Differentially Private Online Submodular Maximization Supplementary Materials

A OMITTED PROOFS

A.1 Proof of Lemma 1

Proof. We prove the lemma by induction on i. The base case of $i = 1$ follows from Proposition 2. For the inductive step, assume the result is true for some $i \geq 1$, and we now prove that it also holds for $i + 1$. That is, we aim to show that $(\mathcal{E}_{i+1}, \dots, \mathcal{E}_1) : \mathcal{F}^T \to U^T \times \cdots \times U^T$ is $((i+1)\varepsilon', (i+1)\delta')$-private, where $\varepsilon' = \varepsilon/k$ and $\delta' = \delta/k$. Let $a \wedge b$ be the minimum of a and b and recall that $\mathcal{M}^{S^i}$ is the behavior of the i-th expert across all T rounds.

Consider the neighboring databases F and F'. Pick any set $S \subseteq U^T$ and a fixed $S^i = (a^i, \dots, a^1) \in (U^T)^i$, then

$$\begin{aligned}
&\Pr(\mathcal{E}_{i+1}(F) \in S \mid (\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) \\
&= \Pr(\mathcal{M}^{S^i}(F) \in S) \\
&\leq (e^{\varepsilon'} \Pr(\mathcal{M}^{S^i}(F') \in S)) \wedge 1 + \delta' && ((\varepsilon', \delta')\text{-DP of } \mathcal{M}^{S^i}) \\
&= (e^{\varepsilon'} \Pr(\mathcal{E}_{i+1}(F') \in S \mid (\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i)) \wedge 1 + \delta'.
\end{aligned}$$

This is true as long as $(\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i$ and $(\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i$ are non-zero probability events, which is ensured to be true since the Hedge algorithm places positive probability on all events.

We can write

$$\Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) = e^{i\varepsilon'} \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i) + \mu(S^i),$$

where $\mu(S^i) = \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) - e^{i\varepsilon'} \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i)$. We have $\mu(\mathcal{S}) \leq i\delta'$ for any $\mathcal{S} \subseteq (U^T)^i$ since $(\mathcal{E}_i, \dots, \mathcal{E}_1)$ is $(i\varepsilon', i\delta')$-DP by the inductive hypothesis.

Now, consider any set $\mathcal{S} \subseteq (U^T)^{i+1}$. Then,

$$\begin{aligned}
&\Pr((\mathcal{E}_{i+1}, \mathcal{E}_i, \dots, \mathcal{E}_1)(F) \in S) \\
&= \sum_{S^i \in \mathcal{S}'} \Pr((\mathcal{E}_{i+1}(F), S^i) \in S \mid (\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) \\
&\leq \sum_{S^i \in \mathcal{S}'} \left((e^{\varepsilon'} \Pr((\mathcal{E}_{i+1}(F'), S^i) \in S \mid \mathcal{E}_1(F') = a^1)) \wedge 1 + \delta' \right) \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) \\
&\leq \sum_{S^i \in \mathcal{S}'} \left((e^{\varepsilon'} \Pr((\mathcal{E}_{i+1}(F'), S^i) \in S \mid (\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i)) \wedge 1 \right) \left(e^{i\varepsilon'} \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i) + \mu(S^i) \right) \\
&\qquad + \sum_{S^i \in \mathcal{S}'} \delta' \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F) = S^i) \\
&\leq e^{(i+1)\varepsilon'} \sum_{S^i \in \mathcal{S}'} \Pr((\mathcal{E}_{i+1}(F'), S^i) \in S \mid (\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i) \Pr((\mathcal{E}_i, \dots, \mathcal{E}_1)(F') = S^i) + \mu(\mathcal{S}'_+) + \delta' \\
&\leq e^{(i+1)\varepsilon'} \Pr((\mathcal{E}_{i+1}, \mathcal{E}_i, \dots, \mathcal{E}_1)(F') \in S) + (i+1)\delta'
\end{aligned}$$

where $\mathcal{S}' = \{S^i \in (U^T)^i : (a^{i+1}, S^i) \in \mathcal{S} \text{ for some } a^i \in U^T\}$ and $\mathcal{S}'_+$ are the elements $S^i \in \mathcal{S}'$ such that $\mu(\mathcal{S}') \geq 0$. This concludes the proof.

□

A.2 Proof of Proposition 3

Proposition 3. *The $(1-1/e)$-regret of Algorithm 2 is bounded by the expected regret of $\mathcal{E}_1,\dots,\mathcal{E}_k$.*

Proof. Fix the choices $S_1,\dots,S_T$ of the experts arbitrarily, and let r_i the overall regret experience by $\mathcal{E}_i$. That is,

$$r_i = \max_{a\in U}\sum_{t=1}^{T} f_t(S_t^{i-1}+a) - f_t(S_t^{i-1}) - \sum_{t=1}^{T} f_t(S_t^{i-1}+a_t^i) - f_t(S_t^{i-1}).$$

Define the new function $F: 2^{[T]\times U}\to\mathbb{R}$ as

$$F(A) = \frac{1}{T}\sum_{t=1}^{T} f_t(A_t),$$

where $A_t = \{x\in U : (t,x)\in A\}$. Clearly, F is submodular, nondecreasing and $F(\emptyset)=0$. Then,

$$\frac{r_i}{T} = \max_{a\in U} F(\widetilde{S}^{i-1}+\widetilde{a}) - F(\widetilde{S}^{i-1}) - (F(\widetilde{S}^{i}) - F(\widetilde{S}^{i-1})),$$

where $\widetilde{S}^i = \bigcup_{t=1}^{T}\{t\}\times S^i$.

Let $OPT\subseteq U$ be the optimal solution of $\max_{|S|\le k}\sum_{t=1}^{T} f_t(S)$ and consider its extension to $[T]\times U$, i.e., $\widetilde{OPT} = \bigcup_{t=1}^{T}\{t\}\times OPT$.

Claim A.1. *For any $i=1,\dots,k$, $\max_{a\in U} F(\widetilde{S}^{i-1}+\widetilde{a}) - F(\widetilde{S}^{i-1}) \ge \frac{F(\widetilde{OPT})-F(S^{i-1})}{k}$.*

Proof of Claim A.1.

$$\begin{aligned}
&F(\widetilde{OPT}) - F(\widetilde{S}^{i-1})\\
&\le F(\widetilde{S}^{i-1}+\widetilde{OPT}) - F(\widetilde{S}^{i-1})\\
&\le \sum_{\widetilde{a}\in\widetilde{OPT}\setminus\widetilde{S}^{i-1}} F(\widetilde{S}^{i-1}+\widetilde{a}) - F(\widetilde{S}^{i-1})\\
&\le k\cdot\left(\max_{a\in U} F(\widetilde{S}^{i-1}+\widetilde{a}) - F(\widetilde{S}^{i-1})\right).
\end{aligned}$$

□

Using this claim, we can see,

$$F(\widetilde{S}^{i}) - F(\widetilde{S}^{i-1}) \ge \frac{F(\widetilde{OPT}) - F(\widetilde{S}^{i-1})}{k} - \frac{r_i}{T}.$$

Unrolling the recursion, we obtain

$$\sum_{t=1}^{T} f_t(S_t) \ge \left(1-\frac{1}{e}\right)\sum_{t=1}^{T} f_t(OPT) - \sum_{i=1}^{k} r_i.$$

□

A.3 Proof of Lemma 2

Lemma 2. *If r_i denotes the regret experience by expert $\mathcal{E}_i$ in Algorithm 3, then*

$$\left(1-\frac{1}{e}\right)\max_{|S|\le k}\sum_{t=1}^{T} f_t(S) - \mathbb{E}\left[\sum_{t=1}^{T} f_t(S_t)\right] \le \sum_{i=1}^{k}\mathbb{E}[r_i] + \gamma T.$$

Proof. Observe that at exploration time-steps τ, i.e, when $b_\tau = 1$, Algorithm 3 plays a set of the form $S_\tau = S_\tau^{i-1} + a$. Right after this, the algorithm samples a new set $S_{\tau+1}$ given by the Hedge algorithms and will keep playing this set until the next exploration time step.

For the sake of analysis, we introduce the following set. Let $t_0 = 0, t_1, \ldots, t_M$ be the times when a new sample set is obtained. Note that besides time t_0, all times $t_1, \ldots, t_M$ are exploration times. Now, let $S'_t = S_{t_i}$ for $t = t_i + 1, \ldots, t_{i+1}$. Note that for times $b_t = 0$, then $S'_t = S_t$; however, for times $b_t = 1$, then S'_t is not necessarily the same as $S_t = S_t^{i-1} + a$. In other words, S'_t corresponds to the real full exploitation scheme. Now, as in the full information setting, we have

$$\left(1 - \frac{1}{e}\right) \max_{|S| \le k} \sum_{t=1}^T f_t(S) - \sum_{t=1}^T f_t(S'_t) \le \sum_{i=1}^k r_i,$$

where $r_i = \max_{a \in U} \sum_{t=1}^T f_t^{i,a} - \sum_{t=1}^T f_t^{i,a_t^i}$. Thus

$$\begin{aligned}
&\left(1 - \frac{1}{e}\right) \max_{|S| \le k} \sum_{t=1}^T f_t(S) - \mathbb{E}\left[\sum_{t=1}^T f_t(S_t)\right] \\
&\le \sum_{i=1}^k \mathbb{E}[r_i] + \mathbb{E}\left[\sum_{t=1}^T f_t(S'_t) - f_t(S_t)\right] \\
&\le \sum_{i=1}^k \mathbb{E}[r_i] + \gamma T,
\end{aligned}$$

since at the end, only the exploration times could contribute to the difference $f_t(S'_t) - f_t(S_t)$ and those are γT in expectation. □

A.4 Proof of Lemma 3

Lemma 3. *If each $\mathcal{E}_i$ is a Hedge algorithm with learning rate $\eta = \frac{\varepsilon}{k\sqrt{32(2\gamma T)\log(k/\delta)}}$ then $\mathbb{E}[r_i] \le 16\frac{k^2|U|\log|U|\sqrt{T\log(k/\delta)}}{\varepsilon\sqrt{\gamma}} + \frac{k|U|}{\gamma}T \cdot e^{-8\gamma^2 T}$.*

Proof. From the perspective of expert $\mathcal{E}_i$, at every time-step t, she sees the vector $\widehat{f}_t^i$ such that

$$\widehat{f}_t^{i,a} = f_t(S_t^{i-1} + a)\mathbf{1}_{\{\text{Explore at time } t,\text{ pick } i,\text{ pick } a\}}$$

in its a-th coordinate. Notice that this vector is 0 if no exploration occurs at time t. The expert $\mathcal{E}_i$ samples a new element in U only after exploitation times. Observe that the feedback of $\mathcal{E}_i$ is independent of choices made by $\mathcal{E}_i$. Indeed, this feedback depends only on the set S_t^{i-1} constructed by $\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}$ and the decision of the learner to explore, which is independent of the learning task. Therefore, the sequence $\widehat{f}^i = (\widehat{f}_1^i, \ldots, \widehat{f}_T^i)$ could be considered oblivious for $\mathcal{E}_i$ and we can apply the guarantee of Hedge over $\widehat{f}_i$. That is, for any $a \in U$,

$$\sum_{t=1}^T \widehat{f}_t^{i,a} - \sum_{t=1}^T \mathbf{x}_t^\top \widehat{f}_t^i \le \eta \sum_{t=1}^T \mathbf{x}_t^\top (\widehat{f}_t^i)^2 + \frac{\log|U|}{\eta},$$

where $\mathbf{x}_t \in \Delta(U)$ is the non-zero distribution used by expert $\mathcal{E}_i$ in the Hedge algorithm and $\Delta(U) = \{\mathbf{x} \in \mathbb{R}^U : \|\mathbf{x}\|_1 = 1, \mathbf{x} \ge 0\}$ is the probability simplex over elements in U. Notice that exploitation times appear in the summation with 0 contribution. This expression is not the same as the regret of $\mathcal{E}_i$ but we can relate these quantities as follows. Conditioned on $S_1^{i-1}, \ldots, S_T^{i-1}$ we obtain,

$$\mathbb{E}[\widehat{f}_t^{i,a} \mid S_1^{i-1}, \ldots, S_T^{i-1}] = \frac{\gamma}{k|U|} f_t^{i,a} + \delta_t,$$

where $f_t^{i,a} = f(S_t^{i-1} + a) - f(S_t^{i-1})$ and $\delta_t^i = \frac{\gamma}{k|U|} f(S_t^{i-1})$. Notice that $S_t^{i-1}, \ldots, S_T^{i-1}$ are independent of actions taken by $\mathcal{E}_i$, so

$$\mathbb{E}[\mathbf{x}_t^\top \widehat{f}_t^i \mid S_1^{i-1}, \ldots, S_T^{i-1}] = \frac{\gamma}{k|U|} \mathbb{E}[\mathbf{x}_t^\top f_t^i \mid S_t^1, \ldots, S_T^{i-1}] + \delta_t$$

and

$$\mathbb{E}[\mathbf{x}_t^\top (\widehat{f}_t^i)^2 \mid S_1^{i-1}, \ldots, S_T^{i-1}] = \mathbb{E}\left[\sum_{a \in U} x_t(a)(\widehat{f}_t^{i,a})^2 \mid S_1^{i-1}, \ldots, S_T^{i-1}\right]$$
$$= \sum_{a \in U} \mathbb{E}[x_t(a) \mid S_1^{i-1}, \ldots, S_T^{i-1}] \frac{\gamma}{k|U|} f(S_t^{i-1} + a)^2$$
$$\leq \frac{\gamma}{k|U|}.$$

Let M be the number of times Algorithm 3 decides to explore. That is, M is distributed as the sum of T Bernoulli random variables with parameter γ. By concentration bounds,

$$\Pr(M > 2\gamma T) \leq e^{-8\gamma^2 T}.$$

Now, let $t_1, \ldots, t_M$ be the times the algorithm decides to explore and let $t_0 = 0$. For $i = 1, \ldots, M$, we can assume that expert $\mathcal{E}_i$ releases the same vector $\mathbf{x}_t \in \Delta_U$ during the time interval $[t_{i-1}, t_i)$ since she does not get any feedback during those times. If we consider $\eta = \frac{\varepsilon}{k\sqrt{32(2\gamma T)\log(k/\delta)}}$, then for any $a \in U$ we have

$$\frac{\gamma}{k|U|}\,\mathbb{E}\left[\sum_{t=1}^T f_t^{i,a} - \sum_{t=1}^T \mathbf{x}_t^\top f_t^i\right] = \mathbb{E}\left[\sum_{t=1}^T \widehat{f}_t^{i,a} - \sum_{t=1}^T \mathbf{x}_t^\top \widehat{f}_t^i\right]$$
$$\leq \left(\eta \sum_{t=1}^T \mathbb{E}\left[\mathbf{x}_t^\top (\widehat{f}_t^i)^2\right] + \frac{\log|U|}{\eta}\right) + T \cdot e^{-8\gamma^2 T}$$
$$\leq \left(\eta \frac{\gamma}{k|U|} T + \frac{\log|U|}{\eta}\right) + T \cdot e^{-8\gamma^2 T}$$

Therefore,

$$\mathbb{E}[r_i] = \max_{a \in U} \sum_{t=1}^T f_t^{i,a} - \mathbb{E}\left[\sum_{t=1}^T \mathbf{x}_t^\top f_t^i\right] \leq 16 \frac{k^2 |U| \log|U| \sqrt{T \log(k/\delta)}}{\varepsilon\sqrt{\gamma}} + \frac{k|U|}{\gamma} T \cdot e^{-8\gamma^2 T}.$$

□

B ADDITIONAL RESULTS IN BANDIT SETTING

B.1 $\mathcal{O}(T^{3/4})$ Regret Bound of Direct Approach in Bandit Setting

In the bandit setting, the direct approach for differential privacy corresponds to sampling a new set from the Hedge algorithms at each time step. As in the full-information setting, to ensure (ε, δ)-DP, a learning rate of $\eta = \frac{\varepsilon}{k\sqrt{32T\log(k/\delta)}}$ is enough.

Similar to Lemma 3, in this setting we have

$$\left(1 - \frac{1}{e}\right) \max_{|S| \leq k} \sum_{t=1}^T f_t(S) - \mathbb{E}\left[\sum_{t=1}^T f_t(S_t)\right] \leq \sum_{t=1}^k \mathbb{E}[r_i] + \gamma T.$$

Since,

$$\mathbb{E}[r_i] \leq \frac{k|U|}{\gamma}\left(\eta \frac{\gamma}{k|U|} T + \frac{\log|U|}{\eta}\right)$$
$$= \frac{k^3 |U| \sqrt{32 T \log(k\delta)}}{\varepsilon\gamma} + \frac{\varepsilon k \sqrt{T}}{\sqrt{32\log(k/\delta)}},$$

then we have,

$$\left(1-\frac{1}{e}\right)\max_{|S|\le k}\sum_{t=1}^{T} f_t(S) - \mathbb{E}\left[\sum_{t=1}^{T} f_t(S_t)\right] \le \frac{k^4|U|\sqrt{32T\log(k\delta)}}{\varepsilon\gamma} + \frac{\varepsilon k^2\sqrt{T}}{\sqrt{32\log(k/\delta)}} + \gamma T.$$

This last bound is minimized when $\gamma = \Theta(T^{-1/4})$ which gives a $(1-1/e)$-regret bound of $\mathcal{O}(T^{3/4})$.

B.2 Trading Off Privacy δ-Term and Space

In this subsection, we show how to trade-off the δ-term $e^{-8T^{1/3}}$ by allowing additional space. For each $t \in T$, select t as an explore round independently with probability γ. Let M be the number of time-steps selected. Note that $\mathbb{E}[M] = \gamma T$. Now, run Algorithm 3 with $\eta = \frac{\varepsilon}{k\sqrt{32(M+1)\log(k/\delta)}}$ and force the algorithm to explore at the M sampled time-steps and utilize the rest of the time-steps to exploit.

In this case, and following the proof of Lemma 3 we obtain:

$$\begin{aligned}
\mathbb{E}[r_i] &\le \frac{k|U|}{\gamma}\,\mathbb{E}\left[\eta M + \frac{\log|U|}{\eta}\right] \\
&\le \frac{k|U|}{\gamma}\,\mathbb{E}\left[6\frac{k\log|U|\sqrt{\log(k/\delta)}}{\varepsilon}\sqrt{M+1}\right] \\
&\le \frac{k|U|}{\gamma}\left(6\frac{k\log|U|\sqrt{\log(k/\delta)}}{\varepsilon}\sqrt{\mathbb{E}[M]+1}\right) && \text{(Jensen's inequality)} \\
&= 8\frac{k^2|U|\log|U|\log(k/\delta)}{\varepsilon}\sqrt{\frac{T}{\gamma}}.
\end{aligned}$$

Using Lemma 2 we obtain the $(1-1/e)$-regret bound of

$$8\frac{k^3|U|\log|U|\log(k/\delta)}{\varepsilon}\sqrt{\frac{T}{\gamma}} + \gamma T.$$

This is minimized at $\gamma = \Theta(1/T^{1/3})$ with a regret bound of $\mathcal{O}(T^{2/3})$ and expected space used $\Theta(T^{2/3})$.

C EXTENSION TO CONTINUOUS FUNCTIONS

In this section we prove Theorem 9. Before this, we present some preliminaries in online convex optimization.

In online convex optimization (OCO), there is compact convex set $\mathcal{X} \subseteq \mathbb{R}^n$ where the learner makes decisions. At time-step t, a convex function $f_t : \mathcal{X} \to \mathbb{R}$ arrives. Without observing this function, the learner has to select a point $\mathbf{x}_t \in \mathcal{X}$ based on previous functions $f_1, \ldots, f_{t-1}$. After the decision has been made, the learner receives the cost $f_t(\mathbf{x}_t)$ and gains oracle access to ∇f_t. The learner's objective is to minimize the regret:

$$\mathcal{R}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}}\sum_{t=1}^{T} f_t(\mathbf{x}).$$

Thakurta and Smith (2013) introduced PFTAL (Private Follow the Approximate Leader) to privately solve the OCO problem.

Theorem 10 (Thakurta and Smith (2013))**.** *PFTAL is $(\varepsilon, 0)$-DP and for any input stream of convex and L-Lipschitz functions $f_1, \ldots, f_T$ has expected regret*

$$\mathbb{E}\left[\mathcal{R}_T\right] \le \mathcal{O}\left(\frac{\sqrt{n\log^{2.5} T}\left(L + \sqrt{\frac{n\log^{2.5} T}{\varepsilon T}}\operatorname{diam}\mathcal{X}\right)^2}{\varepsilon}\sqrt{T}\right).$$

Similar to the Hedge algorithm, we utilize PFTAL as a black-box in Algorithm 4.

Now, we present the proof of Theorem 9 in two parts, and prove each separately.

Lemma 4 (Privacy guarantee)**.** *Algorithm 4 is $(\varepsilon, 0)$-DP.*

Lemma 5 (Regret guarantee)**.** *Let $R = \sup_{\mathbf{x}\in\mathcal{X}} \|x\|_2$, G be a bound on the gradients $\|\nabla f_t(\mathbf{x}_t)\|_2$, and β be the smoothness parameter of $f_1, \ldots, f_T$. Then Algorithm 4 has $(1-1/e)$-regret*

$$\mathbb{E}\left[\left(1-\tfrac{1}{e}\right)\max_{\mathbf{x}\in\mathcal{X}} \textstyle\sum_{t=1}^T f_t(\mathbf{x}) - \sum_{t=1}^T f_t(\mathbf{x}_t)\right] = \mathcal{O}\left(T^{3/4}\sqrt{\log^{2.5} T}\left(\frac{\sqrt{n}\left(G+\sqrt{\frac{n}{\varepsilon T^{3/4}}}\log^{2.5} T \operatorname{diam}\mathcal{X}\right)^2}{\varepsilon} + \beta R^2\right)\right).$$

Proof of Lemma 4 As with the analysis of Algorithm 2, we show that $(\mathcal{E}_{K-1}, \ldots, \mathcal{E}_0)$ is $(\varepsilon, 0)$-DP. If each $\mathcal{E}_k$ were $(\varepsilon/K, 0)$-DP, then the result would immediately follow by simple composition. However, we cannot guarantee that each $\mathcal{E}_k$ is $(\varepsilon/K, 0)$-DP since $\mathcal{E}_k$ obtains as input the privatized output from $\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}$ in the linear function $\ell_k(\mathbf{v}) = \nabla f_t(\mathbf{x}_t^k)^\top \mathbf{v}$, where $\mathbf{x}_t^k$ is computed by $\mathcal{E}_0, \ldots, \mathcal{E}_{k-1}$, while at the same time is accessing again the function f_t (and so the database) via this linear function in the gradient ∇f_t. This clearly breaks the privacy that could have been gained via a simple post-processing argument and therefore and alternative method is needed.

We do not show that each $\mathcal{E}_k$ is $(\varepsilon/K, 0)$-DP but the group $(\mathcal{E}_{K-1}, \ldots, \mathcal{E}_0)$ is $(\varepsilon, 0)$-DP. The proof of the following lemma follows the same steps as the proof of Lemma 1. The proof is slightly simpler since there is no δ-privacy term included but it requires some care since the distributions are continuous in this case.

Lemma 6. *For any $i \geq 1$, the group $(\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0) : \mathcal{F}^T \to (\mathcal{X}^T)^i$ is $i\varepsilon/K$-DP.*

Proof. We proceed by induction in i. The base case $i = 1$ follows immediately from privacy of PFTAL in Thakurta and Smith (2013) because $\mathcal{E}_0$ is the only algorithm that has not its distribution perturbed by any other algorithm. For the inductive step, assume the result is true for some $i \geq 1$ and let us prove it for $i+1$.

Let $\mathbf{x}_0^T, \ldots, \mathbf{x}_{i-1}^T \in \mathcal{X}^T$ and $\mathbf{X}_{i-1} = (\mathbf{x}_{i-1}^T, \ldots, \mathbf{x}_1^T)$. Then, for any $\mathbf{x}_i^T \in \mathcal{X}^T$ we have

$$\Pr(\mathcal{E}_i(F) = \mathbf{x}_i^T \mid (\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F) = \mathbf{X}_{i-1}) \leq e^{\varepsilon/K} \Pr(\mathcal{E}_i(F') = \mathbf{x}_i^T \mid (\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F') = \mathbf{X}_{i-1})$$

by the guarantee of PFTAL. Note that we are referring to the PMF and not the CDF of the distribution. This is because PFTAL utilizes Gaussian noise. With this, for $\mathbf{X}_i = (\mathbf{x}_i^T, \ldots, \mathbf{x}_0^T)$ we have,

$$\begin{aligned}
&\Pr((\mathcal{E}_i, \ldots, \mathcal{E}_0)(F) = \mathbf{X}_i) \\
&= \Pr(\mathcal{E}_i(F) = \mathbf{x}_i^T \mid (\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F) = \mathbf{X}_{i-1}) \Pr((\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F) = \mathbf{X}_{i-1}) \\
&\leq e^{\varepsilon/K} \Pr(\mathcal{E}_i(F') = \mathbf{x}_i^T \mid (\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F') = \mathbf{X}_{i-1}) \cdot e^{i\varepsilon/K} \Pr((\mathcal{E}_{i-1}, \ldots, \mathcal{E}_0)(F') = \mathbf{X}_{i-1}),
\end{aligned}$$

where we utilized induction and the previous inequality. This completes the proof. ☐

Proof of Lemma 5 Let $G = \sup_{\substack{t=1,\ldots,T \\ \mathbf{x}\in\mathcal{X}}} \|\nabla f_t(\mathbf{x})\|_2$. Let r_i be the regret experienced by algorithm $\mathcal{E}_i$ in Algorithm 4.

The following result appears in the proof of Theorem 1 in Chen et al. (2018b).

Lemma 7 (Chen et al. (2018b))**.** *Assume f_t is monotone DR-submodular and β-smooth for every t. Then Algorithm 4 ensures*

$$\left(1-\frac{1}{e}\right)\max_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) - \sum_{t=1}^T f_t(\mathbf{x}_t) \leq \frac{1}{K}\sum_{i=0}^{K-1} r_i + \frac{\beta R^2 T}{2K}.$$

where $R = \sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\|_2$ and r_i is the regret of algorithm $\mathcal{E}_i$.

Using this result, we obtain

$$\mathbb{E}\left[\left(1-\frac{1}{e}\right)\max_{\mathbf{x}\in\mathcal{X}}\sum_{t=1}^{T}f_t(\mathbf{x})-\sum_{t=1}^{T}f_t(\mathbf{x}_t)\right]\leq\frac{1}{K}\sum_{i=0}^{K-1}\mathbb{E}[r_i]+\frac{\beta R^2}{2K}$$
$$\leq\mathcal{O}\left(\frac{\sqrt{n\log^{2.5}T}\left(G+\sqrt{\frac{n\log^{2.5}T}{\varepsilon T/K}}\operatorname{diam}\mathcal{X}\right)^2}{\varepsilon/K}\sqrt{T}+\frac{\beta R^2 T}{2K}\right).$$

We can find the regret by setting $K=\left(\frac{T}{\log^{2.5}T}\right)^{1/4}$.