Supplementary Materials for "Regret-Optimal Filtering"

The main objective of the Supplementary Materials file is to provide detailed derivation and proofs for the main results in the paper. It has three main sections. In the first section, we present the solution to the general Nehari problem. The second section includes the statements and the proofs of the technical lemmas that are summarized in the main body of the paper. The last section utilizes the technical lemmas to prove the main theorems of Section 3 in the main body.

1 The Nehari problem

We start by introducing the solution to the Nehari problem. The problem and its solution are well-known in many cases, e.g., Zhou, Glover and Doyle (1999). The following explicit solution for discrete-time state-space is a simplification of the central solution found in Chapter 12 of Hassibi, Sayed and Kailath (1999). The simplification was derived by (Sabag, 2021).

Theorem 1 (Solution to the general Nehari problem)**.** *Consider the Nehari problem with $T(z) = H(z^{-1}I - F)^{-1}G$ in a minimal form and stable F. The optimal norm is given by*

$$\min_{\text{causal and bounded } L(z)} \|L(z) - T(z)\| = \bar{\sigma}(Z\Pi), \tag{1}$$

where Z and Π are the unique solutions to the Lyapunov equations

$$\begin{aligned} Z &= F^* Z F + H^* H \\ \Pi &= F \Pi F^* + G G^*. \end{aligned} \tag{2}$$

Moreover, for any $\gamma \geq \sqrt{\bar{\sigma}(Z\Pi)}$, a γ-optimal solution to (1) *is given by*

$$L(z) = H\Pi(I + F_\gamma (zI - F_\gamma)^{-1}) K_\gamma, \tag{3}$$

with

$$\begin{aligned} K_\gamma &= (I - F^* Z_\gamma F \Pi)^{-1} F^* Z_\gamma G \\ F_\gamma &= F^* - K_\gamma G^*, \end{aligned} \tag{4}$$

and Z_γ is the solution to the Lyapunov equation

$$Z_\gamma = F^* Z_\gamma F + \gamma^{-2} H^* H. \tag{5}$$

2 Statement and Proof of Technical Lemmas

For completeness, we state the technical lemmas that appeared in the main body of the paper.

Lemma 3. *The transfer function* $I + H(z)H^*(z^{-*})$ *can be factored as*

$$\Delta(z)\Delta^*(z^{-*}) = I + H(z)H^*(z^{-*})$$

with

$$\Delta(z) = (I + H(zI - F)^{-1}K_P)(I + HPH^*)^{1/2} \tag{6}$$

where $(I+HPH^*)^{1/2}(I+HPH^*)^{*/2} = I+HPH^*$, $K_P = FPH^*(I+HPH^*)^{-1}$ *and* P *is the stabilizing solution to the Riccati equation*

$$P = GG^* + FPF^* - FPH^*(I + HPH^*)^{-1}HPF^*.$$

Moreover, the transfer function $\Delta^{-1}(z)$ *is bounded on* $|z| \geq 1$*.*

Proof of Lemma 3. This is a standard factorization, e.g., Chapter 12 in Hassibi et al. (1999). The main idea is to introduce an auxiliary matrix P and to choose it such that the product reduced its rank.

Consider the term $I + H(z)H^*(z^{-*}) = I + H(zI - F)^{-1}GG^*(z^{-1}I - F^*)^{-1}H^*$ in a mtrix form

$$\begin{aligned} &\begin{pmatrix} H(zI-F)^{-1} & I \end{pmatrix} \begin{pmatrix} GG^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{pmatrix} \\ &= \begin{pmatrix} H(zI-F)^{-1} & I \end{pmatrix} \begin{pmatrix} I & \Psi(P) \\ 0 & I \end{pmatrix} \begin{pmatrix} \Gamma(P) & 0 \\ 0 & I + HPH^* \end{pmatrix} \begin{pmatrix} I & 0 \\ \Psi^*(P) & I \end{pmatrix} \begin{pmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{pmatrix}, \end{aligned} \tag{7}$$

where the equality holds for any Hermitian matrix P and

$$\begin{aligned} \Psi(P) &= FPH^*(I + HPH^*)^{-1} \\ \Gamma(P) &= GG^* - P + FPF^* - FPH^*(I + HPH^*)^{-1}HPF^*. \end{aligned}$$

We now choose P to be the stabilizing solution such that the Riccati equation is $\Gamma(P) = 0$, and define $K_P \triangleq \Psi(P)$. Thus, we have

$$\Delta(z) = (I + H(zI - F)^{-1}K_P)(I + HPH^*)^{1/2}. \tag{8}$$

One can see that the inverse is

$$\Delta^{-1}(z) = (I + HPH^*)^{-1/2}(I - H(zI - F_P)^{-1}K_P), \tag{9}$$

and by the detectability of (F, H), the spectral radius of P is strictly smaller than 1. □

In the second factorization, the expression we aim to factor is positive but the order of its causal and anticausal components are in the reversed order. This is resolved with an additional Riccati equation.

Lemma 4. *For any* $\gamma > 0$*, the factorization* $\nabla_\gamma^*(z^{-*})\nabla_\gamma(z) = \gamma^{-2}(I + \gamma^{-2}L(z)(I + H^*(z^{-*})H(z))^{-1}L^*(z^{-*}))$ *holds with*

$$\nabla_\gamma(z) = R_Q^{-1/2}(I - L(zI - F_Q)^{-1}K_Q), \tag{10}$$

where $R_Q = R_Q^{1/2}R_Q^{*/2}$, Q *is a solution to the Riccati equation*

$$Q = -GR_W^{-1}G^* + F_W Q F_W^* - K_Q R_Q K_Q^*,$$

and $K_Q = F_W Q L^* R_Q^{-1}$ *and* $R_Q = \gamma^2 I + LQL^*$ *and the closed-loop system* $F_Q = F_W - K_Q L$*. The constants* (F_W, K_W) *are obtained from the solution* W *to the Riccati equation*

$$W = H^*H + L_\gamma^* L_\gamma + F^*WF - K_W^* R_W K_W, \tag{11}$$

and $K_W = R_W^{-1}G^*WF$ *and* $R_W = I + G^*WG$ *with* $R_W = R_W^{*/2}R_W^{1/2}$ *and* $F_W = F - GK_W$*.*

Proof of Lemma 4. Recall that need to perform the canonical factorization

$$\nabla^*(z^{-*})\nabla(z) = \gamma^{-2}(I + \gamma^{-2}L(z)(I + H^*(z^{-*})H(z))^{-1}L^*(z^{-*})) \tag{12}$$

The transfer matrices are in the reversed order, so we invert both sides of the equation as

$$\nabla_\gamma^{-1}(z)\nabla_\gamma^{-*}(z^{-*}) = \gamma^2 I - L(z)(I + H^*(z^{-*})H(z) + \gamma^{-2}L^*(z^{-*})L(z))^{-1}L^*(z^{-*})).$$

The factorization will be done in two steps. First, we will factorize the inner expression:

$$\Gamma^*(z^{-*})\Gamma(z) = I + H^*(z^{-*})H(z) + \gamma^{-2}L^*(z^{-*})L(z). \tag{13}$$

Then, the proof will be completed by factorizing

$$\nabla_\gamma^{-1}(z)\nabla_\gamma^{-*}(z^{-*}) = \gamma^2 I - L(z)\Gamma^{-1}(z)\Gamma^{-*}(z^{-*})L^*(z^{-*}). \tag{14}$$

The inner factorization can be written as

$$\begin{aligned}\Gamma^*(z^{-*})\Gamma(z) &= I + H^*(z^{-*})H(z) + L_\gamma^*(z^{-*})L_\gamma(z) \\ &= I + G^*(z^{-1}I - F^*)^{-1}(H^*H + L_\gamma^*L_\gamma)(zI - F)^{-1}G.\end{aligned} \tag{15}$$

Using the same argument as in Lemma 3, one can show

$$\Gamma(z) = R_W^{1/2}(I + K_W(zI - F)^{-1}G) \tag{16}$$

where W is the unique solution for

$$W = H^*H + L_\gamma^*L_\gamma + F^*WF - F^*WG(I + G^*WG)^{-1}G^*WF,$$

$K_W = (I + G^*WG)^{-1}G^*WF$ and $R_W = (I + G^*WG)$ with $R_W = R_W^{*/2}R_W^{1/2}$ and $F_W = F - GK_W$.
Before the second factorization, let us simplify the transfer function

$$\begin{aligned}L(z)\Gamma^{-1}(z) &= L(zI - F)^{-1}[(I + GK_W(zI - F)^{-1})]^{-1}GR_W^{-1/2} \\ &= L(zI - F_W)^{-1}GR_W^{-1/2}.\end{aligned} \tag{17}$$

We can now perform the second factorization as

$$\begin{aligned}\nabla_\gamma^{-1}(z)\nabla_\gamma^{-*}(z^{-*}) &= \gamma^2 I - L(z)\Gamma^{-1}(z)\Gamma^{-*}(z^{-*})L^*(z^{-*}) \\ &= \gamma^2 I - L(zI - F_W)^{-1}GR_W^{-1}G^*(z^{-1}I - F_W^*)^{-1}L^* \\ &= (I + L(zI - F_W)^{-1}K_Q)R_Q(I + K_Q^*(z^{-1}I - F_W^*)^{-1}L^*),\end{aligned} \tag{18}$$

where Q is a solution to the Riccati equation

$$Q = -GR_W^{-1}G^* + F_WQF_W^* - F_WQL^*(\gamma^2 I + LQL^*)^{-1}LQF_W^*,$$

and $K_Q = F_WQL^*R_Q^{-1}$ and $R_Q = \gamma^2 I + LQL^*$ and the closed-loop system $F_Q = F_W - K_QL$.
Finally, by (18), we can write

$$\nabla_\gamma^{-1}(z) = (I + L(zI - F_W)^{-1}K_Q)R_Q^{1/2} \tag{19}$$

with $R_Q = R_Q^{1/2}R_Q^{*/2}$ and

$$\begin{aligned}\nabla_\gamma(z) &= R_Q^{-1/2}(I + L(zI - F_W)^{-1}K_Q)^{-1} \\ &= R_Q^{-1/2}(I - L(zI - F_Q)^{-1}K_Q).\end{aligned} \tag{20}$$

□

The following lemma is the required decomposition.

Lemma 5. *The product of the transfer matrices $\nabla_\gamma(z)L(z)H^*(z^{-*})\Delta^{-*}(z^{-*})$ can be written as the sum of an anticausal transfer function*

$$T(z) = R_Q^{-1/2}L(P-U)F_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2}. \tag{21}$$

and a causal transfer function

$$\begin{aligned} S(z) &= \nabla_\gamma(z)L[(zI-F)^{-1}F + I]PH^*(I+HPH^*)^{-*/2} \\ &\quad - R_Q^{-1/2}L[(zI-F_Q)^{-1}F_Q + I]UH^*(I+HPH^*)^{-*/2}, \end{aligned}$$

where U solves $U = K_Q LPF_P^* + F_Q UF_P^*$.

It can be shown that the first line of $S(z)$ is $\nabla_\gamma(z)K_{H_2}(z)\Delta(z)$ where $K_{H_2}(z)$ is the optimal H_2 estimator.

Lastly, we prove the decomposition.

Proof of Lemma 5. Let us write each function:

$$\begin{aligned} \nabla_\gamma(z) &= R_Q^{-1/2}(I - L(zI-F_Q)^{-1}K_Q) \\ L(z) &= L(zI-F)^{-1}G \\ H^*(z^{-*}) &= G^*(z^{-1}I - F^*)^{-1}H^* \\ \Delta^{-*}(z^{-*}) &= (I + K_P^*(z^{-1}I - F^*)^{-1}H^*)^{-1}(I+HPH^*)^{-*/2}. \end{aligned} \tag{22}$$

First, we simplify write

$$H^*(z^{-*})\Delta^{-*}(z^{-*}) = G^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2}. \tag{23}$$

Consider now the product $L(z)H^*(z^{-*})\Delta^{-*}(z^{-*})$:

$$\begin{aligned} L(z)H^*(z^{-*})\Delta^{-*}(z^{-*}) &= L(zI-F)^{-1}GG^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2} \\ &= L[(zI-F)^{-1}FP + PF_P^*(z^{-1}I - F_P^*)^{-1} + P] \\ &\quad \cdot H^*(I+HPH^*)^{-*/2}, \end{aligned} \tag{24}$$

where in the last equality we used a standard decomposition (e.g., Lemma 12.3.3in Hassibi et al. (1999)). Now, we can write the causal part of $L(z)H^*(z^{-*})\Delta^{-*}(z^{-*})$ as:

$$L[(zI-F)^{-1}F + I]PH^*(I+HPH^*)^{-*/2} \tag{25}$$

and its anticausal part as

$$LPF_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2}. \tag{26}$$

Note that $\nabla_\gamma(z)$ is causal and, therefore, all left to decompose is $\nabla_\gamma(z)$ multiplied (from right) with (26)

$$\begin{aligned} &R_Q^{-1/2}(I - L(zI-F_Q)^{-1}K_Q)LPF_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2} \\ &= R_Q^{-1/2}LPF_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2} - R_Q^{-1/2}L[(zI-F_Q)^{-1}F_QU + UF_P^*(z^{-1}I - F_P^*)^{-1} + U] \\ &\cdot H^*(I+HPH^*)^{-*/2} \\ &= R_Q^{-1/2}L(P-U)F_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2} - R_Q^{-1/2}L[(zI-F_Q)^{-1}F_Q + I]UH^*(I+HPH^*)^{-*/2}, \end{aligned} \tag{27}$$

where in the first equality, the decomposition is done with U that solves $U = K_Q LPF_P^* + F_Q UF_P^*$.

To summarize the derivation, the anticausal expression is

$$T(z) = R_Q^{-1/2}L(P-U)F_P^*(z^{-1}I - F_P^*)^{-1}H^*(I+HPH^*)^{-*/2}. \tag{28}$$

The causal transfer function is

$$\begin{aligned} S(z) &= \nabla_\gamma(z)L[(zI-F)^{-1}F + I]PH^*(I+HPH^*)^{-*/2} \\ &\quad - R_Q^{-1/2}L[(zI-F_Q)^{-1}F_Q + I]UH^*(I+HPH^*)^{-*/2} \end{aligned} \tag{29}$$

□

3 Proof of the Main results

We are now ready to prove the main results for the state-space setting.

Proof of Theorem 1. Applying Theorem 1 with the anticasusal operator $T(z)$ in Lemma 5 shows that the minimal norm is precisely $\bar{\sigma}(Z\Pi)$. Thus, in order to get a norm smaller than 1 (Theorem 1) is equivalent to $\bar{\sigma}(Z\Pi) \leq 1$. □

Proof of Lemma 2. The solution to the Nehari problem follows directly by applying Theorem 1 with $T(z)$ in (21). We then convert the results from the frequency-domain representation to the state-space in Lemma 2. □

Proof of Theorem 4. The proof follows directly from Theorem 1 with Lemma 3 that provides explicit characterization for $\Delta(z)$, Lemma 4 that provides explicit characterization for $\nabla(z)$ and Lemma 5 that provide explicit expressions for $S(z)$ and $T(z)$, i.e., the causal and strictly anticausal parts of the operator $\nabla_\gamma \mathcal{K}_0 \Delta$. With some abuse of notation, we write the term $S(z)$ in Theorem 4 without its first line in (22) and use the equation $K_{H_2}(z) = \nabla_\gamma^{-1}(z)(\nabla_\gamma(z) L[(zI-F)^{-1}F+I]PH^*(I+HPH^*)^{-*/2})\Delta^{-1}(z)$. □

Proof of Theorem 3. The proof is a simplification of the frequency-domain representation in Theorem 3. Recall that, by Theorem 3, the estimator is given by

$$K(z) = \nabla_\gamma^{-1}(z)(S(z) + K_N(z))\Delta^{-1}(z) + K_{H_2}(z). \tag{30}$$

with

$$\begin{aligned}
\Delta^{-1}(z) &= (I+HPH^*)^{-1/2}(I+H(zI-F)^{-1}K_P)^{-1} \\
&= (I+HPH^*)^{-1/2}(I-H(zI-F_P)^{-1}K_P) \\
\nabla_\gamma^{-1}(z) &= (I+L(zI-F_W)^{-1}K_Q)R_Q^{1/2} \\
S(z) &= -R_Q^{-1/2} L[(zI-F_Q)^{-1}F_Q+I]UH^*(I+HPH^*)^{-*/2}.
\end{aligned} \tag{31}$$

The H_2 estimator is

$$LPH^*(I+HPH^*)^{-1} + (L - LPH^*(I+HPH^*)^{-1}H)(zI-F_P)^{-1}K_P \tag{32}$$

Consider the product $\nabla_\gamma^{-1}(z)S(z)$

$$\begin{aligned}
&-(I+L(zI-F_W)^{-1}K_Q)LUH^*(I+HPH^*)^{-*/2} - L(I-(zI-F_Q)^{-1}K_Q L)^{-1}(zI-F_Q)^{-1}F_Q UH^*(I+HPH^*)^{-*/2} \\
&= -(I+L(zI-F_W)^{-1}K_Q)LUH^*(I+HPH^*)^{-*/2} - L(zI-F_W)^{-1}F_Q UH^*(I+HPH^*)^{-*/2} \\
&= -L(I+(zI-F_W)^{-1}F_W)UH^*(I+HPH^*)^{-*/2},
\end{aligned}$$

and multiplying the resulted term with $\Delta^{-1}(z)$ gives

$$-L(I+(zI-F_W)^{-1}F_W)UH^*(I+HPH^*)^{-1}(I-H(zI-F_P)^{-1}K_P). \tag{33}$$

The last term results in from the solution to the Nehari problem

$$\begin{aligned}
&\nabla^{-1}(z)\tilde{\Pi}(I+F_N(zI-F_N)^{-1})G_N\Delta^{-1}(z) \\
&= (I+L(zI-F_W)^{-1}K_Q)R_Q^{1/2}\tilde{\Pi}(I+F_N(zI-F_N)^{-1})\, G_N(I+HPH^*)^{-1/2}(I-H(zI-F_P)^{-1}K_P) \\
&= R_Q^{1/2}\tilde{\Pi}(I+F_N(zI-F_N)^{-1})G_N(I+HPH^*)^{-1/2}(I-H(zI-F_P)^{-1}K_P) \\
&+ L(zI-F_W)^{-1}K_Q R_Q^{1/2}\tilde{\Pi}(I+F_N(zI-F_N)^{-1})G_N(I+HPH^*)^{-1/2}(I-H(zI-F_P)^{-1}K_P)
\end{aligned} \tag{34}$$

It can now be verified that the sum of (32)-(34) can be compactly expressed as $K(z) = \tilde{H}(zI-\tilde{F})^{-1}\tilde{G} + \tilde{J}$ with the matrices given in Theorem 3.

□

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