A KNOWN RESULTS

Our proofs depend on the following result.

Lemma 9 (Lemma 11 of Abbasi-Yadkori et al. (2011))**.** Let $\{x_t\}_{t\in[T]}$ be any sequence such that $x_t \in \mathbb{R}^d$ and $\|x_t\|_2 \leq L$ for all $t \in [T]$. Let V be a positive definite matrix and $V_t = V + \sum_{s\in[t]} x_s x_s^\top$. Then, we have

$$\sum_{t\in[T]} \min\left(1, \|x_t\|^2_{V_{t-1}^{-1}}\right) \leq 2(d\log((\mathrm{trace}(V) + L^2T)/d) - \log\det(V)).$$

B MISSING PROOFS

B.1 Proof of Lemma 1

Proof of Lemma 1. We fix $t \in [T]$ and $s \geq 1$ arbitrarily. For all $t' \in \Psi_{t,s}$ we have $\|x_{t'}(i_{t'})\|_{V^{-1}_{t'-1,s}} > c^{-s}$ by definition of $i_{t'}$. Thus, from $c > 1$ and Lemma 9, we obtain

$$\begin{aligned}|\Psi_{t,s}|c^{-2s} &\leq \sum_{t'\in\Psi_{t,s}} \min\left(1, \|x_{t'}(i_{t'})\|^2_{V^{-1}_{t'-1,s}}\right)\\ &\leq 2d\log(1 + L^2|\Psi_{t,s}|/(d\lambda)).\end{aligned}$$

□

B.2 Proof of Lemma 3

To prove Lemma 3, we use the following concentration inequality.

Lemma 10. Let $\{\mathcal{F}_t\}_{t=1}^\infty$ be a filtration. Let $\{\eta_t\}_{t=1}^\infty$ be a real-valued stochastic process such that η_t is $\mathcal{F}_t$-measurable. Assume that η_t is conditionally R_t-sub-Gaussian for all t. Then, for any $t > 0$ and $a > 0$,

$$\mathbb{P}\left(\sum_{s\in[t]} \eta_s > a\right) \leq \exp\left(-\frac{a^2}{2\sum_{s\in[t]} R_s^2}\right).$$

Proof. Using Markov's inequality, for any $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{s\in[t]} \eta_s > a\right) &= \mathbb{P}\left(\exp\left(\lambda\sum_{s\in[t]} \eta_s > \exp(\lambda a)\right)\right)\\ &\leq \exp(-\lambda a)\mathbb{E}\left(\exp\left(\lambda\sum_{s\in[t]}\eta_s\right)\right).\end{aligned}$$

For the second term on the right-hand side, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda\sum_{s\in[t]}\eta_s\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\lambda\sum_{s\in[t]}\eta_s\right) \mid \mathcal{F}_{t-1}\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\prod_{s\in[t]}\exp\left(\lambda\eta_s\right) \mid \mathcal{F}_{t-1}\right]\right].\end{aligned}$$

Since η_s is measurable with respect to $\mathcal{F}_{t-1}$ for all $t > 0$ and $s \in [t-1]$, we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[\prod_{s\in[t]}\exp\left(\lambda\eta_s\right) \mid \mathcal{F}_{t-1}\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\lambda\eta_t\right) \mid \mathcal{F}_{t-1}\right]\prod_{s\in[t-1]}\exp\left(\lambda\eta_s\right)\right] \\ &\leq \exp\left(\lambda^2 R_t^2/2\right)\mathbb{E}\left[\prod_{s\in[t-1]}\exp\left(\lambda\eta_s\right)\right] \\ &\leq \exp\left(\lambda^2\sum_{s\in[t]}R_s^2/2\right).\end{aligned}$$

Thus, we obtain

$$\mathbb{P}\left(\sum_{s\in[t]}\eta_s > a\right) \leq \exp(-\lambda a)\exp\left(\lambda^2\sum_{s\in[t]}R_s^2/2\right).$$

Choosing $\lambda = a/\sum_{s\in[t]}R_t^2$, we have the desired result. □

Proof of Lemma 3. Recall that $\tilde{\theta}_{t,s} = V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}(\theta^\top x_\tau(i_\tau) + \eta_\tau)x_\tau(i_\tau)$. We arbitrarily fix $s \in [S]$, $t \in [T]$, and $i \in I_{t,s}$. From the definition of $\tilde{\theta}_{t,s}$, we have

$$\begin{aligned}(\tilde{\theta}_{t,s} - \theta)^\top x_t(i) &= \left(V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}(\theta^\top x_\tau(i_\tau) + \eta_\tau)x_\tau(i_\tau) - \theta\right)^\top x_t(i) \\ &= x_t(i)^\top V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}\eta_\tau x_\tau(i_\tau) + x_t(i)^\top V_{t-1,s}^{-1}\left(\sum_{\tau\in\Psi_{t,s}}x_\tau(i_\tau)x_\tau(i_\tau)^\top - V_{t-1,s}^{-1}\right)\theta \\ &= x_t(i)^\top V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}\eta_\tau x_\tau(i_\tau) - \lambda x_t(i)^\top V_{t-1,s}^{-1}\theta. \qquad (10)\end{aligned}$$

Let $\alpha = R\sqrt{2\log(2/\delta)}$. For the first term on the right-hand side of (10), from Lemma 14 of Auer (2002) and Lemma 10, we have

$$\begin{aligned}&\mathbb{P}\left(\left|x_t(i)^\top V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}\eta_\tau x_\tau(i_\tau)\right| > \alpha\|x_t(i)\|_{V_{t-1,s}^{-1}}\right) \\ &= \mathbb{P}\left(\left|\sum_{\tau\in\Psi_{t,s}}x_t(i)^\top V_{t-1,s}^{-1}x_\tau(i_\tau)\eta_\tau\right| > \alpha\|x_t(i)\|_{V_{t-1,s}^{-1}}\right) \\ &\leq 2\exp\left(-\frac{\alpha^2\|x_t(i)\|_{V_{t-1,s}^{-1}}^2}{2R^2\sum_{\tau\in\Psi_{t,s}}(x_t(i)^\top V_{t-1,s}^{-1}x_\tau(i_\tau))^2}\right) \\ &= 2\exp\left(-\frac{\alpha^2\|x_t(i)\|_{V_{t-1,s}^{-1}}^2}{2R^2 x_t(i)^\top V_{t-1,s}^{-1}(\sum_{\tau\in\Psi_{t,s}}x_\tau(i_\tau)x_\tau(i_\tau)^\top)V_{t-1,s}^{-1}x_t(i)^\top}\right) \\ &\leq 2\exp\left(-\frac{\alpha^2}{2R^2}\right) \\ &= \delta.\end{aligned}$$

Thus, replacing δ with $\delta/(KST)$, we have

$$\left|x_t(i)^\top V_{t-1,s}^{-1}\sum_{\tau\in\Psi_{t,s}}\eta_\tau x_\tau(i_\tau)\right| < R\sqrt{2\log(2KST/\delta)}\|x_t(i)\|_{V_{t-1,s}^{-1}}$$

with probability at least $1 - \delta/(KST)$. For the second term on the right-hand side of (3), we have

$$\begin{aligned}\lambda x_t(i)^\top V_{t-1,s}^{-1}\theta &\leq \lambda \|\theta\|_{V_{t-1,s}^{-1}} \|x_t(i)\|_{V_{t-1,s}^{-1}} \\ &\leq \sqrt{\lambda}\|\theta\|_2 \|x_t(i)\|_{V_{t-1,s}^{-1}} \\ &\leq \sqrt{\lambda} M \|x_t(i)\|_{V_{t-1,s}^{-1}}.\end{aligned}$$

□

B.3 Proof of Lemma 7

Proof of Lemma 7. We arbitrarily fix $t \in \Psi_0$ and $i \in I_{t,s_t}$. By the same line of calculation in the proof for Lemma 2, we obtain

$$\begin{aligned}|(\hat{\theta}_{t,s} - \theta)^\top x_t(i)| &\leq \left| \left(V_{t-1,s_t}^{-1} \sum_{\tau \in \Psi_0} r_\tau(i_\tau) x_\tau(i_\tau) - \theta \right)^\top x_t(i) \right| \\ &\leq \left| \left(V_{t-1,s_t}^{-1} \sum_{\tau \in \Psi_0} (\theta^\top x_\tau(i_\tau) + \eta_\tau) x_\tau(i_\tau) - \theta \right)^\top x_t(i) \right| \qquad (11) \\ &+ \left| \left(V_{t-1,s_t}^{-1} \sum_{\tau \in \Psi_0} \varepsilon_\tau(i_\tau) x_\tau(i_\tau) \right)^\top x_t(i) \right|. \qquad (12)\end{aligned}$$

Applying Lemma 3 to the term (11), we have

$$\left| \left(V_{t-1,s_t}^{-1} \sum_{\tau \in \Psi_0} (\theta^\top x_\tau(i_\tau) + \eta_\tau) x_\tau(i_\tau) - \theta \right)^\top x_t(i) \right| \leq \beta_t(\delta) \|x_t(i)\|_{V_{t-1,s_t}^{-1}}$$

with probability at least $1 - \delta/(KST)$. Taking the union bound over the rounds and arms, the above inequality holds with probability at least $1 - \delta/S$ for all $t \in \Psi_0$ and $i \in I_{t,s_t}$. For the term (12), from the same line of calculation in the proof for Lemma 2, we have

$$\left| \left(V_{t-1,s_t}^{-1} \sum_{\tau \in \Psi_0} \varepsilon_\tau(i_\tau) x_\tau(i_\tau) \right)^\top x_t(i) \right| \leq \varepsilon \sqrt{|\Psi_0| x_t(i)^\top V_{t-1,s_t}^{-1} x_t(i)}.$$

From the definition of Ψ_0, we have $\|x_t(i)\|_{V_{t-1,s_t}^{-1}} \leq \sqrt{d/T}$. Since $|\Psi_0| \leq T$, we have

$$\varepsilon \sqrt{|\Psi_0| x_t(i)^\top V_{t-1,s_t}^{-1} x_t(i)} \leq \varepsilon \sqrt{d}.$$

□

B.4 Proof of Lemma 8

Proof of Lemma 8. We arbitrarily fix $t \in \Psi_0$. From Assumption 2, we have

$$\mu_t(i_{t,s_t}^*) - \mu_t(i_t) \leq \theta^\top (x_t(i_{t,s_t}^*) - x_t(i_t)) + 2\varepsilon.$$

Using Lemma 7, we have

$$\theta^\top (x_t(i_{t,s_t}^*) - x_t(i_t)) + 2\varepsilon \leq \hat{\theta}_{t,s_t}^\top x_t(i_{t,s_t}^*) + \beta(\delta) \|x_t(i_{t,s_t}^*)\|_{V_{t-1,s_t}^{-1}} + \varepsilon\sqrt{d} - \theta^\top x_t(i_t) + 2\varepsilon.$$

From the fact that $i_t \in \operatorname{argmax}_{i \in I_{t,s_t}} (\hat{r}_{t,s}(i) + w_{t,s}(i))$, we obtain

$$\hat{\theta}_{t,s_t}^\top x_t(i_{t,s_t}^*) + \beta(\delta) \|x_t(i_{t,s_t}^*)\|_{V_{t-1,s_t}^{-1}} \leq \hat{\theta}_{t,s_t}^\top x_t(i_t) + \beta(\delta) \|x_t(i_t)\|_{V_{t-1,s_t}^{-1}}.$$

Since $\|x_t(i_t)\|_{V_{t-1,s_t}^{-1}} \leq \sqrt{d/T}$, we have

$$\hat{\theta}_{t,s_t}^\top x_t(i_t) + \beta(\delta)\|x_t(i_t)\|_{V_{t-1,s_t}^{-1}} \leq \hat{\theta}_{t,s_t}^\top x_t(i_t) + \beta(\delta)\sqrt{d/T}.$$

Therefore, we obtain

$$\begin{aligned}
&\hat{\theta}_{t,s_t}^\top x_t(i_{t,s_t}^*) + \beta(\delta)\|x_t(i_{t,s_t}^*)\|_{V_{t-1,s_t}^{-1}} + \varepsilon\sqrt{d} - \theta^\top x_t(i_t) + 2\varepsilon \\
&\leq (\hat{\theta}_{t,s_t} - \theta)^\top x_t(i_t) + \beta(\delta)\sqrt{d/T} + \varepsilon(2 + \sqrt{d}).
\end{aligned}$$

Using Lemma 7 again, we have

$$(\hat{\theta}_{t,s_t} - \theta)^\top x_t(i_t) + \beta(\delta)\sqrt{d/T} + \varepsilon(2 + \sqrt{d}) \leq 2\beta(\delta)\sqrt{d/T} + 2\varepsilon(1 + \sqrt{d}).$$

$\square$