Supplementary Material for
LARNet: Lie Algebra Residual Network for Face Recognition

This supplementary material is used to explain the details of the theoretical proof in the third section *Methodology* of the paper: *LARNet: Lie Algebra Residual Network for Face Recognition*.

1. The existence of Lie algebra

Claim: Given any rotation matrix $\mathbf{R} \in \mathbb{R}^{3\times 3}$, $\exists\, \boldsymbol{\phi} \in \mathbb{R}^3$ and a skew-symmetric operator $^\wedge$, there exists an exponential mapping:

$$\mathbf{R} = \exp(\boldsymbol{\phi}^\wedge). \tag{1}$$

Proof: Without loss of generality, $\mathbf{R}$ is orthogonal and $\mathbf{R} \in SO(3)$, and we have $\mathbf{R} \cdot \mathbf{R}^\top = \mathbf{I}$. We assume that there is a continuous transformation from the frontal face to the profile face, so the time parameter t is introduced and we derive:

$$\begin{aligned} &\dot{\mathbf{R}}(t)\mathbf{R}(t)^\top + \mathbf{R}(t)\dot{\mathbf{R}}(t)^\top = 0, \\ &\dot{\mathbf{R}}(t)\mathbf{R}(t)^\top = -\mathbf{R}(t)\dot{\mathbf{R}}(t)^\top = -(\dot{\mathbf{R}}(t)\mathbf{R}(t)^\top)^\top. \end{aligned} \tag{2}$$

The above equation has a skew-symmetric form, denoted as:

$$\begin{aligned} &\dot{\mathbf{R}}(t)\mathbf{R}(t)^\top = \boldsymbol{\phi}(t)^\wedge, \\ &\dot{\mathbf{R}}(t) = \boldsymbol{\phi}(t)^\wedge \cdot \mathbf{R}(t), \end{aligned} \tag{3}$$

where $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3)^\top \in \mathbb{R}^3$ and its skew-symmetric matrix is:

$$\boldsymbol{\phi}^\wedge = \begin{bmatrix} 0 & -\boldsymbol{\phi}_3 & \boldsymbol{\phi}_2 \\ \boldsymbol{\phi}_3 & 0 & -\boldsymbol{\phi}_1 \\ -\boldsymbol{\phi}_2 & \boldsymbol{\phi}_1 & 0 \end{bmatrix}. \tag{4}$$

Therefore, taking the derivative once is equivalent to multiplying $\boldsymbol{\phi}^\wedge$ on the left side. And Eq. (3) is an ordinary differential equation with parameter variables:

$$\frac{d\mathbf{R}(t)}{dt} = \boldsymbol{\phi}(t)^\wedge \cdot \mathbf{R}(t),$$

$$\frac{d\mathbf{R}(t)}{\mathbf{R}(t)} = \boldsymbol{\phi}(t)^\wedge \cdot dt. \tag{5}$$

Integrating both sides of Eq. (5) leads to:

$$\begin{aligned} \int d\ln \mathbf{R}(t) &= \int \boldsymbol{\phi}(t)^\wedge dt + C, \\ \mathbf{R}(t) &= \exp^{\int \boldsymbol{\phi}(t)^\wedge dt + C}, \end{aligned} \tag{6}$$

where C is a constant, determined by the initial value. Based on Riemann integral, we have

$$\begin{aligned} \int \boldsymbol{\phi}(t)^\wedge dt \approx \sum_{i=0}^{n-1} \boldsymbol{\phi}(t')^\wedge \cdot (t_{i+1} - t_i),\ t' \in [t_i, t_{i+1}], \\ (0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = t) \end{aligned} \tag{7}$$

when there is a sufficiently small partition $T = \{t_i\}_{i=0}^n$ such that for any $\delta > 0$ and all i, there exists $|t_{i+1} - t_i| < \delta$, and then approximation $\approx$ can be equivalent $=$. In order to facilitate the calculation, we assume that the initial value $\mathbf{R}(0) = I$, and we have:

$$\begin{aligned} \mathbf{R}(t) &= \exp^{\sum_{i=0}^{n-1} \boldsymbol{\phi}(t')^\wedge \cdot (t_{i+1} - t_i)} \\ &= \lim_{\Delta t_i \to 0} \prod_{i=0}^{n-1} \exp^{\boldsymbol{\phi}(t')^\wedge \cdot \Delta t_i} \\ &= \prod_{t=0}^{n-1} \exp^{\boldsymbol{\phi}(t')^\wedge dt'}. \end{aligned} \tag{8}$$

For any t of rotation $\mathbf{R}(t)$, there exists another sufficiently small partition $S = \{s_i\}_{i=0}^n$ such that

$$\begin{aligned} \mathbf{R}(t) = \prod_{i=0}^{n-1} \mathbf{R}(s),\ s \in [s_i, s_{i+1}]. \\ (0 = s_0 < s_1 < s_2 < \ldots < s_{n-1} < s_n = t) \end{aligned} \tag{9}$$

Thus, we construct a new partition $M = T \bigcap S = \{m_i\}_{i=0}^n$, and for all $m \in [m_i, m_{i+1}]$, $\Delta m_i = |m_{i+1} - m_i| \to 0$, we have:

$$\mathbf{R}(m) = \exp(\boldsymbol{\phi}(m)^\wedge). \tag{10}$$

Referring to product integral theory (Antonin, 2007), including its Lebesgue type II-geometric integral (Michael & Katz, 1972), it provides a guarantee from discrete back to continuous: when $\Delta m_i \to 0$, we construct a new function $f(m)$ and let $(\exp(\boldsymbol{\phi}(m)^\wedge)^{dm} = 1 + f(m)dm$. Based on the properties of equivalent infinitesimal: $x \simeq \ln(1+x)$, the following equation holds:

$$\ln(\exp \boldsymbol{\phi}(m)^\wedge)dm = \ln(1 + f(m)dm) = f(m)dm.$$

Furthermore , we have the following equation holding true:

$$\begin{aligned}\prod_{i=0}^{n-1} \exp^{\boldsymbol{\phi}(m)^\wedge dm} &= \prod_{i=0}^{n-1}(1 + f(m)dm) \\ &= \exp(\int_0^\top f(t)dt) \\ &= \exp(\int_0^\top \ln(\exp^{\boldsymbol{\phi}(t')^\wedge})dt') \\ &= \exp(\int_0^\top \boldsymbol{\phi}(t')^\wedge dt').\end{aligned} \tag{11}$$

Hence, for a fixed observation moment t, this equation holds:

$$\mathbf{R}(t) = \exp(\boldsymbol{\phi}(t)^\wedge) = \exp(\int_0^\top \boldsymbol{\phi}(m)^\wedge dm). \tag{12}$$

□

2. The properties of Lie algebra

Every matrix Lie group is smooth manifolds and has a corresponding Lie algebra. Lie algebra consists of a vector space $\mathbb{G}$ expanded on the number field $\mathbb{F}$ and a binary operator, which is called as Lie bracket $[\cdot,\cdot]$ and defined by the cross product $[X,Y] = X \times Y$ when $\mathbb{G} = \mathbb{R}^3$ (McKenzie, 2015). Now we only need to check that $\boldsymbol{\phi}$ satisfies the four basic properties of Lie algebra:

1. Closure

$$[\boldsymbol{\phi}_1^\wedge, \boldsymbol{\phi}_2^\wedge] = \boldsymbol{\phi}_1^\wedge \boldsymbol{\phi}_2^\wedge - \boldsymbol{\phi}_2^\wedge \boldsymbol{\phi}_1^\wedge = (\underbrace{\boldsymbol{\phi}_1^\wedge \boldsymbol{\phi}_2}_{\in \mathbb{R}^3})^\wedge \in \mathfrak{so}(3).$$

2. Alternativity

$$[\boldsymbol{\phi}^\wedge, \boldsymbol{\phi}^\wedge] = \boldsymbol{\phi}^\wedge \cdot \boldsymbol{\phi}^\wedge - \boldsymbol{\phi}^\wedge \cdot \boldsymbol{\phi}^\wedge = 0 \in \mathfrak{so}(3).$$

3. Jacobi identity can be verified by substituting and applying the definition of Lie bracket.

4. Bilinearity follows directly from the fact that $(\cdot)^\wedge$ is a linear operator.

Informally, we will refer to $\mathfrak{so}(3)$ as the Lie algebra, although technically this is only the associated vector space. Furthermore, the derivative of rotation space is $\boldsymbol{\phi}$. From the above **Claim** and Eq. (3), we have $\dot{\mathbf{R}}(t) = \boldsymbol{\phi}^\wedge \cdot \mathbf{R}(t)$. And also setting $t_0 = 0$ and $\mathbf{R}(t_0) = I$, we perform the first-order Taylor expansion:

$$\mathbf{R} \approx \mathbf{R}(t_0) + \dot{\mathbf{R}}(t_0)(t - t_0) = \mathbf{I} + \boldsymbol{\phi}(t_0)^\wedge \cdot (t). \tag{13}$$

$\boldsymbol{\phi}$ reflects properties of the derivative of $\mathbf{R}$. Mathematically, we call it on the tangent space near the origin of $SO(3)$. □

3. The exponential mapping of Lie algebra

From the above section, we can clarify the composition of Lie algebra:

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\phi} \in \mathbb{R}^3 | \boldsymbol{\Gamma} = \boldsymbol{\phi}^\wedge \in \mathbb{R}^{3\times 3} \right\}. \tag{14}$$

Let perform Taylor expansion on it, but we note that the expansion can only be solved when it converges, and that the result is still a matrix:

$$\exp(\boldsymbol{\phi}^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{\phi}^\wedge)^n. \tag{15}$$

As mentioned in our main paper, vector $\boldsymbol{\phi}$ can be denoted as $\boldsymbol{\phi} = \theta\psi$. Using its properties: (odd power) $\psi^\wedge\psi^\wedge\psi^\wedge = -\psi^\wedge$ and (even power) $\psi^\wedge\psi^\wedge = \psi\psi^\top - I$, we have:

$$\begin{aligned}\exp(\boldsymbol{\phi}^\wedge) = \exp(\theta\psi^\wedge) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\theta\psi^\wedge)^n \\ &= \cos\theta \mathbf{I} + (1 - \cos\theta)\psi\psi^\top + \sin\theta\psi^\wedge.\end{aligned} \tag{16}$$

This formula is exactly the same as Rodriguez' rotation. Therefore, we can consider solving the rotation vector through the trace of the matrix:

$$\begin{aligned} tr(\mathbf{R}) &= tr(\cos\theta \mathbf{I} + (1-\cos\theta)\psi\psi^\top + \sin\theta\psi^\wedge) \\ &= \cos\theta tr(\mathbf{I}) + (1-\cos\theta)tr(\psi\psi^\top) + \sin\theta tr(\psi^\wedge) \\ &= 2\cos\theta + 1. \end{aligned} \tag{17}$$

By solving the above equation, it can be found that the exponential map is surjective. But there exist multiple $\mathfrak{so}(3)$ elements $\theta + 2k\pi, k \in \mathbb{Z}$ corresponding to the same SO(3). If the rotation angle θ is fixed at $[-\pi/2, +\pi/2]$, then the elements in the Lie group and Lie algebra have a one-to-one correspondence (bijection). This means that our Lie algebra can completely replace rotation and will not produce adversarial examples. Besides, for $\mathbf{R}\psi = \psi$, ψ is the eigenvector of $\mathbf{R}$, the corresponding eigenvalue is $\lambda = 1$, and it is very convenient to solve $\boldsymbol{\phi} = \theta\psi$. □

4. The optimization of Lie algebra

To compound two matrix exponentials, we use the *Baker-Campbell-Hausdorff* (BCH) formula (Wulf, 2002; Brian,

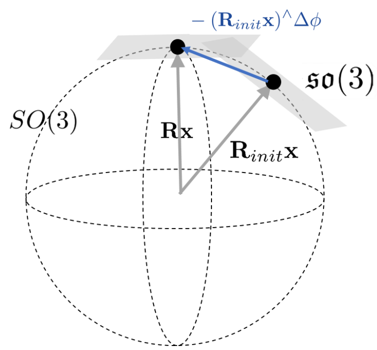

Figure 1. Nonlinear optimization of Lie algebra. During optimization, we keep our nominal rotation in the Lie group and consider a perturbation to take place in the Lie algebra, which is locally the tangent space of the group.

2015) and *Friedrichs'* theorem (Jacobson, 1966; Wilhelm, 1954) :

$$\ln(\exp(\mathbf{A})\exp(\mathbf{B})) = \mathbf{A} + \sum_{n=0}^{\infty}(-1)^n \frac{B_n}{n!} \underbrace{[\mathbf{A}, [\mathbf{A}, \ldots [\mathbf{A}, \mathbf{B}]}_{n} \ldots]], \tag{18}$$

where B_n are $Bernoulli\ numbers$ and the Lie bracket is the usual $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ for $\mathbf{A}, \mathbf{B} \in SO(3)$. Since the derivation can be seen as a change brought about by small increments, we have a more concise calculation model with vector representation using the approximate BCH :

$$\ln\left(\exp\left(\boldsymbol{\phi}_1^\wedge\right)\exp\left(\boldsymbol{\phi}_2^\wedge\right)\right)^\vee \approx \begin{cases} \mathbf{J}_\ell\left(\boldsymbol{\phi}_2\right)^{-1}\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 \\ \quad \text{if } \boldsymbol{\phi}_1 \text{ small} \\ \boldsymbol{\phi}_1 + \mathbf{J}_r\left(\boldsymbol{\phi}_1\right)^{-1}\boldsymbol{\phi}_2 \\ \quad \text{if } \boldsymbol{\phi}_2 \text{ small} \end{cases}. \tag{19}$$

$\mathbf{J}_\ell$ and $\mathbf{J}_r$ are referred to as the $left$ and $right\ Jacobians$ of $SO(3)$, respectively. Now for Lie algebra $\boldsymbol{\phi}$ and increment $\Delta\boldsymbol{\phi}$, we have:

$$\begin{aligned} \exp\left(\Delta\boldsymbol{\phi}^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right) &= \exp\left(\left(\boldsymbol{\phi} + \mathbf{J}_l(\boldsymbol{\phi})^{-1}\Delta\boldsymbol{\phi}\right)^\wedge\right), \\ \exp\left((\boldsymbol{\phi} + \Delta\boldsymbol{\phi})^\wedge\right) &= \exp\left((\mathbf{J}_l\Delta\boldsymbol{\phi})^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right) \\ &= \exp\left(\boldsymbol{\phi}^\wedge\right)\exp\left((\mathbf{J}_r\Delta\boldsymbol{\phi})^\wedge\right). \end{aligned} \tag{20}$$

Considering that the compound of rotation is left multiplication, we will work with the left increment and Jacobian. By comparing the derivative model on Lie algebra and the perturbation scheme on Lie group, we choose the latter for more conciseness, as follows:

$$\begin{aligned} \frac{\partial \mathbf{Rx}}{\partial \Delta\boldsymbol{\phi}} &= \lim_{\Delta\boldsymbol{\phi}\to 0} \frac{\exp\left(\Delta\boldsymbol{\phi}^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)\mathbf{x} - \exp\left(\boldsymbol{\phi}^\wedge\right)\mathbf{x}}{\Delta\boldsymbol{\phi}} \\ &= -(\mathbf{Rx})^\wedge, \end{aligned} \tag{21}$$

where $\mathbf{x} \in \mathbb{R}^3$ is an arbitrary three-dimensional point. When we take the product between rotation and a point with the perturbation scheme, we can get an approximation:

$$\mathbf{Rx} = \exp(\Delta\boldsymbol{\phi}^\wedge)\mathbf{R}_{init}\mathbf{x} \approx \mathbf{R}_{init}\mathbf{x} - (\mathbf{R}_{init}\mathbf{x})^\wedge\Delta\boldsymbol{\phi}. \tag{22}$$

This is depicted graphically in Fig. 1. The above formula has the form, which makes sense to nonlinear optimization like Gauss-Newton algorithm, and is adapted to work with the matrix Lie group by exploiting the surjective-only property of the exponential map. Furthermore, our scheme guarantees that it will iterate to convergence and $\mathbf{R}_{init} \in SO(3)$ at each iteration. □

5. Lie algebra of $SE(3)$

We have proven that Euclidean transformation itself still has the same properties as $SO(3)$ with a more complex form, and we give its structure as:

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} | \mathbf{R} \in SO(3), \mathbf{t} \in \mathbb{R}^3 \right\}. \tag{23}$$

Each transformation matrix has six degrees of freedom, so the corresponding Lie algebra is in $\mathbb{R}^6$, that is,

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\xi}^\wedge \in \mathbb{R}^{4\times 4} | \ \boldsymbol{\xi} \in \mathbb{R}^6 \right\}, \tag{24}$$

where $^\wedge$ is not a skew-symmetric operator and has a new definition:

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^\top & 0 \end{bmatrix}, \ \phi, \rho \in \mathbb{R}^3. \tag{25}$$

Same as Eq. (3), we also have an ordinary differential equation:

$$\dot{\mathbf{T}}(t) = \boldsymbol{\xi}^\wedge(t)\mathbf{T}(t). \tag{26}$$

Similarly, we can get its solution and the exponential mapping as:

$$\begin{aligned} \mathbf{T} &= \exp\left(\boldsymbol{\xi}^\wedge\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\boldsymbol{\xi}^\wedge\right)^n \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\phi^\wedge\right)^n & \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\phi^\wedge\right)^n\right)\rho \\ 0^\top & 1 \end{bmatrix}. \end{aligned} \tag{27}$$

Similarly, we can define addition operation, multiplication operation, and the perturbation model. Because it has nothing to do with our work, we won't delve into details here.

Finally, we give detailed calculation tables on the next page for Lie algebra, Lie group, and Jacobian. All our proofs and definitions are made up of the equations in the tables.

$SO(3)$ Identities and Approximations

Lie Algebra

$$\mathbf{u}^\wedge = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$$(\alpha\mathbf{u} + \beta\mathbf{v})^\wedge \equiv \alpha\mathbf{u}^\wedge + \beta\mathbf{v}^\wedge$$

$$\mathbf{u}^{\wedge^T} \equiv -\mathbf{u}^\wedge$$

$$\mathbf{u}^\wedge\mathbf{v} \equiv -\mathbf{v}^\wedge\mathbf{u}$$

$$\mathbf{u}^\wedge\mathbf{u} \equiv \mathbf{0}$$

$$(\mathbf{W}\mathbf{u})^\wedge \equiv \mathbf{u}^\wedge\left(\operatorname{tr}(\mathbf{W})\,\mathbf{1} - \mathbf{W}\right) - \mathbf{W}^T\mathbf{u}^\wedge$$

$$\mathbf{u}^\wedge\mathbf{v}^\wedge \equiv -(\mathbf{u}^T\mathbf{v})\,\mathbf{1} + \mathbf{v}\mathbf{u}^T$$

$$\begin{aligned}\mathbf{u}^\wedge\mathbf{W}\,\mathbf{v}^\wedge \equiv & -\left(-\operatorname{tr}(\mathbf{v}\mathbf{u}^T)\,\mathbf{1} + \mathbf{v}\mathbf{u}^T\right) \\ & \times\left(-\operatorname{tr}(\mathbf{W})\,\mathbf{1} + \mathbf{W}^T\right) \\ & + \operatorname{tr}(\mathbf{W}^T\mathbf{v}\mathbf{u}^T)\,\mathbf{1} - \mathbf{W}^T\mathbf{v}\mathbf{u}^T\end{aligned}$$

$$\mathbf{u}^\wedge\mathbf{v}^\wedge\mathbf{u}^\wedge \equiv \mathbf{u}^\wedge\mathbf{u}^\wedge\mathbf{v}^\wedge + \mathbf{v}^\wedge\mathbf{u}^\wedge\mathbf{u}^\wedge + (\mathbf{u}^T\mathbf{u})\,\mathbf{v}^\wedge$$

$$(\mathbf{u}^\wedge)^3 + (\mathbf{u}^T\mathbf{u})\,\mathbf{u}^\wedge \equiv 0$$

$$\mathbf{u}^\wedge\mathbf{v}^\wedge\mathbf{v}^\wedge - \mathbf{v}^\wedge\mathbf{v}^\wedge\mathbf{u}^\wedge \equiv (\mathbf{v}^\wedge\mathbf{u}^\wedge\mathbf{v})^\wedge$$

$$[\mathbf{u}^\wedge, \mathbf{v}^\wedge] \equiv \mathbf{u}^\wedge\mathbf{v}^\wedge - \mathbf{v}^\wedge\mathbf{u}^\wedge \equiv (\mathbf{u}^\wedge\mathbf{v})^\wedge$$

$$\underbrace{[\mathbf{u}^\wedge, [\mathbf{u}^\wedge, \ldots [\mathbf{u}^\wedge,}_{n} \mathbf{v}^\wedge] \ldots]] \equiv \left((\mathbf{u}^\wedge)^n\,\mathbf{v}\right)^\wedge$$

Lie Group

$$\mathbf{C} = \exp(\boldsymbol{\phi}^\wedge) \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n$$

$$\equiv \cos\phi\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi\mathbf{a}^\wedge$$

$$\approx \mathbf{1} + \boldsymbol{\phi}^\wedge$$

$$\mathbf{C}^{-1} \equiv \mathbf{C}^T \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(-\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\phi}^\wedge$$

$$\boldsymbol{\phi} = \phi\mathbf{a}$$

$$\mathbf{a}^T\mathbf{a} \equiv 1$$

$$\mathbf{C}^T\mathbf{C} \equiv \mathbf{1} \equiv \mathbf{C}\mathbf{C}^T$$

$$\operatorname{tr}(\mathbf{C}) \equiv 2\cos\phi + 1$$

$$\det(\mathbf{C}) \equiv 1$$

$$\mathbf{C}\mathbf{a} \equiv \mathbf{a}$$

$$\mathbf{C}\boldsymbol{\phi} = \boldsymbol{\phi}$$

$$\mathbf{C}\mathbf{a}^\wedge \equiv \mathbf{a}^\wedge\mathbf{C}$$

$$\mathbf{C}\boldsymbol{\phi}^\wedge \equiv \boldsymbol{\phi}^\wedge\mathbf{C}$$

$$(\mathbf{C}\mathbf{u})^\wedge \equiv \mathbf{C}\mathbf{u}^\wedge\mathbf{C}^T$$

$$\exp\left((\mathbf{C}\mathbf{u})^\wedge\right) \equiv \mathbf{C}\exp(\mathbf{u}^\wedge)\,\mathbf{C}^T$$

(left) Jacobian

$$\mathbf{J} = \int_0^1 \mathbf{C}^\alpha\,d\alpha \equiv \sum_{n=0}^{\infty}\tfrac{1}{(n+1)!}\left(\boldsymbol{\phi}^\wedge\right)^n$$

$$\equiv \tfrac{\sin\phi}{\phi}\mathbf{1} + \left(1 - \tfrac{\sin\phi}{\phi}\right)\mathbf{a}\mathbf{a}^T + \tfrac{1-\cos\phi}{\phi}\mathbf{a}^\wedge$$

$$\approx \mathbf{1} + \tfrac{1}{2}\boldsymbol{\phi}^\wedge$$

$$\mathbf{J}^{-1} \equiv \sum_{n=0}^{\infty}\tfrac{B_n}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n$$

$$\equiv \tfrac{\phi}{2}\cot\tfrac{\phi}{2}\mathbf{1} + \left(1 - \tfrac{\phi}{2}\cot\tfrac{\phi}{2}\right)\mathbf{a}\mathbf{a}^T - \tfrac{\phi}{2}\mathbf{a}^\wedge$$

$$\approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\phi}^\wedge$$

$$\exp\left((\boldsymbol{\phi} + \delta\boldsymbol{\phi})^\wedge\right) \approx \exp\left((\mathbf{J}\,\delta\boldsymbol{\phi})^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right)$$

$$\mathbf{C} \equiv \mathbf{1} + \boldsymbol{\phi}^\wedge\mathbf{J}$$

$$\mathbf{J}(\boldsymbol{\phi}) \equiv \mathbf{C}\,\mathbf{J}(-\boldsymbol{\phi})$$

$$\left(\exp\left(\delta\boldsymbol{\phi}^\wedge\right)\mathbf{C}\right)^\alpha \approx \left(\mathbf{1} + \left(\mathbf{A}(\alpha, \boldsymbol{\phi})\,\delta\boldsymbol{\phi}\right)^\wedge\right)\mathbf{C}^\alpha$$

$$\mathbf{A}(\alpha, \boldsymbol{\phi}) = \alpha\,\mathbf{J}(\alpha\boldsymbol{\phi})\mathbf{J}(\boldsymbol{\phi})^{-1} = \sum_{n=0}^{\infty}\tfrac{F_n(\alpha)}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n$$

$$\alpha, \beta \in \mathbb{R},\ \mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{W}, \mathbf{A}, \mathbf{J} \in \mathbb{R}^{3\times 3},\ \mathbf{C} \in SO(3)$$

$SE(3)$ Identities and Approximations

Lie Algebra

$$\mathbf{x}^\wedge = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\wedge = \begin{bmatrix} \mathbf{v}^\wedge & \mathbf{u} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

$$\mathbf{x}^\curlywedge = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\curlywedge = \begin{bmatrix} \mathbf{v}^\wedge & \mathbf{u}^\wedge \\ \mathbf{0} & \mathbf{v}^\wedge \end{bmatrix}$$

$$(\alpha\mathbf{x} + \beta\mathbf{y})^\wedge \equiv \alpha\mathbf{x}^\wedge + \beta\mathbf{y}^\wedge$$

$$(\alpha\mathbf{x} + \beta\mathbf{y})^\curlywedge \equiv \alpha\mathbf{x}^\curlywedge + \beta\mathbf{y}^\curlywedge$$

$$\mathbf{x}^\curlywedge\mathbf{y} \equiv -\mathbf{y}^\curlywedge\mathbf{x}$$

$$\mathbf{x}^\curlywedge\mathbf{x} \equiv \mathbf{0}$$

$$(\mathbf{x}^\wedge)^4 + (\mathbf{v}^T\mathbf{v})\,(\mathbf{x}^\wedge)^2 \equiv \mathbf{0}$$

$$(\mathbf{x}^\curlywedge)^5 + 2(\mathbf{v}^T\mathbf{v})\,(\mathbf{x}^\curlywedge)^3 + (\mathbf{v}^T\mathbf{v})^2\,(\mathbf{x}^\curlywedge) \equiv \mathbf{0}$$

$$[\mathbf{x}^\wedge, \mathbf{y}^\wedge] \equiv \mathbf{x}^\wedge\mathbf{y}^\wedge - \mathbf{y}^\wedge\mathbf{x}^\wedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\wedge$$

$$[\mathbf{x}^\curlywedge, \mathbf{y}^\curlywedge] \equiv \mathbf{x}^\curlywedge\mathbf{y}^\curlywedge - \mathbf{y}^\curlywedge\mathbf{x}^\curlywedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\curlywedge$$

$$\underbrace{[\mathbf{x}^\wedge, [\mathbf{x}^\wedge, \ldots [\mathbf{x}^\wedge,}_{n} \mathbf{y}^\wedge] \ldots]] \equiv \left((\mathbf{x}^\curlywedge)^n\,\mathbf{y}\right)^\wedge$$

$$\underbrace{[\mathbf{x}^\curlywedge, [\mathbf{x}^\curlywedge, \ldots [\mathbf{x}^\curlywedge,}_{n} \mathbf{y}^\curlywedge] \ldots]] \equiv \left((\mathbf{x}^\curlywedge)^n\,\mathbf{y}\right)^\curlywedge$$

$$\mathbf{p}^\odot = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\odot = \begin{bmatrix} \eta\mathbf{1} & -\boldsymbol{\varepsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}$$

$$\mathbf{p}^\circledcirc = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\circledcirc = \begin{bmatrix} \mathbf{0} & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon}^\wedge & \mathbf{0} \end{bmatrix}$$

$$\mathbf{x}^\wedge\mathbf{p} \equiv \mathbf{p}^\odot\mathbf{x}$$

$$\mathbf{p}^T\mathbf{x}^\wedge \equiv \mathbf{x}^T\mathbf{p}^\circledcirc$$

Lie Group

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}$$

$$\mathbf{T} = \exp(\boldsymbol{\xi}^\wedge) \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(\boldsymbol{\xi}^\wedge\right)^n$$

$$\equiv \mathbf{1} + \boldsymbol{\xi}^\wedge + \left(\tfrac{1-\cos\phi}{\phi^2}\right)(\boldsymbol{\xi}^\wedge)^2 + \left(\tfrac{\phi-\sin\phi}{\phi^3}\right)(\boldsymbol{\xi}^\wedge)^3$$

$$\approx \mathbf{1} + \boldsymbol{\xi}^\wedge$$

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{C} & \mathbf{J}\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{\xi}^\curlywedge \equiv \operatorname{ad}\left(\boldsymbol{\xi}^\wedge\right)$$

$$\boldsymbol{\mathcal{T}} = \exp(\boldsymbol{\xi}^\curlywedge) \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$$\begin{aligned}\equiv \mathbf{1} & + \left(\tfrac{3\sin\phi - \phi\cos\phi}{2\phi}\right)\boldsymbol{\xi}^\curlywedge + \left(\tfrac{4 - \phi\sin\phi - 4\cos\phi}{2\phi^2}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^2 \\ & + \left(\tfrac{\sin\phi - \phi\cos\phi}{2\phi^3}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^3 + \left(\tfrac{2 - \phi\sin\phi - 2\cos\phi}{2\phi^4}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^4\end{aligned}$$

$$\approx \mathbf{1} + \boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{T}} = \operatorname{Ad}(\mathbf{T}) \equiv \begin{bmatrix} \mathbf{C} & (\mathbf{J}\boldsymbol{\rho})^\wedge\,\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

$$\operatorname{tr}(\mathbf{T}) \equiv 2\cos\phi + 2, \quad \det(\mathbf{T}) \equiv 1$$

$$\operatorname{Ad}(\mathbf{T}_1\mathbf{T}_2) = \operatorname{Ad}(\mathbf{T}_1)\operatorname{Ad}(\mathbf{T}_2)$$

$$\mathbf{T}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\wedge\right) \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(-\boldsymbol{\xi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\wedge$$

$$\mathbf{T}^{-1} \equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T\mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{\mathcal{T}}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\curlywedge\right) \equiv \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(-\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{T}}^{-1} \equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T(\mathbf{J}\boldsymbol{\rho})^\wedge \\ \mathbf{0} & \mathbf{C}^T \end{bmatrix}$$

$$\boldsymbol{\mathcal{T}}\boldsymbol{\xi} \equiv \boldsymbol{\xi}$$

$$\mathbf{T}\boldsymbol{\xi}^\wedge \equiv \boldsymbol{\xi}^\wedge\mathbf{T}, \quad \boldsymbol{\mathcal{T}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{T}}$$

$$(\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge \equiv \mathbf{T}\mathbf{x}^\wedge\mathbf{T}^{-1}, \quad (\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge \equiv \boldsymbol{\mathcal{T}}\mathbf{x}^\curlywedge\boldsymbol{\mathcal{T}}^{-1}$$

$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge\right) \equiv \mathbf{T}\exp(\mathbf{x}^\wedge)\,\mathbf{T}^{-1}$$

$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge\right) \equiv \boldsymbol{\mathcal{T}}\exp(\mathbf{x}^\curlywedge)\,\boldsymbol{\mathcal{T}}^{-1}$$

$$(\mathbf{T}\mathbf{p})^\odot \equiv \mathbf{T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$

$$(\mathbf{T}\mathbf{p})^{\odot^T}(\mathbf{T}\mathbf{p})^\odot \equiv \boldsymbol{\mathcal{T}}^{-T}\mathbf{p}^{\odot^T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$

(left) Jacobian

$$\boldsymbol{\mathcal{J}} = \int_0^1 \boldsymbol{\mathcal{T}}^\alpha\,d\alpha \equiv \sum_{n=0}^{\infty}\tfrac{1}{(n+1)!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$$\begin{aligned}= \mathbf{1} & + \left(\tfrac{4 - \phi\sin\phi - 4\cos\phi}{2\phi^2}\right)\boldsymbol{\xi}^\curlywedge + \left(\tfrac{4\phi - 5\sin\phi + \phi\cos\phi}{2\phi^3}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^2 \\ & + \left(\tfrac{2 - \phi\sin\phi - 2\cos\phi}{2\phi^4}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^3 + \left(\tfrac{2\phi - 3\sin\phi + \phi\cos\phi}{2\phi^5}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^4\end{aligned}$$

$$\approx \mathbf{1} + \tfrac{1}{2}\boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{J}} \equiv \begin{bmatrix} \mathbf{J} & \mathbf{Q} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}$$

$$\boldsymbol{\mathcal{J}}^{-1} \equiv \sum_{n=0}^{\infty}\tfrac{B_n}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{J}}^{-1} \equiv \begin{bmatrix} \mathbf{J}^{-1} & -\mathbf{J}^{-1}\mathbf{Q}\mathbf{J}^{-1} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix}$$

$$\mathbf{Q} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\tfrac{1}{(n+m+2)!}\left(\boldsymbol{\phi}^\wedge\right)^n\boldsymbol{\rho}^\wedge\left(\boldsymbol{\phi}^\wedge\right)^m$$

$$\begin{aligned}\equiv & \tfrac{1}{2}\boldsymbol{\rho}^\wedge + \left(\tfrac{\phi - \sin\phi}{\phi^3}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ & + \left(\tfrac{\phi^2 + 2\cos\phi - 2}{2\phi^4}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge - 3\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ & + \left(\tfrac{2\phi - 3\sin\phi + \phi\cos\phi}{2\phi^5}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)\end{aligned}$$

$$\exp\left((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\wedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)$$

$$\exp\left((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\curlywedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\curlywedge\right)\exp\left(\boldsymbol{\xi}^\curlywedge\right)$$

$$\boldsymbol{\mathcal{T}} \equiv \mathbf{1} + \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$

$$\boldsymbol{\mathcal{J}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$

$$\boldsymbol{\mathcal{J}}(\boldsymbol{\xi}) \equiv \boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{J}}(-\boldsymbol{\xi})$$

$$\left(\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\mathbf{T}\right)^\alpha \approx \left(\mathbf{1} + \left(\boldsymbol{\mathcal{A}}(\alpha, \boldsymbol{\xi})\,\delta\boldsymbol{\xi}\right)^\wedge\right)\mathbf{T}^\alpha$$

$$\boldsymbol{\mathcal{A}}(\alpha, \boldsymbol{\xi}) = \alpha\,\boldsymbol{\mathcal{J}}(\alpha\boldsymbol{\xi})\boldsymbol{\mathcal{J}}(\boldsymbol{\xi})^{-1} = \sum_{n=0}^{\infty}\tfrac{F_n(\alpha)}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$$\alpha, \beta \in \mathbb{R},\ \mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{p} \in \mathbb{R}^4,\ \mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \delta\boldsymbol{\xi} \in \mathbb{R}^6,\ \mathbf{C} \in SO(3),\ \mathbf{J}, \mathbf{Q} \in \mathbb{R}^{3\times 3},\ \mathbf{T}, \mathbf{T}_1, \mathbf{T}_2 \in SE(3),\ \boldsymbol{\mathcal{T}} \in \operatorname{Ad}(SE(3)),\ \boldsymbol{\mathcal{J}}, \boldsymbol{\mathcal{A}} \in \mathbb{R}^{6\times 6}$$

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