega_i^c \right] = P \left[ \sup_{\hat{x} \in \mathcal{S}_\triangle} \frac{1}{k} \sum_{i=1}^{k} \|r\| \leq \Delta \sup_{\|r\| \leq \Delta} \left| \sin(Q(w_i^T \hat{x} + w_i^T r)) - \sin(Q(w_i^T \bar{x})) \right| \geq L_Q^x \gamma \Delta + \epsilon_1 \right]. \]

Since the terms in the summation, \( \sup_{\|r\| \leq \Delta} |\sin(Q(w_i^T \hat{x} + w_i^T r)) - \sin(Q(w_i^T \bar{x}))| \), are i.i.d. random variables, for any \( \hat{x} \in \mathcal{S}_\Delta \) the expectation admits

\[ \mathbb{E} \left[ \sup_{\|r\| \leq \Delta} \left| \sin(Q(w_i^T \hat{x} + w_i^T r)) - \sin(Q(w_i^T \bar{x})) \right| \right] \leq L_Q^x \gamma \Delta, \]

due to mean smoothness of \( Q \). By Hoeffding’s inequality on bounded variables, we get \( \forall \hat{x} \in \mathcal{S}_\Delta \)

\[ P \left[ \frac{1}{k} \sum_{i=1}^{k} \|r\| \leq \Delta \left| \sin(Q(w_i^T \hat{x} + w_i^T r)) - \sin(Q(w_i^T \bar{x})) \right| \geq L_Q^x \gamma \Delta + \epsilon_1 \right] \leq e^{-2k^2\epsilon_1^2/4k} = e^{-k^2\epsilon_1^2/2}. \]

Applying union bound over all \( \hat{x} \in \mathcal{S}_\Delta \), we obtain

\[ P \left[ \Omega_i^c \right] \leq |\mathcal{S}_\Delta| e^{-k\epsilon_1^2/2} \leq \left( \frac{2}{\Delta} + 1 \right)^d e^{-k\epsilon_1^2/2}, \]
where the last inequality is due to the bound on covering number of the unit sphere (Corollary 4.2.13 in Vershynin (2018)). When \( k \geq \frac{8d_1}{\epsilon_1} \log(\frac{2}{\Delta} + 1) \), we have \( P[\Omega_1] \leq e^{-\frac{k\epsilon_1^2}{4}} \). For \( \Omega_2 \), applying Hoeffding’s inequality yields a point-wise bound, where for \( \forall \tilde{x}, \tilde{y} \in \tilde{S}_\Delta \),

\[
P\left[ |\hat{K}^\Delta_Q(\tilde{x}, \tilde{y}) - K^\Delta_Q(\tilde{x}, \tilde{y})| \geq \epsilon_2 \right] = P\left[ \frac{1}{k} \sum_{i=1}^{k} \sin(Q(\tilde{x})) \sin(Q(\tilde{y})) - K^\Delta_Q(\tilde{x}, \tilde{y}) \right] \leq 2e^{-\frac{k\epsilon_2^2}{2}}.
\]

Casting an union bound over \((\tilde{x}, \tilde{y}) \in \tilde{S}_\Delta \times \tilde{S}_\Delta \) yields

\[
P\left[ \Omega_2 \right] = P\left[ \sup_{\tilde{x}, \tilde{y} \in \tilde{S}_\Delta} |\hat{K}^\Delta_Q(\tilde{x}, \tilde{y}) - K^\Delta_Q(\tilde{x}, \tilde{y})| \geq \epsilon_2 \right] \leq 2\left( \frac{|\tilde{S}_\Delta|}{2} \right) e^{-\frac{k\epsilon_2^2}{2}} \leq \left( \frac{2}{\Delta} + 1 \right)^{2d} e^{-\frac{k\epsilon_2^2}{2}}.
\]

Consequently, \( P[\Omega_2] \leq e^{-\frac{k\epsilon_2^2}{4}} \) when \( k \geq \frac{8d_1}{\epsilon_1} \log(\frac{2}{\Delta} + 1) \). Therefore, we obtain that when \( k \geq 4d_1 \log(\frac{2}{\Delta} + 1) \max\{\epsilon_1^{-2}, 2\epsilon_2^{-2}\} \),

\[
P\left[ \Omega_1 \cup \Omega_2 \right] \leq e^{-\frac{k\epsilon_1^2}{4}} + e^{-\frac{k\epsilon_2^2}{4}}.
\]

Now by letting \( \epsilon_1 = \epsilon_2 = \epsilon/8 \), and choosing \( \Delta = \frac{\epsilon}{32L^\Delta_Q \gamma} \), we have proved that when \( k \geq \frac{512d_1}{\epsilon_1^2} \log(\frac{64L^\Delta_Q \gamma}{\epsilon} + 1) \), the error of sine part is bounded as

\[
|\hat{K}^\Delta_Q(x, y) - K^\Delta_Q(x, y)| \leq \epsilon/2,
\]

with probability at least \( 1 - 2e^{-\frac{k\epsilon_2^2}{256}} \). Similarly analysis can be used to bound the cosine part. For conciseness we omit the detailed proof. It is true that when \( k \geq \frac{512d_1}{\epsilon_2^2} \log(\frac{64L^\Delta_Q \gamma}{\epsilon} + 1) \), with probability \( 1 - 2e^{-\frac{k\epsilon_2^2}{256}} \) we have

\[
|\hat{K}^\Delta_Q(x, y) - K^\Delta_Q(x, y)| \leq \epsilon/2.
\]

Therefore, by (11) and union bound we know that when \( k \geq \frac{512d_1}{\epsilon_2^2} \log(\frac{64\max\{L^\Delta_Q, L^\Delta_c\} \gamma}{\epsilon} + 1) \), the kernel approximation error is uniformly bounded by

\[
|\hat{K}_Q(x, y) - K_Q(x, y)| \leq \epsilon,
\]

for \( \forall x, y \in S \), with probability \( 1 - 4e^{-\frac{k\epsilon^2}{256}} \). This completes the proof. \( \square \)