Learning Shared Subgraphs in Ising Model Pairs

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Abstract

Probabilistic graphical models (PGMs) are effective for capturing the statistical dependencies in stochastic databases. In many domains (e.g., working with multimodal data), one faces multiple information layers that can be modeled by structurally similar PGMs. While learning the structures of PGMs in isolation is well-investigated, the algorithmic design and performance limits of learning from multiple coupled PGMs are not well-investigated. This paper considers learning the structural similarities shared by a pair of Ising PGMs. The objective is to learn only the shared structure with no regard for the structures exclusive to either of the graphs. This is significantly different from the existing approaches that focus on learning the entire structures of the graphs. This paper proposes an algorithm for learning the shared structure, evaluates its performance empirically and analytically, and compares the performance with that of the existing approaches.

1 INTRODUCTION

Probabilistic graphical models (PGMs) are commonly used for capturing the conditional interdependence in probabilistic databases or random fields (Lauritzen, 1996; Pearl, 2009). Each vertex in a PGM represents a random variable (RV), and the edges encode the interdependence among the RVs. The complete structure of a PGM is captured by the joint probability measure of all the random variables involved. PGMs offer a rich context for designing effective and tractable algorithms for various inferential and decision-making objectives in a broad range of technological, social, or biological networks. Some examples include computer vision (Won and Derin, 1992), genetics (Chen et al., 2013; Fang et al., 2016; Dobra et al., 2004), and social networks (Jacob et al., 2014).

There are various domains in which we face multiple layers of information networks. Specifically, due to the proliferation of sensing technologies, there is a growing trend of facing multimodal databases. Graphically modeling such databases, as a result, involves multiple PGMs, each corresponding to one data mode. For instance, consider localizing anomalous events in networks. When a network's behavior undergoes abnormal changes, localizing the changes requires determining the discrepancies between the pre- and post-change network models. In another example, consider diffusion tensor imaging (DTI) and functional magnetic resonance imaging (fMRI) for brain imaging. DTI and fMRI images of a brain represent different structures of the underlying brain network (Honey et al., 2007), and their conformity of the two images can be leveraged to assess a brain's cognitive health (Buckner and DiNicola, 2019). Finding the maximum common subgraph in graph modeling of molecular structures in biology is another application important for drug discovery (Okamoto, 2020).

In this paper, we focus on learning the structural similarities between a pair of PGMs. We emphasize that our focus is on learning *only the shared* subgraphs without any interest in the structures exclusive to the individual graphs. A naive approach to this problem is learning each of the graphs independently and then searching for similarities. This approach is highly inefficient since it learns the graphs *completely*, resulting in unnecessarily learning the exclusive structures of the graphs too. This compromises the efficiency (e.g., sample complexity), especially in large-scale graphical models that have relatively small shared structures.

Learning the structure of a single graph, while being NP-hard (Chickering, 1996) in general, becomes tractable under various restrictions on graph struc-

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tures (Yuan and Lin, 2007; Rothman et al., 2008; Ravikumar et al., 2010; Banerjee et al., 2008; Anandkumar et al., 2010; Klivans and Meka, 2017). Relevant studies to the scope of this paper are the structure learning algorithms for tree-structured Ising models in (Bresler et al., 2020); for loosely connected Ising models with correlation decay in (Wu et al., 2013); and for degree-bounded Ising models in (Anandkumar et al., 2010; Bresler, 2015; Vuffray et al., 2016; Klivans and Meka, 2017). A greedy learning approach for structure learning is also proposed in (Bresler, 2015). A convex optimization framework for structure learning is formalized in (Vuffray et al., 2016). An online learning-based algorithm driven by the principles of prediction with expert advice is proposed in (Klivans and Meka, 2017).

While there exists rich literature on structure learning of a single graph, achievable and fundamental limits of structure learning from a group of related graphs are not as well-investigated. These existing studies focus on joint learning of all graphs in their *entirety*. The existing studies include strategies for joint learning of multiple graphical models that leverage the side information about the similarities among different graphs. In (Chen et al., 2013), an empirical Bayes method is studied to identify interactions that are unique to each class of cancers and that are common across all classes. In (Fang et al., 2016; Guo et al., 2011; Danaher et al., 2014; Mohan et al., 2014; Yang et al., 2015), joint inference of Gaussian graphical models is studied using graphical Lasso-based algorithms. Joint estimation of the graph structures based on discrete data is studied in (Guo et al., 2015). Information-theoretic bounds on the sample complexity of joint learning of graphical models with their similarity as side information are studied in (Sihag and Tajer, 2019a,b).

In a relevant problem, there exist studies on directly estimating the difference between graphs. (Zhao et al., 2014; Liu et al., 2014) study this problem for undirected graphs, where (Wang et al., 2019; Ghoshal and Honorio, 2019) consider causal linear structural equation models. This literature focuses on sparse discrepancies between the graph pairs, i.e., the pair are overwhelmingly similar.

In contrast to the studies mentioned above on learning the entire structures of structurally similar graphs, and complementary to learning their differences, in this paper, our objective is to learn only the *common* edge structure between two graphs using their data samples. Our focus is on Ising models with similarly labeled vertices and freely distinct structures. The subgraphs' size and its location are unknown, and we propose an algorithm for determining the size, identifying the location, and learning the structure of a subgraph shared by two distinct Ising models. In the face of lack of any information about the shared structure's size or location, a direct approach to learning the shared structure involves learning both graphs entirely and then finding their alignment. This approach will inevitably learn the unshared parts, which translates into severe inefficiency when our objective is learning only the shared structure. To circumvent this, we devise an adaptive algorithm that progressively prunes the vertices deemed highly unlikely to belong to the shared subgraph. We evaluate the effectiveness of the proposed algorithm by experiments on synthesized datasets and compare the performance of joint shared structure learning with that achieved by the direct approach of learning the two graphs independently to determine their shared structure in different regimes of interest such as correlation decay (Anandkumar et al., 2010) and bounded model width (Klivans and Meka, 2017; Wu et al., 2019).

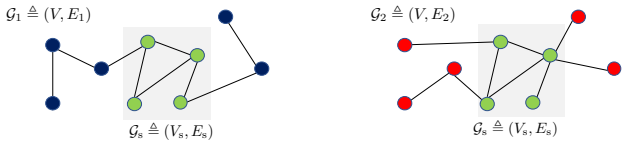


Figure 1: Green vertices and the edges between them represent the shared subgraph $\mathcal{G}_s$.

2 GRAPHICAL MODELS

Consider two distinct undirected graphs $\mathcal{G}_1 \triangleq (V, E_1)$ and $\mathcal{G}_2 \triangleq (V, E_2)$ over a set of vertices $V \triangleq \{1, \dots, p\}$. The graphs are formed by two distinct collections of undirected edges denoted by $E_1 \subseteq V \times V$ and $E_2 \subseteq V \times V$. When there exists an edge between vertices $u, v \in V$ in graph $\mathcal{G}_i$, we denote it by $(u, v) \in E_i$. The set of neighbors of vertex u in graph $\mathcal{G}_i$ is denoted by

$$\mathcal{N}_i(u) \triangleq \{w \in V : (u, w) \in E_i\} \ . \tag{1}$$

The graphs are degree-bounded, with their maximum degrees being upper bounded by $d \in \mathbb{N}$. We say that the edge (u, v) is shared by both graphs if $(u, v) \in E_i$ for $i \in \{1, 2\}$. The *unknown* set of edges that are shared by both graphs is denoted by $E_s = E_1 \cap E_2$. Accordingly, we define V_s as the set of vertices that the edges in E_s cover, and denote its size by $q \triangleq |V_s|$. Finally, the graph formed by the vertices in V_s and the edges in E_s is denoted by $\mathcal{G}_s \triangleq (V_s, E_s)$.

Ising model: We assume Ising models for both graphs, and define $X_i^u \in \mathcal{X} \triangleq \{-1, 1\}$ as the random variable associated with the vertex $u \in V$ in graph $\mathcal{G}_i$, and define the random vector $\mathbf{X}_i \triangleq [X_i^1, \dots, X_i^p]$ as the collection of the random variables in $\mathcal{G}_i$. We define

$\lambda \in \mathbb{R}^+$ to capture the dependence between the random variables associated with the vertices in a graph. The joint probability mass function (pmf) of random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$, denoted by $f(\mathbf{X}_1, \mathbf{X}_2)$, is given by

$$f(\mathbf{X}_1, \mathbf{X}_2) = \frac{1}{Z_{12}} \exp\left(\sum_{(u,v)\in E_{\mathrm{s}}} \lambda(X_1^u X_1^v + X_2^u X_2^v) + \sum_{(u,v)\in \tilde{E}_1} \lambda X_1^u X_1^v + \sum_{(u,v)\in \tilde{E}_2} \lambda X_2^u X_2^v \right), \tag{2}$$

where Z_{12} is the partition function that ensures $f(\mathbf{X}_1, \mathbf{X}_2)$ is a valid pmf, and we have defined $\tilde{E}_1 \triangleq E_1 \backslash E_{\mathrm{s}}$ and $\tilde{E}_2 \triangleq E_2 \backslash E_{\mathrm{s}}$. The class of graphs associated with $\mathcal{G}_{\mathrm{s}}$ is given by $\mathcal{I}_p^{\mathrm{s}}$ and it is formally defined as follows.

Definition 1 ($\mathcal{I}_p^{\mathrm{s}}$ class)**.** *Define $\mathcal{I}_p$ as the class of Ising models with p vertices, and define $\mathcal{I}_p^{\mathrm{s}} \subseteq \mathcal{I}_p$ as the class of all valid subgraphs $\mathcal{G}_{\mathrm{s}}$ that can be shared by two distinct Ising models with p vertices. A valid edge structure refers to any instance of a collection of edges that form E_{s}.*

Furthermore, we define $\mathcal{I}_p^{\mathrm{s}}(\mathcal{G}_{\mathrm{s}}) \subseteq \mathcal{I}_p \times \mathcal{I}_p$ as the class of all possible pairs of Ising models whose shared structure is given by $\mathcal{G}_{\mathrm{s}}$. We also denote the set of random variables associated with V_{s} in $\mathcal{G}_i$ by $\mathbf{X}_i^{\mathrm{s}}$ and those with $V \backslash V_{\mathrm{s}}$ by $\mathbf{X}_i^{\mathrm{c}}$. The marginal joint pmf of the random variables $\mathbf{X}_i^{\mathrm{s}}$ is given by

$$\begin{aligned}\tilde{f}(\mathbf{X}_1^{\mathrm{s}}, \mathbf{X}_2^{\mathrm{s}}) &\triangleq \frac{1}{|\mathcal{I}_p^{\mathrm{s}}(\mathcal{G}_{\mathrm{s}})|} \exp\left(\sum_{(u,v)\in E_{\mathrm{s}}} \lambda(X_1^u X_1^v + X_2^u X_2^v) \right) \\ &\times \left(\sum_{\mathbf{X}_1^{\mathrm{c}}, \mathbf{X}_2^{\mathrm{c}}} \sum_{(\tilde{E}_1, \tilde{E}_2)\in \mathcal{I}_p^{\mathrm{s}}(\mathcal{G}_{\mathrm{s}})} \frac{1}{Z_{\tilde{E}_1, \tilde{E}_2}} \right. \\ &\left. \times \exp\left(\sum_{(u,v)\in \tilde{E}_1} \lambda X_1^u X_1^v + \sum_{(u,v)\in \tilde{E}_2} \lambda X_2^u X_2^v \right) \right),\end{aligned} \tag{3}$$

where $Z_{\tilde{E}_1, \tilde{E}_2}$ is a partition function associated with pmf of the pair of Ising models with edge structures $\tilde{E}_1$ and $\tilde{E}_2$ unique to $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Clearly, finding a closed-form for $\tilde{f}(\mathbf{X}_1^{\mathrm{s}}, \mathbf{X}_2^{\mathrm{s}})$ is intractable in general (except for specific cases, such as connected $\mathcal{G}_{\mathrm{s}}$ in tree-structured graphs or isolated graph $\mathcal{G}_{\mathrm{s}}$) and performing marginal inference on Ising models is an open problem. We next provide the objectives and performance metrics for learning the structure of $\mathcal{G}_{\mathrm{s}}$.

3 OBJECTIVES AND METRICS

Graph decoding. In this section, we formalize the criterion for learning the structure of the shared subgraph and the attendant performance metrics. Our objective is to learn only the structure of $\mathcal{G}_{\mathrm{s}}$ based on a collection of n samples generated by each graph. The collection of n independent samples from $\mathcal{G}_i$ is denoted by $\mathbf{x}_i^n \in \mathcal{X}^{n\times p}$. We formally define the graph decoder to represent the algorithmic framework that uses samples from both graphs and estimates the shared structure.

Definition 2 (Graph Decoder)**.** *We define a graph decoder ψ_{s} as a function that maps the data samples to a subgraph in $\mathcal{I}_p^{\mathrm{s}}$, i.e.,*

$$\psi_{\mathrm{s}} : \mathcal{X}^{n\times p} \times \mathcal{X}^{n\times p} \to \mathcal{I}_p^{\mathrm{s}} \,. \tag{4}$$

Our objective is to *perfectly* learn the structure of the shared subgraph $\mathcal{G}_{\mathrm{s}}$. Hence, an error event occurs when $\psi(\mathbf{x}_1^n, \mathbf{x}_2^n) \neq E_{\mathrm{s}}$. In this paper, we characterize the graph decoder ψ by providing an adaptive algorithmic framework and empirically evaluating its performance. In order to specify relevant performance metrics for ψ in terms of number of samples, we first provide a brief overview of the algorithm with its details specified in Section 4.

Algorithm Overview. We design an adaptive algorithm based on the following premise. Given that the size, the location, and structure of the subgraph $\mathcal{G}_{\mathrm{s}} = (V_{\mathrm{s}}, E_{\mathrm{s}})$ are all unknown, an attempt for *directly* identifying it will inevitably require learning significant fractions of the structures of both graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, well beyond only their shared graphs, thus, potentially penalizing the efficiency (e.g., sample complexity). Motivated by this, we devise an algorithm that is focused on learning only $\mathcal{G}_{\mathrm{s}}$. We initialize our algorithm by considering all vertices in V as candidates for being in the shared subgraph. The set of vertices of interest at the k-th iteration of the algorithm is denoted by $\hat{V}_{\mathrm{s}}(k)$, where $\hat{V}_{\mathrm{s}}(0) = V$. At every iteration, we evaluate the empirical pairwise correlation between the data samples of the pair of vertices that could potentially be in $\mathcal{G}_{\mathrm{s}}$. If all pairwise empirical correlations for a vertex fall below a certain threshold, that vertex ceases to be of interest, and we stop collecting new data samples from it. In parallel to pairwise correlation-based pruning of the vertices of interest in the graphs, we also run an online structure learning algorithm with similar principles as adopted in (Klivans and Meka, 2017), but with appropriate modifications to facilitate joint learning using samples from two graphs.

Performance metrics. First, we collect n_{L} graph samples $\{(\mathbf{x}_1^k, \mathbf{x}_2^k) : k \in \{1, \ldots, n_{\mathrm{L}}\}\}$ one-at-a-time and

evaluate $\hat{V}_{\rm s}(k) \subseteq V$ and $(E_1^k \cap E_2^k)$ to form estimates of $\mathcal{G}_s = (V_{\rm s}, E_{\rm s})$. We denote the probability of error in recovering shared structures from a class of graphs $\mathcal{I}_p^{\rm s}$ by

$$\mathsf{P}_{\rm L}(\mathcal{I}_p^{\rm s}) \triangleq \max_{\mathcal{G}_1,\mathcal{G}_2 \in \mathcal{I}_p} \mathbb{P}(|E_{\rm s} \Delta \hat{E}_{\rm s}| \neq 0) \ , \tag{5}$$

where Δ denotes the difference between two edge sets, i.e., $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$. For a structure learning process, we evaluate the performance of our algorithm for the exact recovery of shared subgraph in two ways:

- The total number of algorithm iterations, denoted by $n_{\rm T}$.
- The total number of measurements collected over $n_{\rm T}$ iterations, i.e.,

$$\mathsf{N}(n_{\rm T}) \triangleq \sum_{k=1}^{n_{\rm T}} |\hat{V}_{\rm s}(k)| \ . \tag{6}$$

If we collect samples from all vertices in each iteration, we have $\mathsf{N}(n_{\rm T}) = pn_{\rm T}$. However, since the number of vertices in $\hat{V}_{\rm s}(k)$ potentially decreases over the iterations, we expect $\mathsf{N}(n_{\rm T})$ to be considerably smaller than $pn_{\rm T}$ for graph pairs with $V_{\rm s} \subset V$.

4 STRUCTURE LEARNING

In this section, we provide Algorithm 1 for shared structure learning, provide an insight into the logic of the included steps, and characterize its sample complexity under certain assumptions. Our algorithm mainly relies on two ideas: pruning the set of vertices of interest and jointly learning the edge structure spanned by vertices of interest in the two graphs.

4.1 Similarity-based Pruning

We focus on localizing the vertices in $V_{\rm s}$ during the structure learning, and hence, pruning the vertices that are highly likely not to be a part of the shared subgraph. These decisions are made in an online manner, and their objective is to form *coarse* decisions about the vertices that are promising candidates for being members of $V_{\rm s}$, and therefore, are retained for further scrutiny. In this part of the algorithm, as more data is collected, the sampling resources progressively shift to the vertices that are deemed to have a higher chance of belonging to the shared subgraph. This facilitates a significant reduction in the sampling complexity, as observed in the experiments.

For k samples from vertices u and v, we define the empirical pairwise mean of X_i^u and X_i^v as follows:

$$\bar{\mathbb{E}}_k(X_i^u X_i^v) \triangleq \frac{1}{k} \sum_{\ell=1}^{k} x_i^u(\ell) x_i^v(\ell) \ , \tag{7}$$

where $x_i^u(\ell)$ represents the ℓ-th data sample from vertex u in $\mathcal{G}_i$. We note that the pairwise correlation in (7) is a sufficient statistic for capturing the structure of Ising models in certain correlation-decay regimes (Bento and Montanari, 2011). Using the empirical pairwise correlations, we design the thresholding rule for controlling false negative rates. Specifically, at iteration k, we include vertices u, v in $\hat{V}_{\rm s}(k)$ if they meet the criterion:

$$\min_{i \in \{1,2\}} \bar{\mathbb{E}}_k[X_i^u X_i^v] \ > \ h(\lambda, \alpha, p, k) \ , \tag{8}$$

where we have defined

$$h(\lambda, \alpha, p, k) \triangleq \tanh \lambda - \sqrt{\frac{\alpha \log p}{2k}} \ , \tag{9}$$

and $\alpha > 2$ controls the rate of false negatives. The rationale for the above choice of $h(\lambda, \alpha, p, k)$ will become clear with the following lemma.

Lemma 1. *The pruning rule specified in* (8) *with k samples and setting $h(\lambda, \alpha, p, k)$ according to* (9) *ensures that $\mathbb{P}(V_{\rm s} \subseteq \hat{V}_{\rm s}(k)) \geq 1 - 2p^{2-\alpha}$.*

Proof. Note that for a vertex pair (u, v) that is connected via an edge in an Ising model, we have $\mathbb{E}[X^u X^v] \geq \tanh \lambda$ (Daskalakis et al., 2019). Using the Hoeffding's inequality (Hoeffding, 1994) to establish the the concentration bound around the true pairwise means, for any $\epsilon > 0$, we have

$$\mathbb{P}[|\bar{\mathbb{E}}_k[X_i^u X_i^v] - \mathbb{E}[X_i^u X_i^v]| \leq \epsilon] \geq 1 - \frac{2}{p^\alpha} \ , \tag{10}$$

for $k = \frac{\alpha \log p}{2\epsilon^2}$ number of samples. This ensures that by plugging $\epsilon = \sqrt{\alpha \log p / 2k}$ in (10) and using (9) for an edge (u, v), the relationship

$$\bar{\mathbb{E}}_k[X_i^u X_i^v] \geq \mathbb{E}[X_i^u X_i^v] - \sqrt{\frac{\alpha \log p}{2k}} \geq h(\lambda, \alpha, p, k) \ , \tag{11}$$

holds with a probability at least $1 - 2p^{-\alpha}$. Taking a union bound over all possible pairs in both graphs, (11) holds for all such pairs with probability at least $(1 - 2p^{2-\alpha})$. This implies that this decision rule (8) guarantees that $V_{\rm s} \subseteq \hat{V}_{\rm s}(k)$ with a probability not smaller than $1 - 2p^{2-\alpha}$. □

Algorithm 1 Learning Shared subgraph $\mathcal{G}_{\rm s}$

Input α, β, $\hat{V}_{\rm s}(0) = V$, $\hat{E}_{\rm s}(0) = \emptyset$, $n_{\rm L} = n_{\rm T} + n_{\rm M}$ pairs of data samples
Initialize $\kappa_i^{uv}(1) = 1/(|V|-1)$, $\tilde{\kappa}_i^{uv}(1) = 1/(|V|-1)$ and $w_i^{uv}(1) = 0$ for all $u \neq v$ and $i \in \{1, 2\}$
for a new pair of data sample $k \in \{1, \ldots, n_{\rm T}\}$ **do**
  for each pair $u, v \in \hat{V}_{\rm s}(k)$, $u \neq v$ **do**
    Calculate $\bar{\mathbb{E}}_k[X_1^u X_1^v]$ and $\bar{\mathbb{E}}_k[X_2^u X_2^v]$
    if (8) is satisfied **then**
      Add (u, v) to $\hat{E}_{\rm s}(k)$
    end if
    Update the weights $\tilde{\kappa}_i^{uv}(k+1) = \tilde{\kappa}_i^{uv}(k) \exp(\beta/2)$
  end for
  Pruning step: Construct $\hat{V}_{\rm s}(k) = \{u | \exists v, (u, v) \in \hat{E}_{\rm s}(k)\}$
  Get samples from $\hat{V}_{\rm s}(k)$ vertices on both graphs and compute the losses $\ell_i^{uv}(k)$
  Weight update step:
  for each pair $(u, v) \in \hat{V}_{\rm s}(k)$, $u \neq v$ **do**
    Update the weights jointly $\kappa_i^{uv}(k+1)$ according to (14)
  end for
  for each pair $(u, v) \notin \hat{V}_{\rm s}(k)$, $u \neq v$ **do**
    Update the weights $\kappa_i^{uv}(k+1) = \kappa_i^{uv}(k) \exp(\beta/2)$
  end for
  for each pair $u \neq v$ **do**
    Compute $w_i^{uv}(k+1)$ via normalizing $\kappa_i^{uv}(k+1)$
  end for
  Compute estimates E_1^k and E_2^k such that for every pair $u \neq v$, $(u, v) \in E_i^k$ if $w_i^{uv} \geq \lambda/2$
  Compute empirical risks ϵ_i^k using $n_{\rm M}$ samples
end for
Compute $\mathrm{m}_1 = \operatorname{argmin}_k \epsilon_1^k$ and $\mathrm{m}_2 = \operatorname{argmin}_k \epsilon_2^k$
return $\hat{E}_{\rm s} = E_1^{\mathrm{m}_1} \cap E_2^{\mathrm{m}_2}$

4.2 Prediction-guided Structure Learning

Given the coarse estimate $\hat{V}_{\rm s}(k)$ at the k-th iteration, we then focus on learning the internal graph structures of vertices in $\hat{V}_{\rm s}(k)$. For this purpose, we provide a prediction-guided learning algorithm where we leverage the data samples only from the vertices in $\hat{V}_{\rm s}(k)$. It is noteworthy that since the estimate $\hat{V}_{\rm s}(k)$ may be imperfect and the internal structure of $\hat{V}_{\rm s}(k)$ may have additional edges, the subgraphs spanned by these vertices in the two graphs are not necessarily the same.

To formalize the learning process, corresponding to each random variable $X_i^u \in \{-1, 1\}$, we define the Bernoulli random variable $B_i^u \triangleq \frac{1}{2}(1 - X_i^u)$ and instead of analyzing X_i^u we equivalently analyze B_i^u. Accordingly, we denote the the k-th realization of B_i^u by $b_i^u(k) = \frac{1}{2}(1 - x_i^u(k))$. The prediction-guided algorithm consists of two steps. Step 1 collects $n_{\rm T} < n_{\rm L}$ samples to form prediction-guided estimates for $E_{\rm s}$ at each iteration k. We note that since the estimate $\hat{V}_{\rm s}(k)$ becomes more accurate with an increasing number of samples, we need measurements from a gradually decreasing number of vertices. Once the predictions are formed, we use the remaining $n_{\rm M} \triangleq n_{\rm L} - n_{\rm T}$ samples to form a final estimate for $\mathcal{G}_{\rm s}$ in Step 2.

Step 1: Predicting $E_{\rm s}$. We start with the premise that any pair of vertices in $\hat{V}_{\rm s}(k)$ can be potential neighbors. Each vertex can act as an *expert* and predicts the value of its neighbors. In the k-th iteration, at vertex u in $\mathcal{G}_i$, we form a prediction for $b_i^u(k)$ by aggregating the predictions provided by other vertices in the graph for this vertex according to

$$\hat{b}_i^u(k) = \sigma\Big(\sum_{v \neq u} \kappa_i^{uv}(k) x_i^v(k) \Big), \quad \forall k \in \{1, \ldots, n_{\rm T}\}, \tag{12}$$

where $\{\kappa_i^{uv}\}$ are the weights to be selected properly and σ is the standard sigmoid function.

Loss function. To quantify the quality of the predictions, for every pair $u, v \in \hat{V}_{\rm s}$ we evaluate the pairwise loss function

$$\ell_i^{uv}(k) \triangleq \frac{1}{2}\Big(1 + [\hat{b}_i^u(k) - b_i^u(k)] x_i^v(k)\Big) . \tag{13}$$

Predictor Update. Note that $\hat{V}_{\rm s}(k)$ is formed with the decision rule (8). Hence, for all pairs $u, v \in \hat{V}_{\rm s}(k)$, we allow the transfer of loss functions between the graphs for updating the multiplicative weights

$$\kappa_i^{uv}(k+1) = \kappa_i^{uv}(k) \cdot \exp\left(\frac{\beta}{2} [\ell_1^{uv}(k) + \ell_2^{uv}(k)] \right) , \tag{14}$$

where β is an appropriately set hyperparameter, for instance $\beta = \log\Big(1/(1 + \sqrt{(\log p / n_{\rm T})})\Big)$ [1]. Otherwise, if $u, v \notin \hat{V}_{\rm s}(k)$, $\kappa_i^{uv}(k)$ is updated according to:

$$\kappa_i^{uv}(k+1) = \kappa_i^{uv}(k) \cdot \exp\left(\beta/2\right) . \tag{15}$$

Before continuing to the next step, we comment on the multiplicative updates in (14) which facilitate joint learning in our algorithm. We note that, independently learning each graph corresponds to the algorithm in (Klivans and Meka, 2017) that uses the following update for any pair $u, v \in V$:

$$\kappa_i^{uv}(k+1) = \kappa_i^{uv}(k) \cdot \exp\left(\beta \ell_i^{uv}(k)\right) . \tag{16}$$

[1] This choice allows us to leverage the standard Hedge algorithm regret in proof of Theorem 1.

If (u, v) lies in the shared subgraph $\mathcal{G}_{\mathrm{s}}$, then (14) corresponds to processing two graph samples at one step rather than processing 1 graph sample independently as in (16).

We illustrate the improvement of performance by using the mean of losses in the graphs for $u, v \in V_{\mathrm{s}}$ in joint structure learning over the algorithm for learning a single Ising model in (Klivans and Meka, 2017) in Section 5, and provide additional empirical results in Appendix D.

Normalizing weights κ_i^{uv}. Note that weight updates in (14) and (15) do not ensure that the maximum neighborhood weight of any vertex is bounded by λd (since d is the maximum degree). Therefore, we introduce normalized weights:

$$w_i^{uv}(k+1) = \frac{\lambda d \kappa_i^{uv}(k+1)}{\sum\limits_{x \neq u} \left(\kappa_i^{ux}(k+1) + \tilde{\kappa}_i^{ux}(k+1)\right)}, \quad (17)$$

where $\tilde{\kappa}_i^{uv}$ are pseudo-weights to handle vertices with a degree less than d. The theoretical justification for these pseudo-weights is discussed in Appendix A.

Predictions for E_{s}. Using the coefficients $\{w_i^{uv}(k) : k \in \{1, \ldots, n_{\mathrm{T}}\}\}$ we perform thresholding tests to estimate the internal structure of $\hat{V}_{\mathrm{s}}(k)$. Specifically, we form two estimates, one corresponding to each graph:

$$E_i^k \triangleq \left\{(u, v) \ : \ w_i^{uv}(k) \geq \frac{\lambda}{2}\right\}, \quad \forall i \in \{1, 2\}, \quad (18)$$

for $k \in \{1, \ldots, n_{\mathrm{T}}\}$. The processes described above continue sequentially until all the n_{T} samples are exhausted.

Step 2: Estimating E_{s}**.** Finally, to determine the correct subgraph $\mathcal{G}_{\mathrm{s}}$, we collect additional n_{M} samples for vertices in $\hat{V}_{\mathrm{s}}(n_{\mathrm{T}})$ and based on these, we assign a risk metric to each predicted set E_i^k according to:

$$\epsilon_i^k = \frac{1}{n_{\mathrm{M}}} \sum_{k=n_{\mathrm{T}}+1}^{n_{\mathrm{L}}} \sum_{u \in \hat{V}_{\mathrm{s}}(k)} \left[\sigma\left(\sum_{v \in E_i^k} -2 w_i^{uv}(k) x_i^v(k)\right) - b_i^u(k)\right]^2. \quad (19)$$

We select the predictions with the lowest empirical risks. By setting $m_i \triangleq \arg\min_k \epsilon_i^k$ for $i \in \{1, 2\}$, we form the final estimate as $\hat{E}_{\mathrm{s}} = E_1^m \cap E_2^m$.

4.3 Sample Complexity

We remark that in specific scenarios such as when $\mathcal{G}_{\mathrm{s}}$ is an isolated subgraph or in the asymptote of $\mathcal{G}_{\mathrm{s}} = \mathcal{G}_1 = \mathcal{G}_2$, the pmf $\tilde{f}$ becomes tractable and has closed-forms that conform to distributions for an Ising model, which facilitate theoretically characterizing the performance of the proposed algorithm. The sample complexity of the algorithm in a restricted setting is provided in the following theorem.

Theorem 1 (Learning the shared subgraph)**.** *When the graph $\mathcal{G}_{\mathrm{s}}$ is isolated and marginal distribution of $\mathcal{G}_{\mathrm{s}}$ admits Ising model distribution, the sample complexity of Algorithm 1 for ensuring $\mathsf{P}(\mathcal{I}_p^{\mathrm{s}}) \leq (1 - \frac{2}{\rho})(1 - 2p^{2-\alpha})$ is*

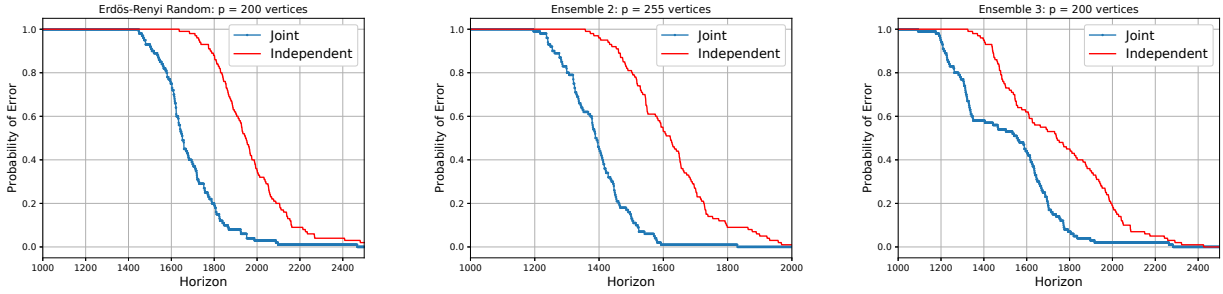
$$O\left(\frac{1}{\lambda^2} \exp(\lambda d) \log \frac{\rho q}{\lambda}\right) + O\left(\frac{1}{\lambda^2} \log p\right). \quad (20)$$

We remark that the condition on the marginal distribution of $\mathcal{G}_{\mathrm{s}}$ is too stringent in practical settings. Therefore, to emphasize upon the algorithm's applicability in wider settings, the rest of the paper focuses on empirical comparisons of Algorithm 1 with the existing structure learning algorithms in different regimes, such as correlation decay and bounded model width.

Remark 1. *The assumption that λ is uniform and known is made for clarity in presentation. It can be readily generalized to assume that (i) the edge weights are non-uniform, (ii) they are not fully known, and are only assumed to belong to a known range, i.e., $\lambda_{ij} \in \lambda \in [\lambda_{\min}, \lambda_{\max}]$. All analytical guarantees will still hold valid with proper adjustments. In Lemma 1, we replace λ with $\lambda_{\min}$ and in Theorem 1, we replace $\exp(\lambda d)$ with $\exp(\lambda_{\max} d)$ and other λ terms with $\lambda_{\min}$.*

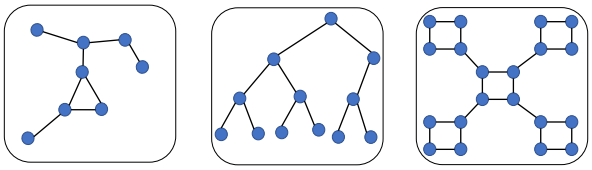
5 EXPERIMENTS

In this section, we evaluate the performance of our algorithm on synthetic data. Our experiments are evaluated on a broad set of graph structures to demonstrate their applicability beyond the stringent assumptions made in Theorem 1. We compare Algorithm 1 against several baseline approaches that are characterized by independently learning two graphs and identifying their shared structure afterwards. We consider three ensembles of graphs: (i) S_1: sparse Erdős-Rényi random graphs $\mathcal{G}(p, c/p)$; (ii) S_2: graphs with tree structures; and (iii) S_3: graphs with cyclic building blocks with degree $d \leq 3$. The figurative representations of these ensembles are shown in Fig. 3. For our experiments, we generate graph pairs with 200 vertices per graph for ensembles S_1 and S_3, and 255 vertices per graph for S_2. The edge structures are generated such that the subgraph E_{s} spans $q = 20$ vertices for the graph pairs in S_1 and S_3, and $q = 25$ for the graph pairs in S_2. The results are reported for 100 random realizations of the graph pairs.



(a) Ensemble 1 (b) Ensemble 2 (c) Ensemble 3

Figure 2: Probability of error P_{L} for recovering E_{s} with n_{L} iterations.



a. Ensemble 1 b. Ensemble 2 c. Ensemble 3

Figure 3: Representative examples of a graph in three ensembles used for analyzing the performance on simulated data.

Joint versus independent structure learning. First, we study the final probability of error P_{L} defined in (5) in graph pairs with bounded width. Note that an error may occur when either the pruning step of estimating $\hat{V}_{\mathrm{s}}(k)$ fails to include a vertex $u \in V_{\mathrm{s}}$ in $\hat{V}_{\mathrm{s}}$, or the shared structure learning part fails to recover the exact E_{s}. We noted in Section 4.2 that removing the pruning step and the update rule in (14) reduces our algorithm to the algorithm in (Klivans and Meka, 2017) with update rule in (16) for a single graph. To compare *joint versus independent learning*, we run the algorithm in (Klivans and Meka, 2017) to learn structures of two graphs separately and then evaluate the error in the shared subgraph of the estimates. Figure 2 illustrates this comparison in terms of the probability of error versus increasing horizon, where horizon refers to number of iterations. We observe that our algorithm significantly outperforms the baseline approach of (Klivans and Meka, 2017) in all three ensembles.

Comparison with correlation thresholding algorithms. Computing and using empirical pairwise means in our algorithm bear similarities to structure learning algorithms that work based on *correlation thresholding* in the *correlation decay* regimes. Thus, we compare our method's performance to that of the correlation thresholding (CT) Algorithm presented in (Anandkumar et al., 2010). In all ensembles, average degree and λ are chosen appropriately for the correlation decay regime.

In Fig. 4, we show this comparison by plotting the error with respect to the number of vertex measurements. Our joint learning of the shared subgraph outperforms the dual approach that uses CT algorithm independently on two graphs and identifies the shared structure at the end. We note that the slow start of our method is due to the initialization of the multiplicative weight updates in the algorithm. Since each vertex is initialized as a candidate neighbor for all the other vertices, it usually takes some training for weights to converge to the proper values. Eventually, we observe significant gains in terms of sample complexity measure $\mathsf{N}(n_{\mathrm{T}})$ defined in (6) as due to the adaptive aspect of our algorithm, the algorithm gradually focuses on sampling from the vertices that are more likely to be in $\mathcal{G}_{\mathrm{s}}$.

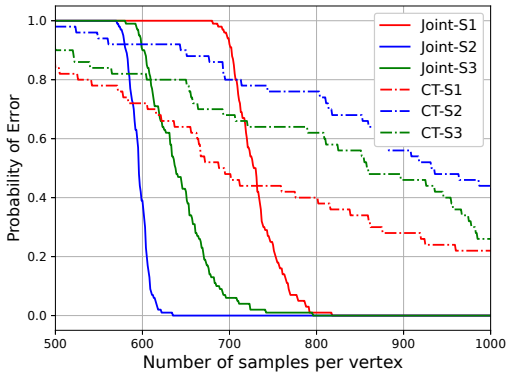


Figure 4: Joint shared subgraph learning and CT algorithm in (Anandkumar et al., 2010) with $\mathsf{N}(n_{\mathrm{T}})$ vertex samples.

Comparison with sparse logistic regression algorithms. (Wu et al., 2019) proposed an algorithm to learn the structure of a single graph by solving an ℓ_1-constrained logistic regression problem for the models with bounded width. We apply this algorithm to learn

two graphs independently and identify their shared structure afterward. To speed up the convergence of their mirror descent updates, we have modified this algorithm with stochastic mirror descent. We run the simulations on random graph pairs from $\mathcal{G}(p, 2/p)$ that has $|V_s| = p/10$ and $\lambda = 0.2$.

Figure 5 offers a number of observations. First, we note that time complexity of the algorithm in (Wu et al., 2019) scales with Tp^2, where T is the number of iterations for mirror descent updates and T scales with $1/\epsilon^4$, where ϵ is the bound for the norm of the error on parameter estimates (see (Wu et al., 2019) for details). This indicates that for a successful structure learning, time complexity grows quickly, especially in large graphs. To illustrate this effect, we run the algorithm of (Wu et al., 2019) for $\{1000, 1250, 1500, 2000, 3000\}$ samples with $T = 100$ (red dots) and $T = 250$ (green dots) iterations. We plot the number of samples for our algorithm both in terms of graph samples and vertex samples per graph, where the latter is significantly smaller. Notice that $\mathsf{N}(n_T)/p$ is plotted in Fig. 5 to illustrate the comparison with (Wu et al., 2019) that uses p vertex measurements for a graph sample.

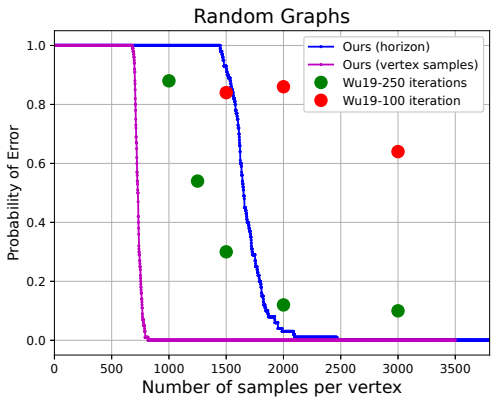


Figure 5: Comparison with (Wu et al., 2019) with respect to n_T and $\mathsf{N}(n_T)$.

Varying the edge parameter λ on sparse random graphs. Next, we study the performance for different edge parameters λ on random graphs $\mathcal{G}(p, 2/p)$. Note that the average degree is chosen to be small so that we will stay in the bounded width regime. Figure 6 shows that for moderately large enough values of λ, our algorithm can achieve a small probability of error. On the other hand, we note that sample complexity of our algorithm scales with $1/\lambda^2$ for a fixed model width. Hence, we expect a performance drop when λ becomes too small.

Varying the average degree of sparse random graphs. We note that our pruning decision rule in (8) is degree independent, and the multiplicative weight updates for structure learning use the maximum degree value only for normalizing the weights. More details can be bound in Appendix A. Next, we create random graphs $\mathcal{G}(p, c/p)$ with $p = 200$ vertices, $\lambda = 0.2$ and $c \in \{1, 2, 3, 4, 5\}$, where c denotes the average degree. Figure 7 shows that our algorithm recovers the shared structure with similar error rates for sparse random graphs with reasonably small average degree values.

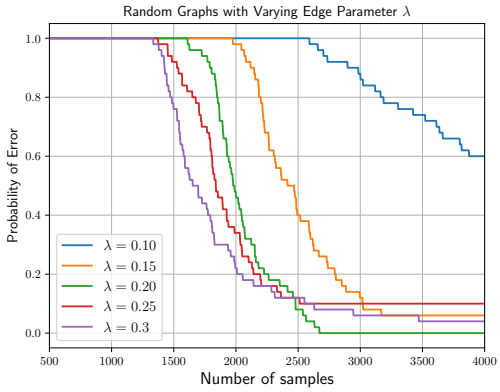


Figure 6: Performance of our method for various λ values.

Finally, we remark that our sample complexity is exponential in model width λd. For random graphs with large values of p, the degree of a vertex follows a Poisson distribution. Moreover, the maximum of these p random variables grows quickly (see (Briggs et al., 2009) for details). Hence, either model width becomes large and sample complexity grows exponentially, or λ happens to be too small to stay in the bounded width regime, which in turns scales the sample complexity with $1/\lambda^2$ and still requires a large number of samples. Therefore, we observe that our algorithm does not perform well for the exact recovery of the shared structure when the random graphs become denser (higher average degree).

Additional experiments. Evaluation of additional aspects of the algorithm, such as the effect of subgraph size, the effect of errors in pruning stage, comparison with a joint learning algorithm that aims to learn the graphs in their entirety, and an application for analysis of the U.S. Senate voting records data are provided in Appendix D.

6 CONCLUSION

The novel problem of learning the shared structure between two Ising models has been considered. An

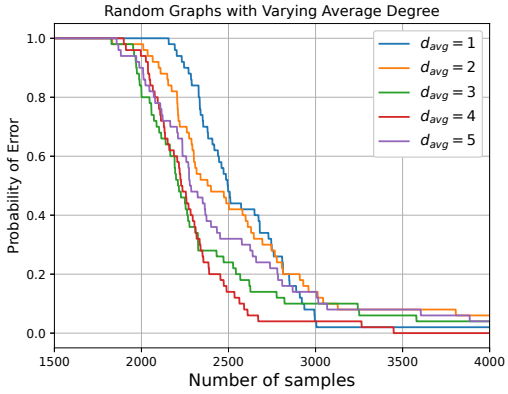


Figure 7: Performance of our method for various average degree values

algorithmic framework has been proposed, which, in contrast to the existing works, is focused on learning the structure of only the shared subgraph between the two graphs. The sample complexity of the framework has been characterized for specific scenarios. The performances of the framework have been numerically evaluated in various ensembles of graphs and shown to outperform naive approach of separately learning two graphs for three baseline approaches.

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