Model-Free Reinforcement Learning: from Clipped Pseudo-Regret to Sample Complexity

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Abstract

In this paper we consider the problem of learning an ϵ-optimal policy for a discounted Markov Decision Process (MDP). Given an MDP with S states, A actions, the discount factor $\gamma \in (0, 1)$, and an approximation threshold $\epsilon > 0$, we provide a model-free algorithm to learn an ϵ-optimal policy with sample complexity $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^{5.5}})$ [1] and success probability $(1-p)$. For small enough ϵ, we show an improved algorithm with sample complexity $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^3})$. While the first bound improves upon all known model-free algorithms and model-based ones with tight dependence on S, our second algorithm beats all known sample complexity bounds and matches the information theoretic lower bound up to logarithmic factors.

1. Introduction

Reinforcement learning (RL) (Burnetas & Katehakis, 1997) studies the problem of how to make sequential decisions to learn and act in unknown environments (which is usually modeled by a Markov Decision Process (MDP)) and maximize the collected rewards. There are mainly two types of algorithms to approach the RL problems: model-based algorithms and model-free algorithms. Model-based RL algorithms keep explicit description of the learned model and make decisions based on this model. In contrast, model-free algorithms only maintain a group of value functions instead of the complete model of the system dynamics. Due to their space- and time-efficiency, model-free RL algorithms have been getting popular in a wide range of practical tasks (e.g., DQN (Mnih et al., 2015), TRPO (Schulman et al., 2015), and A3C (Mnih et al., 2016)).

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[1]In this work, the notation $\tilde{O}(\cdot)$ hides poly-logarithmic factors of $S, A, 1/(1-\gamma)$, and $1/\epsilon$.

In RL theory, model-free algorithms are explicitly defined to be the ones whose space complexity is always sublinear relative to the space required to store the MDP parameters (Jin et al., 2018). For tabular MDPs (i.e., MDPs with finite number of states and actions, usually denoted by S and A respectively), this requires that the space complexity to be $o(S^2A)$. Motivated by the empirical effectiveness of model-free algorithms, the intriguing question of whether model-free algorithms can be rigorously proved to perform as well as the model-based ones has attracted much attention and been studied in the settings such as regret minimization for episodic MDPs (Azar et al., 2017; Jin et al., 2018; Zhang et al., 2020)).

In this work, we study the PROBABLY-APPROXIMATELY-CORRECT-RL (PAC-RL) problem, i.e., to designing an algorithm for learning an approximately optimal policy. We will focus on designing the model-free algorithms, and under the model of discounted tabular MDPs with a discount factor γ. The RL algorithm runs for infinitely many time steps. At each time step t, the RL agent learns a policy π_t based on the information collected before time t, observes the current state s_t, makes an action $a_t = \pi_t(s_t)$, receives the reward r_t and transits to the next state s_{t+1} according to the underlying environments. The goal of the agent is to learn the policy π_t at each time t so as to maximize the γ-*discounted accumulative reward* $V^{\pi_t}(s_t)$. More concretely, we wish to minimize the *sample complexity* for the agent to learn an ϵ-optimal policy, which is defined to be the number of time steps that $V^{\pi_t}(s_t) < V^*(s_t) - \epsilon$, where V^* is the optimal discounted accumulative reward that starts with s_t, and the formal definitions of both V^π and V^* can be found in Section 2.

The PAC-RL addresses the important problem about how many trials are required to learn a good policy. We also note that in the PAC-RL definition, the exploration at each time step has to align with the learned policy (i.e., $a_t = \pi_t(s_t)$). This is stronger than the usual PAC learning definition in other online learning settings such as multi-armed bandits (see, e.g., (Even-Dar et al., 2006)) and PAC-RL with a simulator (see Section 1.2), where the exploration actions can be arbitrary and may incur a large regret compared to the optimum.

Quite a few algorithms have been proposed over the past nearly two decades for the PAC-RL problem. For model-based algorithms, MoRmax (Szita & Szepesvari, 2010) achieves the $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^6})$ sample complexity, and UCRL-γ (Lattimore & Hutter, 2012) achieves $\tilde{O}(\frac{S^2A\ln(1/p)}{\epsilon^2(1-\gamma)^3})$. It is also worthwhile to mention that R-max (Brafman & Tennenholtz, 2003) was designed for learning the more general stochastic games and achieves the $\tilde{O}(\frac{S^2A\ln(1/p)}{\epsilon^3(1-\gamma)^6})$ sample complexity in our setting (as analyzed in (Kakade, 2003)). Unfortunately, none of these algorithms matches the information theoretical lower bound $\Omega(\frac{SA}{\epsilon^2(1-\gamma)^3})$ proved by (Lattimore & Hutter, 2012). On the model-free side, known bounds are even less optimal – the delayed Q-learning algorithm proposed by (Strehl et al., 2006) achieves the sample complexity of $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^4(1-\gamma)^8})$, and recent work (Dong et al., 2019) made an improvement to $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^7})$ via a more carefully designed Q-learning variant. Besides the results above, (Pazis et al., 2016) provided $\tilde{O}\left(\frac{S^2A}{\epsilon^2(1-\gamma)^4}\right)$ sample complexity. However, their algorithm consumes $\tilde{O}(\frac{SA}{\epsilon^2(1-\gamma)^4})$ space cost and $\tilde{O}\left(\frac{SA^2}{\epsilon^2(1-\gamma)^4}\right)$ computational cost each step, which is far beyond the cost of both model-based and model-free algorithms when ϵ is small.

1.1. Our Results

We design a model-free algorithm that achieves asymptotically optimal sample complexity, as follows.

Theorem 1. *By the model-free algorithm* UCB-MULTISTAGE-ADVANTAGE, *for any discounted MDP with S states, A actions, and the discount factor γ, any approximation threshold $\epsilon \in (0, \frac{(1-\gamma)^{14}}{S^2A^2})$ and failure probability parameter p, with probability $(1-p)$, the sample complexity to learn an ϵ-optimal policy with* UCB-MULTISTAGE-ADVANTAGE *is bounded by* $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^3})$.

In the theorem statement, $\text{poly}(S, A, 1/(1-\gamma))$ stands for a universal polynomial that is independent of the MDP. Our UCB-MULTISTAGE-ADVANTAGE algorithm is model-free, which uses only $O(SA)$ space , and its time complexity per time step is $O(1)$. In contrast, the model-based algorithms have to consume $\Omega(S^2A)$ space. For asymptotically small ϵ, the sample complexity of UCB-MULTISTAGE-ADVANTAGE matches the information theoretic lower bound of $\Omega(\frac{SA}{\epsilon^2(1-\gamma)^3})$ up to poly-logarithmic terms, and improves upon all known algorithms in literature, even including the model-based ones. In Appendix A, we present a tabular view of the comparison between our algorithms and the previous works.

To prove Theorem 1, we make two main technical contributions. The first one is a novel relation between sample complexity and the so-called *clipped pseudo-regret*, which can also be viewed as the clipped Bellman error of the learned value function and policy at each time step. This relation enables us to reduce the sample complexity analysis to bounding the clipped pseudo-regret. Our second technique is a *multi-stage update rule*, where the visits to each state-action pair are partitioned according to two types of stages. An update to the Q-function is triggered only when a stage of either type has concluded. The lengths of the two types of stages are set by different choices of parameters so that we can reduce the clipped pseudo-regret while still maintaining a decent rate to learn the value function. Finally, we also spend much technical effort to incorporate the variance reduction technique for RL via *reference-advantage decomposition* introduced in the recent work (Zhang et al., 2020).

A more detailed overview of our techniques is available in Section 4. Since the proof of Theorem 1 is rather involved, we will first provide a proof of the following weaker statement, and defer the full proof of Theorem 1 to Appendix D.

Theorem 2. *By the model-free algorithm* UCB-MULTISTAGE, *for any approximation threshold $\epsilon \in (0, \frac{1}{1-\gamma}]$ and any failure probability parameter p, with probability $(1-p)$, the sample complexity to learn an ϵ-policy with* UCB-MULTISTAGE *is bounded by* $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^{5.5}})$.

We highlight that the sample complexity bound in Theorem 2 holds for every possible $\epsilon \in (0, \frac{1}{1-\gamma}]$. Although the dependency on γ becomes $(1-\gamma)^{-5.5}$, UCB-MULTISTAGE still beats all known model-free and model-based algorithms with tight dependence on S. The proof of Theorem 2 does not rely on the variance reduction technique based on reference-advantage decomposition (Zhang et al., 2020), but is sufficient to illustrate both of our main technical contributions.

1.2. Additional Related Works

The PAC-RL problem has also been extensively studied under the setting of finite-horizon episodic MDPs (Dann & Brunskill, 2015; Dann et al., 2017; 2019), where the sample complexity is defined as the number of episodes in which the policy is not ϵ-optimal. Assuming H is the length of an episode, the optimal sample complexity bound is $\tilde{O}(\frac{SAH^2\ln(1/p)}{\epsilon^2})$, proved by (Dann et al., 2019). Note that the sample complexity bounds for finite-horizon episodic MDP do not imply sample complexity bounds for infinite-horizon discounted MDP because one ϵ-optimal episode may contain non-ϵ-optimal steps. Also we note that existing algorithms for the finite-horizon case are model-based. It is still an open problem whether model-free algorithm can achieve near-optimal sample complexity bound for the finite-horizon case.

Much effort has also been made to study the PAC learning problem for discounted infinite-horizon MDPs, with the access to a generative model (a.k.a., a simulator). In this problem, the agent can query the simulator to draw a sample $s' \sim P(\cdot|s,a)$ for any state-action pair (s,a), and the goal is to output an ϵ-optimal policy (with probability $(1-p)$) at the end of the algorithm. This problem has been studied in (Even-Dar & Mansour, 2003; Azar et al., 2011; Gheshlaghi et al., 2012; Sidford et al., 2018b;a), and (Sidford et al., 2018a) achieves the almost tight sample complexity $\tilde{O}(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^3})$.

2. Preliminaries

A discounted Markov Decision Process is given by the five-tuple $M = \langle \mathcal{S}, \mathcal{A}, P, r, \gamma \rangle$, where $\mathcal{S} \times \mathcal{A}$ is the state-action space, P is the transition probability matrix, r is the deterministic reward function[2] and $\gamma \in (0,1)$ is the discount factor.

The RL agent interacts with the environment for infinite number of times. At the t-th time step, the agent learns a policy π_t based on the samples collected before time t, observes s_t, executes $a_t = \pi_t(s_t)$, receives the reward $r(s_t, a_t)$, and then transits to s_{t+1} according to $P(\cdot|s_t, a_t)$.

Given a deterministic[3] stationary policy $\pi : \mathcal{S} \to \mathcal{A}$, the value function and Q function are defined as

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi(s_t)) \Big| s_1 = s, a_t = \pi(s_t))\right]$$
$$Q^\pi(s,a) = r(s,a) + \gamma P(\cdot|s,a)^\top V^\pi = r(s,a) + \gamma P_{s,a} V^\pi,$$

where we use xy to denote $x^\top y$ for x and y of the same dimension and use $P_{s,a}$ to denote $P(\cdot|s,a)$ for simplicity.

The optimal value function is given by $V^*(s) = \sup_\pi V^\pi(s)$ and the optimal Q-function is defined to be $Q^*(s,a) = r(s,a) + \gamma P_{s,a} V^*$ for any $(s,a) \in \mathcal{S} \times \mathcal{A}$.

We present below the formal definitions for sample complexity and PAC-RL .

Definition 1 (ϵ-sample complexity)**.** *Given an algorithm $\mathcal{G}$ and $\epsilon \in (0, \frac{1}{1-\gamma}]$, the ϵ-sample complexity for $\mathcal{G}$ is* $\sum_{t \geqslant 1} \mathbb{I}\left[V^*(s_t) - V^{\pi_t}(s_t) > \epsilon\right]$.

Definition 2 ((ϵ, p)-PAC-RL)**.** *An algorithm $\mathcal{G}$ is said to be (ϵ,p)-**PAC-RL** (Probably Approximately Correct in RL) if for any $\epsilon \in (0, \frac{1}{1-\gamma}], p > 0$, with probability $1-p$, the sample complexity of $\mathcal{G}$ is bounded by some polynomial in* $(S, A, \frac{1}{\epsilon}, \frac{1}{1-\gamma}, \ln(\frac{1}{p}))$.

When ϵ and p are clear in the context, we simply write (ϵ,p)-PAC-RL and ϵ-sample complexity as PAC-RL and sample complexity respectively. The goal is to propose an PAC-RL algorithm to minimize the sample complexity.

3. The UCB-MULTISTAGE Algorithm

In this section, we introduce the UCB-MULTISTAGE algorithm. The algorithm takes $\mathcal{S}, \mathcal{A}, \gamma, \epsilon$, sets $H = \max\{\frac{\ln(8/((1-\gamma)\epsilon))}{\ln(1/\gamma)}, \frac{1}{1-\gamma}\}$ and $B = \sqrt{H}$. Throughout the paper, we set $\iota = \ln(2/p)$. The algorithm is described in Algorithm 1. For each state-action pair (s,a), the samples are partitioned into consecutive stages. When a stage is filled, we update $Q(s,a)$ and $V(s)$ according to the samples in the stage via the usual value iteration method. The most interesting aspect about our method is that two types of stages, namely the *type-I and type-II stages*, are introduced. More concretely, the length of the j-th type-I stage is roughly $\check{e}_j \approx H(1+1/H)^{j/B}$ and the length of the j-th type-II stage is roughly $\bar{e}_j \approx H(1+1/H)^j$.

We note that the recent work (Zhang et al., 2020) designed a (single-)stage-based model-free RL algorithm for regret minimization. Our type-II stage is similar to their work, and its goal is to make sure that the value function is learned at a decent rate. In contrast, our type-I stage is new: it is shorter than the type-II stage, so that triggers more frequent updates and helps to reduce the difference between the value functions learned in neighboring type-I stages. The hyper-parameter B is used to adjust the frequency of type-I updates (i.e., updates triggered by type-I stage). The two types of stages work together to reduce the clipped pseudo-regret, and therefore achieve low sample complexity.

The precise definition of the stages. Let $d_1 = H$, $d_{j+1} = \lfloor (1+\frac{1}{H})d_j \rfloor$ for all $j \geqslant 1$. The sizes of the j-th type-I and type-II stage are given by $\check{e}_j = d_{\lceil j/B \rceil}$ and $\bar{e}_j = d_j$ respectively.

Let $N_0 = c_1 \cdot \frac{S^3AH^5\ln(4H^2S/\epsilon)\iota}{\epsilon^2}$ for some large enough constant c_1. We stop updating $Q(s,a)$ if the number of visits to (s,a) is greater than N_0, since the value functions will be sufficiently learned by that time.

Therefore, the time steps when an update is triggered by the type-I and type-II stages are respectively given by $\check{\mathcal{L}} = \{\sum_{i=1}^{j} \check{e}_i | 1 \leqslant j \leqslant \check{J}\}$ and $\bar{\mathcal{L}} = \{\sum_{i=1}^{j} \bar{e}_i | 1 \leqslant j \leqslant \bar{J}\}$, where $\check{J} = \max\{j | \sum_{i=1}^{j-1} \check{e}_i \leqslant N_0\}$ and $\bar{J} = \max\{j | \sum_{i=1}^{j-1} \bar{e}_i \leqslant N_0\}$. Without loss of generality, we assume that $\sum_{i=1}^{\check{J}} \check{e}_i = N_0$.

[2] It is easy to generalize our results to stochastic reward functions.

[3] In this work, we mainly consider deterministic policies since the optimal value function can be achieved by a deterministic policy.

The statistics. We maintain the following statistics during the algorithm: for each (s,a), we use $N(s,a)$, $\check{N}(s,a)$, and $\bar{N}(s,a)$ to respectively denote the total visit number, the visit number in the current type-I stage and the visit number in the current type-II stage of (s,a). We also maintain $\check{\mu}(s,a)$ and $\bar{\mu}(s,a)$, which are respectively the accumulators for state values $V(s')$ (where s' is the next state observed after (s,a)) during the current type-I and type-II stages.

We also remark that throughout the paper we will use '$\check{\ }$' to denote the quantities related to the type-I stage, and use '$^{-}$' to denote the quantities related to the type-II stage.

Algorithm 1 UCB-MULTISTAGE

Initialize: $\forall (s,a) \in \mathcal{S} \times \mathcal{A}$: $Q(s,a) \leftarrow \frac{1}{1-\gamma}$,
$N(s,a), \check{N}(s,a), \bar{N}(s,a), \check{\mu}(s,a), \bar{\mu}(s,a) \leftarrow 0$;
for $t = 1,2,3,\ldots$ **do**
  Observe s_t;
  Take action $a_t = \arg\max_a Q(s_t,a)$ and observe s_{t+1};
  \\ *Maintain the statistics*
  $(s,a,s') \leftarrow (s_t,a_t,s_{t+1})$;
  $n := N(s,a) \leftarrow N(s,a) + 1$;
  $\check{n} := \check{N}(s,a) \leftarrow \check{N}(s,a) + 1$;
  $\check{\mu} := \check{\mu}(s,a) \leftarrow \check{\mu}(s,a) + V(s')$;
  $\bar{n} := \bar{N}(s,a) \leftarrow \bar{N}(s,a) + 1$;
  $\bar{\mu} := \bar{\mu}(s,a) \leftarrow \bar{\mu}(s,a) + V(s')$;
  \\ *Update triggered by a type-I stage*
  if $n \in \check{\mathcal{L}}$ **then**

$$\check{b} \leftarrow \min\{2\sqrt{H^2\iota/\check{n}}, 1/(1-\gamma)\}; \tag{1}$$

$$Q(s,a) \leftarrow \min\{r(s,a) + \gamma\big(\check{\mu}/\check{n}\big) + \check{b}, Q(s,a)\}; \tag{2}$$

$$\check{N}(s,a) \leftarrow 0;$$
$$\check{\mu}(s,a) \leftarrow 0;$$
$$V(s) \leftarrow \max_a Q(s,a);$$

  end if
  \\ *Update triggered by a type-II stage*
  if $n \in \bar{\mathcal{L}}$ **then**

$$\bar{b} \leftarrow \min\{2\sqrt{H^2\iota/\bar{n}}, 1/(1-\gamma)\};$$

$$Q(s,a) \leftarrow \min\{r(s,a) + \gamma\big(\bar{\mu}/\bar{n}\big) + \bar{b}, Q(s,a)\}; \tag{3}$$

$$\bar{N}(s,a) \leftarrow 0;$$
$$\bar{\mu}(s,a) \leftarrow 0;$$
$$V(s) \leftarrow \max_a Q(s,a);$$

  end if
end for

4. Technical Overview

Both of the algorithms introduced in this paper are variants of Q-learning, where the optimistic value function V and the Q-function are maintained. For each time t, we use V_t and Q_t to denote the corresponding functions at the beginning of the time step. The learned policy π_t will always be the greedy policy based on Q_t, i.e., $\pi_t(s) = \arg\max_a Q_t(s,a)$ for all $s \in \mathcal{S}$. Below we explain the main techniques used in UCB-MULTISTAGE as well as UCB-MULTISTAGE-ADVANTAGE.

Reducing Sample Complexity to Bounding the Clipped Pseudo-Regret. For any time t, define the *pseudo-regret* vector ϕ_t to be the vector such that for any $s \in \mathcal{S}$,

$$\phi_t(s) = V_t(s) - (r(s,\pi_t(s)) + \gamma P_{s,\pi_t(s)} V_t).$$

We now outline our first technical idea that the sample complexity can be bounded by the total clipped pseudo-regret, approximately in the form of (5) (up to a ϵ^{-1} factor and an additive error term).

Note that ϕ_t can also be viewed as the Bellman error vector of the value function V_t and the policy π_t. Let $\mathtt{P}_{\pi_t}$ be the transition matrix such that $\mathtt{P}_{\pi_t}(s) = P_{s,\pi_t(s)}$ for any $s \in \mathcal{S}$. By Bellman equation we have that

$$\begin{aligned}
&V_t - V^{\pi_t} \\
&= \gamma \mathtt{P}_{\pi_t}(V_t - V^{\pi_t}) + \phi_t \\
&= (\gamma \mathtt{P}_{\pi})^2 (V_t - V^{\pi_t}) + \gamma \mathtt{P}_{\pi_t}\phi_t + \phi_t \\
&= \ldots \\
&= \sum_{i=0}^{\infty} (\gamma \mathtt{P}_{\pi_t})^i \phi_t.
\end{aligned}$$

Define $\mathrm{clip}(x,y) = x\mathbb{I}\left[x \geqslant y\right]$ for $x,y \in \mathbb{R}$ and

$$\mathrm{clip}(x,y) = [\mathrm{clip}(x_1,y),\ldots,\mathrm{clip}(x_n,y)]^\top$$

for $x = [x_1,\ldots,x_n]^\top \in \mathbb{R}^n$.

Therefore, if $V_t(s_t) - V^{\pi_t}(s_t) > \epsilon$, then for some constant $M > 1$,

$$\begin{aligned}
&\mathbf{1}_{s_t}^\top \sum_{i=0}^{\infty} (\gamma \mathtt{P}_{\pi_t})^i \mathrm{clip}(\phi_t, \frac{\epsilon(1-\gamma)}{M}) \\
&\geqslant \mathbf{1}_{s_t}^\top \sum_{i=0}^{\infty} (\gamma \mathtt{P}_{\pi_t})^i \left(\phi_t - \frac{\epsilon(1-\gamma)}{M}\right) \\
&= \mathbf{1}_{s_t}^\top \sum_{i=0}^{\infty} (\gamma \mathtt{P}_{\pi_t})^i \phi_t - \frac{1}{1-\gamma} \cdot \frac{\epsilon(1-\gamma)}{M} \\
&= V_t(s_t) - V^{\pi_t}(s_t) - \frac{\epsilon}{M} \\
&> \frac{(M-1)\epsilon}{M},
\end{aligned}$$

where $\mathbf{1}_{s_t}$ is the unit vector with the only non-zero entry at s_t and the first inequality is by the fact $\mathrm{clip}(x,y) \geqslant x-y$ for $x,y \geqslant 0$. For any $H = \Theta(\ln(((1-\gamma)\epsilon)^{-1})/(1-\gamma))$, it then follows that

$$\begin{aligned}&\mathbb{I}\left[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon\right]\epsilon \\ &\leqslant O\left(\mathbf{1}_{s_t}^\top \sum_{i=0}^{H-1} (\gamma \mathsf{P}_{\pi_t})^i \mathrm{clip}(\phi_t, \epsilon(1-\gamma)/M)\right). \end{aligned} \tag{4}$$

We now sum up (4) over all time steps t. If we can carefully design the algorithm so that π_t, V_t (and therefore ϕ_t) do not change frequently, we have $\pi_t = \pi_{t+i}$ and $\phi_t = \phi_{t+i}$ for small enough i and most t, and therefore we can upper bound $\sum_{t\geqslant 1} \mathbb{I}\left[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon\right]\epsilon$ by the order of

$$\begin{aligned}&\sum_{t\geqslant 1} \mathbf{1}_{s_t}^\top \sum_{i=0}^{H-1} (\gamma \mathsf{P}_{\pi_{t+i}})^i \mathrm{clip}(\phi_{t+i}, \epsilon(1-\gamma)/M) \\ &\leqslant \sum_{t\geqslant 1} \mathbf{1}_{s_t}^\top \sum_{i=0}^{H-1} (\mathsf{P}_{\pi_{t+i}})^i \mathrm{clip}(\phi_{t+i}, \epsilon(1-\gamma)/M) \\ &\approx O(H) \cdot \sum_{t\geqslant 1} \mathrm{clip}(\phi_t(s_t), \epsilon(1-\gamma)/M), \end{aligned} \tag{5}$$

where the approximation (5) also uses the assumption that $\pi_t = \pi_{t+i}$ and $\phi_t = \phi_{t+i}$ hold for most t and i. In Lemma 5, we formalize this intuition and show that if we set $M = 8H(1-\gamma)$, the sample complexity $\sum_{t\geqslant 1} \mathbb{I}\left[V_t(s_t) - V^{\pi_t}(s_t) > \epsilon\right]$ can be upper bounded by $O(H/\epsilon) \cdot \sum_{t\geqslant 1} \mathrm{clip}(\phi_t(s_t), \epsilon(1-\gamma)/M)$ (plus an additive error), and therefore we only need to upper bound the total clipped pseudo-regret.

The Multi-Stage Update Rule. As stated before, the design of type-I stage is our main technical contribution. To better explain the intuition and motivate the type-I stage, let us consider a fixed state-action pair (s,a). Suppose at time step $(t-1)$, (s,a) is visited and the visit number reaches the end of a type-I stage, then the following update is triggered:

$$Q_t(s,a) \leftarrow \min\{r(s,a) + \check{b} + \frac{\gamma}{\check{n}} \sum_{i=1}^{\check{n}} V_{\check{l}_i}(s_{\check{l}_i+1}), Q_{t-1}(s,a)\},$$

where $\check{n}$ is the number of samples in this stage, $\check{l}_i$ is time of the i-th sample in the stage, and $\check{b}$ denotes the exploration bonus. Thanks to the update rule, V_t and Q_t are non-increasing in t. By concentration inequalities and the proper design of $\check{b}$, we get

$$\begin{aligned}&Q_t(s,a) \\ &\leqslant r(s,a) + 2\check{b} + P_{s,a}(\frac{\gamma}{\check{n}} \sum_{i=1}^{\check{n}} V_{\check{l}_i}) \\ &\leqslant r(s,a) + 2\check{b} + \gamma P_{s,a} V_t + \gamma P_{s,a}\left(\frac{1}{\check{n}} \sum_{i=1}^{\check{n}} V_{\check{l}_i} - V_t\right) \quad (6) \\ &\leqslant r(s,a) + 2\check{b} + \gamma P_{s,a} V_t + \gamma P_{s,a}(V_{\underline{t}} - V_{\bar{t}}), \quad (7)\end{aligned}$$

where $\underline{t} = \min_i \check{l}_i$ is the start time of the stage and $\bar{t}$ is the start time of the next stage. Let $a = \pi_t(s)$. By the definition of $\phi_t(s)$ and optimism of V_t, when $Q_t(s,a) - Q^*(s,a) < \epsilon(1-\gamma)/M$, we have that

$$\begin{aligned}&\mathrm{clip}(\phi_t(s), \epsilon(1-\gamma)/M) \\ &\leqslant \mathrm{clip}(Q_t(s,a) - Q^*(s,a), \epsilon(1-\gamma)/M) = 0 \end{aligned} \tag{8}$$

In the case $Q_t(s,a) - Q^*(s,a) \geqslant \epsilon(1-\gamma)/M$, with an averaging argument we have that

$$\begin{aligned}&\mathrm{clip}(\phi_t(s), \epsilon(1-\gamma)/M) \\ &\leqslant \mathrm{clip}(2\check{b} + \gamma P_{s,a}(V_{\underline{t}} - V_{\bar{t}}), \epsilon(1-\gamma)/M) \\ &\leqslant 2\mathrm{clip}(2\check{b}, \epsilon(1-\gamma)/(2M)) \\ &\qquad + O(\gamma) \cdot P_{s,a}\mathrm{clip}(V_{\underline{t}} - V_{\bar{t}}, \epsilon(1-\gamma)/(2M)). \end{aligned} \tag{9}$$

On the benefit of type-II stages, $N_t(s,a) \geqslant N_0$ implies $Q_t(s,a) - Q^*(s,a) < \epsilon(1-\gamma)/M$. So it suffices to bound

$$\begin{aligned}&\mathbb{I}[N_t(s,a) < N_0] P_{s,a}\mathrm{clip}(V_{\underline{t}} - V_{\bar{t}}, \epsilon(1-\gamma)/(2M)) \\ &+ \mathbb{I}[N_t(s,a) < N_0]\mathrm{clip}(2\check{b}, \epsilon(1-\gamma)/M) \end{aligned} \tag{10}$$

.

We now discuss how to deal with the two terms and how the parameter B affects the bounds.

<u>*Bounding the first term of* (10).</u> We first focus on the second term $(\mathbb{I}[N_t(s_t,a_t) < N_0] P_{s,a}\mathrm{clip}(V_{\underline{t}} - V_{\bar{t}}, \epsilon(1-\gamma)/(2M)))$ in (10). For each j, let $t_j = t_j(s,a)$ be the start time of the j-th stage of (s,a). The total contribution of the second term in (10) is bounded by the order of

$$\sum_{s,a}\sum_j \check{e}_j P_{s,a}\mathrm{clip}\left((V_{t_{j-1}(s,a)} - V_{t_{j+1}(s,a)}), \epsilon(1-\gamma)/(2M)\right). \tag{11}$$

Thanks to the updates triggered by the type-II stages, V_t converges to V^* at a rate that is independent of B. Increasing B will shorten the length of the type-I stages, making $V_{t_{j-1}(s,a)}$ closer to $V_{t_{j+1}(s,a)}$, and reduce the magnitude of (11). In Lemma 8, we formalize this intuition and show that when $M = 8H(1-\gamma)$, (11) can be upper bounded by $\tilde{O}(SAH^5 \ln(1/p)/(\epsilon B))$. Therefore, choosing a large enough B will eliminate the H factors in the numerator.

Bounding the second term of (10). On the other hand, however, a larger B means smaller number of samples in the type-I stages, leads to a bigger estimation variance, and therefore forces us to choose a greater exploration bonus $\check{b}$. We have to choose $B = \Theta(\sqrt{H})$ to achieve the optimal balance between the two terms in (10).

To utilize the full power of our multi-stage update rule, we would like to set $B = \Theta(H^3)$. However, the second term in (10) becomes much bigger. In the next subsection, we discuss how to deal with this problem via the variance reduction method, which leads to the asymptotically near-optimal bound in Theorem 1.

Variance Reduction via Reference-Advantage Decomposition. This technique is only used in UCB-MULTISTAGE-ADVANTAGE and the proof of Theorem 1, which is deferred to Appendix D due to space constraints. We explain the technique as follows.

As discussed above, when B is set large, we suffer bigger estimation variance, as fewer samples are allowed in the type-I stages. In model-free regret minimization tasks, similar problem arises where the algorithm (e.g., (Jin et al., 2018)) can only use the recent tiny fraction of the samples and incurs sub-optimal dependency on the episode length. Recent work (Zhang et al., 2020) resolves this problem via the *reference-advantage decomposition* technique.

The high-level idea is that, assuming we have a δ-accurate estimation of V^*, namely the *reference value function* V^{ref}, such that $\|V^{\text{ref}} - V^*\|_\infty \leqslant \delta$, we only need to use the samples to estimate the difference $V^{\text{ref}} - V^*$, which is called the *advantage*. Therefore, the estimation error (incurred in places such as (6)) will be much smaller when δ is small. Choosing $\delta = 1/\sqrt{B}$, and together with the Bernstein-type exploration bonus (see, e.g., (Azar et al., 2017; Jin et al., 2018)), we are able to bound the total contribution of the first term in (9) [4] by $\tilde{O}(SA/(\epsilon(1-\gamma)^2)$, which (together with the H factor in (5)) aligns with the $(1-\gamma)^{-3}$ factor in the bound of Theorem 1. The discussion till now is based on the access of the reference value function V^{ref}. In reality, however, we need to learn the reference value function on the fly. This will incur an additive warm-up cost that polynomially depends on $1/\delta$. However, since δ is independent of ϵ, the extra cost is only a lower-order term.

[4] More precisely, we refer to the total contribution related to the exploration bonus, which is actually in a different form from the first term in (9). This is because $\check{b}$ has to be re-designed using the Bernstein-type exploration bonus technique and evolves to a more complex expression. Please refer to Appendix D for more explanation.

5. Analysis of Sample Complexity

In this section, we prove Theorem 2 for UCB-MULTISTAGE. We start with a few notations: we use $N_t(s,a)$, $\check{N}_t(s,a)$,$\bar{N}_t(s,a)$, $Q_t(s,a)$, $V_t(s)$ to denote respectively the values of $N(s,a)$, $\check{N}(s,a)$, $\bar{N}(s,a)$, $Q(s,a)$, $V(s)$ before the t-th time step. Let $\check{n}_t(s,a)$, $\check{\mu}_t(s,a)$ and $\check{b}^t(s,a)$ be the values of $\check{n}(s,a)$, $\check{\mu}(s,a)$ and $\check{b}(s,a)$ (respectively) in the latest type-I update of $Q(s,a)$ before the t-th time step. In other words, $\check{n}_t(s,a)$ is the length of the type-I stage immediately before the current type-I stage with respect to (s,a); $\check{b}_t(s,a) = \min\{2\sqrt{H^2\iota/\check{n}_t(s,a)}, 1/(1-\gamma)\}$; and

$$\check{\mu}_t(s,a) = \sum_{i=1}^{\check{n}_t(s,a)} V_{\check{l}_{t,i}(s,a)}(s_{\check{l}_{t,i}(s.a)+1}), \tag{12}$$

where $\check{l}_{t,i}(s,a)$ is the time step of the i-th visit among the $\check{n}_t(s,a)$ visits mentioned above. When t belongs to the first type-I stage of (s,a), we define $\check{n}_t(s,a) = 0$, $\check{\mu}_t(s,a) = 0$, and $\check{b}_t(s,a) = 1/(1-\gamma)$.

Given (s,a) and a time step t such that $(s_t, a_t) = (s,a)$, we use $j_t(s,a)$ to denote the index of the type-I which (the beginning of) the t-th time step belongs to with respect to (s,a). For $1 \leqslant j \leqslant \check{J}$, we use $\rho(j,s,a)$ to denote the start time of the j-th type-I with respect to (s,a). Besides, we define $\rho(\check{J}+1,s,a)$ to be the time t such that $N_t(s,a) = N_0$. We also define $\underline{\rho}_t(s,a) := \rho(j_t(s,a)-1,s,a)$ if $j_t(s,a) \geqslant 2$ and 0 otherwise, and $\overline{\rho}_t(s,a) := \rho(j_t(s,a)+1,s,a)$.

5.1. The Good Event

Let (s,a) and j be fixed. With a slight abuse of notation, we define $\check{l}_i$ to be the time when the i-th visit in the j-th type-I stage of (s,a) occurs. Define $\check{b}^{(j)} = \min\{2\sqrt{\frac{H^2\iota}{\check{e}_j}}, \frac{1}{1-\gamma}\}$ for $j \geqslant 2$. Define $\check{E}^{(j)}(s,a)$ be the event where the inequalities below hold

$$\frac{1}{\check{e}_j}\sum_{i=1}^{\check{e}_j} V^*(s_{\check{l}_i+1}) + \check{b}^{(j)} \geqslant P_{s,a}V^*;$$

$$\left|\frac{1}{\check{e}_j}\sum_{i=1}^{\check{e}_j}\left(V_{\check{l}_i}(s_{\check{l}_i+1}) - P_{s,a}V_{\check{l}_i}\right)\right| \leqslant \check{b}^{(j)}.$$

Similarly, let $\bar{l}_i$ be the time when the i-th visit in the j-th type-II stage of (s,a) occurs and $\bar{b}^{(j)} = \min\{2\sqrt{\frac{H^2\iota}{\bar{e}_j}}, \frac{1}{1-\gamma}\}$ for $j \geqslant 1$. Define $\bar{E}^j(s,a)$ be the event where

$$\frac{1}{\bar{e}_j}\sum_{i=1}^{\bar{e}_j} V^*(s_{\bar{l}_i+1}) + \bar{b}^{(j)} \geqslant P_{s,a}V^*;$$

$$\left|\frac{1}{\bar{e}_j}\sum_{i=1}^{\bar{e}_j}\left(V_{\bar{l}_i}(s_{\bar{l}_i+1}) - P_{s,a}V_{\bar{l}_i}\right)\right| \leqslant \bar{b}^{(j)}.$$

hold.

The total good event E_1 is then given by

$$E_1 = \left(\bigcap_{s,a,1\leqslant j\leqslant \check{J}} \check{E}^{(j)}(s,a)\right)\bigcap\left(\bigcap_{s,a,1\leqslant j'\leqslant \bar{J}} \bar{E}^{(j')}(s,a)\right). \tag{13}$$

We claim that E_1 happens with large probability.

Lemma 3. $\mathbb{P}[E_1] \geqslant (1 - SAH(\check{J} + \bar{J})p)$.

The following statement shows that $\{Q_t\}$ is a sequence of non-increasing optimistic estimates of Q^*.

Proposition 4. *Conditioned on the event E_1, it holds that $Q_t(s,a) \geqslant Q^*(s,a)$ and $Q_{t+1}(s,a) \leqslant Q_t(s,a)$ for all $t \geqslant 1$ and (s,a).*

The proofs of Lemma 3, Proposition 4 and all the lemmas in the remaining part of this section can be found in Appendix C. Throughout the rest of this section, the analysis will be done assuming the successful event E_1.

5.2. Using Clipped Pseudo-Regret to Bound Sample Complexity

By the update rule (2), for any $t \geqslant 1$ and s, letting $a = \pi_t(s)$, we have that

$$\begin{aligned}
&V_t(s) - V^{\pi_t}(s)\\
&\leqslant \check{b}_t(s,a) + \frac{\gamma}{\check{n}_t(s,a)}\sum_{u=1}^{\check{n}_t(s,a)} V_{\check{l}_{t,u}(s,a)}(s_{\check{l}_{t,u}(s,a)+1})\\
&\qquad - \gamma P_{s,a}V^{\pi_t}\\
&\leqslant 2\check{b}_t(s,a) + \gamma P_{s,a}\left(\frac{1}{\check{n}_t(s,a)}\sum_{u=1}^{\check{n}_t(s,a)} V_{\check{l}_{t,u}(s,a)} - V^{\pi_t}\right) && (14)\\
&\leqslant 2\check{b}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V^{\pi_t}) && (15)\\
&= 2\check{b}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t) + \gamma P_{s,a}(V_t - V^{\pi_t}). && (16)
\end{aligned}$$

where Inequality (14) is due to the concentration inequality, which is part of the successful event E_1 defined in (41), and Inequality (15) holds because $\underline{\rho}_t(s,a) \leqslant \check{l}_{t,u}(s,a)$ for any $1 \leqslant u \leqslant \check{n}_t(s,a)$ and the fact V_t is non-increasing in t (Proposition 4).

On the other hand, we also have

$$\begin{aligned}
&V_t(s) - V^{\pi_t}(s)\\
&= Q_t(s,a) - Q^*(s,a) + Q^*(s,a) - Q^{\pi_t}(s,a)\\
&= Q_t(s,a) - Q^*(s,a) + \gamma P_{s,a}(V^* - V^{\pi_t})\\
&\leqslant Q_t(s,a) - Q^*(s,a) + \gamma P_{s,a}(V_t - V^{\pi_t}). && (17)
\end{aligned}$$

Combining (16) and (17), we have that

$$\begin{aligned}
&V_t(s) - V^{\pi_t}(s)\\
&\leqslant \min\big\{2\check{b}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t),\\
&\qquad Q_t(s,a) - Q^*(s,a)\big\} + \gamma P_{s,a}(V_t - V^{\pi_t}). && (18)
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\phi_t(s) = V_t(s) - (r(s,a) + \gamma P_{s,a}V_t)\\
&= V_t(s) - V^{\pi_t}(s) - \gamma P_{s,a}(V_t - V^{\pi_t})\\
&\leqslant \min\big\{2\check{b}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t),\\
&\qquad Q_t(s,a) - Q^*(s,a)\big\}. && (19)
\end{aligned}$$

Define κ_t by setting $\kappa_t(s)$ as the **RHS** of (19). Recall that $\mathtt{P}_{\pi_t}$ is the matrix such that $\mathtt{P}_{\pi_t}(s) = P_{s,\pi_t(s)}$ for any $s \in \mathcal{S}$. By Bellman equation we have that

$$\begin{aligned}
V^*(s_t) - V^{\pi_t}(s_t) &\leqslant V_t - V^{\pi_t}\\
&= \sum_{i=0}^{\infty}(\gamma\mathtt{P}_{\pi_t})^i\phi_t\\
&\leqslant \sum_{i=0}^{H-1}(\gamma\mathtt{P}_{\pi_t})^i\phi_t + \frac{\epsilon}{8} && (20)\\
&\leqslant \sum_{s,a}(\gamma\mathtt{P}_{\pi_t})^i\kappa_t + \frac{\epsilon}{8}.
\end{aligned}$$

By definition of $\kappa_t(s)$, and noting that $x \leqslant \mathrm{clip}(x,y) + y$ for any $x, y > 0$, we further have that

$$\begin{aligned}
&V^*(s_t) - V^{\pi_t}(s_t)\\
&\leqslant \sum_{s,a} w_t(s,a)\Big(\min\big\{2\check{b}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t),\\
&\qquad Q_t(s,a) - Q^*(s,a)\big\}\Big) + \frac{\epsilon}{8} && (21)\\
&\leqslant \sum_{s,a} w_t(s,a)\Big(\min\big\{\mathrm{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}),\\
&\qquad 2\mathrm{clip}(\check{b}_t(s,a), \frac{\epsilon}{8H}) + \gamma P_{s,a}\mathrm{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{8H}),\big\}\Big)\\
&\qquad + \sum_{s,a} w_t(s,a)\max\{\frac{3\epsilon}{4H}, \frac{\epsilon}{4H} + \gamma P_{s,a}\mathbf{1}\cdot\frac{\epsilon}{8H}\} + \frac{\epsilon}{8}\\
&\leqslant \sum_{s,a} w_t(s,a)\Big(\min\big\{\mathrm{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}),\\
&\qquad 2\mathrm{clip}(\check{b}_t(s,a), \frac{\epsilon}{8H}) + \gamma P_{s,a}\mathrm{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{8H}),\big\}\Big)\\
&\qquad + \frac{7\epsilon}{8} && (22)
\end{aligned}$$

where $w_t(s,a) = \mathbb{I}[\pi_t(s) = a]\cdot\sum_{i=0}^{H-1}\mathbf{1}_{s_t}^{\top}(\gamma\mathtt{P}_{\pi_t})^i\mathbf{1}_s$ is the expected discounted visit number of (s,a) in the next H

steps following π_t; and Inequality (22) is due to an averaging argument and the fact that $\sum_{s,a} w_t(s,a) \leqslant H$.

Let

$$\beta_t := \sum_{s,a} w_t(s,a) \min\Big\{\mathrm{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}), \big(2\mathrm{clip}(\check{b}_t(s,a), \frac{\epsilon}{8H}) + \gamma P_{s,a}\mathrm{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{8H})\big)\Big\}. \tag{23}$$

Define $\mathcal{T} = \{t \geqslant 1 | \beta_t > \frac{1}{8}\epsilon\}$. By (22) we have that the sample complexity of UCB-MULTISTAGE is bounded by

$$\sum_{t\geqslant 1} \mathbb{I}\left[V^*(s_t) - V^{\pi_t}(s_t) > \epsilon\right] \leqslant \sum_{t\geqslant 1} \mathbb{I}\left[\beta_t > \frac{1}{8}\epsilon\right] = |\mathcal{T}|.$$

To bound $|\mathcal{T}|$, we consider bounding $\sum_{t\in\mathcal{T}} \beta_t$ instead, since $\sum_{t\in\mathcal{T}} \beta_t \geqslant \frac{|\mathcal{T}|\epsilon}{8}$ and therefore $|\mathcal{T}| \leqslant (8/\epsilon)\cdot\sum_{t\in\mathcal{T}}\beta_t$. Let

$$\tilde{\beta}_t := \min\Big\{\mathrm{clip}(Q_t(s_t,a_t) - Q^*(s_t,a_t), \frac{3\epsilon}{4H}) \; 2\mathrm{clip}(\check{b}_t(s_t,a_t), \frac{\epsilon}{8H}) + \gamma P_{s_t,a_t}\mathrm{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t, \frac{\epsilon}{8H})\Big\}, \tag{24}$$

If π_t does not change very frequently, we have the approximation that $\beta_t \approx \sum_{i=0}^{H-1} \tilde{\beta}_{t+i}$. More formally, we prove the following statement (see Appendix C.3 for the proof).

Lemma 5. *For any $K \geqslant 1$, it holds that*

$$\mathbb{P}\Big[\sum_{t\in\mathcal{T}} \beta_t \geqslant 12KH^3\iota + 24SAH^4B\ln(N_0), \sum_{t\geqslant 1}\tilde{\beta}_t < 3KH^2\iota\Big] \leqslant Hp.$$

By Lemma 5 and the discussion above, if we are able to bound $\sum_{t\geqslant 1}\tilde{\beta}_t \leqslant X$ (for $X \geqslant 3H^2\iota$), then with high probability, the sample complexity of UCB-MULTISTAGE is bounded by roughly $O(H/\epsilon)\cdot X$.

5.3. Bounding the Clipped Pseudo-Regret

We now turn to bound $\sum_{t\geqslant 1}\tilde{\beta}_t$. By (24),for t such that $N_t(s_t,a_t) < N_0$, we have that

$$\tilde{\beta}_t \leqslant \Big(2\mathrm{clip}(\check{b}_t(s_t,a_t), \frac{\epsilon}{8H}) + \gamma P_{s_t,a_t}\mathrm{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t, \frac{\epsilon}{8H})\Big), \tag{25}$$

and for $N_t(s,a) \geqslant N_0$, we have

$$\tilde{\beta}_t \leqslant \mathrm{clip}(Q_t(s_t,a_t) - Q^*(s_t,a_t), \frac{3\epsilon}{4H}). \tag{26}$$

The first term in (25) is exploration bonus for the type-I stage. For this term, we have the following lemma (see Appendix C.4 for proof).

Lemma 6.

$$\sum_{t\geqslant 1}\mathrm{clip}(\check{b}_t(s_t,a_t), \frac{\epsilon}{8H}) \leqslant O\left(\frac{SAB\iota}{\epsilon(1-\gamma)^4}\right).$$

The exploration bonus is increasing in B because more frequent updates implies fewer available samples in a single update due to the limitation in model-free RL.

For the second term in (25), let $\alpha_t = \mathbb{I}[N_t(s_t,a_t) < N_0]P_{s_t,a_t}\mathrm{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t, \frac{\epsilon}{8H})$ for short. On benefit of type-II updates, we can ensure a decent convergence rate for Q_t (see Appendix C.7 for proof).

Lemma 7. *Conditioned on the successful event of E_1 defined in* (41)*, for any $\epsilon_1 \in [\epsilon, \frac{1}{1-\gamma}]$ it holds that*

$$\sum_{t=1}^{\infty}\mathbb{I}\left[V_t(s_t) - V^*(s_t)) \geqslant \epsilon_1\right] \leqslant \sum_{t=1}^{\infty}\mathbb{I}\left[Q_t(s_t,a_t) - Q^*(s_t,a_t)) \geqslant \epsilon_1\right] \leqslant O\left(\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon_1^2}\right). \tag{27}$$

By the basic convergence rate provided by Lemma 7, we have that (see Appendix C.5 for proof)

Lemma 8. *With probability $1-(1+2SAH(\check{J}+\bar{J}))p$, it holds that*

$$\sum_{t\geqslant 1}\alpha_t \leqslant O\left(\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon B} + SABH^3 + SAH\ln(N_0)\right).$$

The term α_t reflects the difference of the value functions between the neighboring updates. As mentioned in Section 4, we can reduces this term by increasing B as long as $\frac{SAH^2\ln(\frac{4H}{\epsilon})\iota}{\epsilon B}$ is larger $SABH^3$. We highlight that Lemma 7 is necessary to derive Lemma 8 even when B is large. This is due to the nature of model-free RL algorithms: more frequent updates would incur large variances (and thus greater exploration bonuses) due to fewer available samples between updates. As a result, without type-II updates, simply increasing B would not guarantee a decent convergence rate. In contrast, the type-II updates use more available samples, incurring a smaller exploration bonus, and thus guarantees a decent convergence rate.

Moreover, by Lemma 7, we have the lemma below to bound the term in (26) (see Appendix C.8 for proof).

Lemma 9. *With probability $1-(1+2SAH(\check{J}+\bar{J}))p$, for any $t \geqslant 1$ such that $N_t(s_t,a_t) \geqslant N_0$, it holds that*

$$\mathrm{clip}\left(Q_t(s_t,a_t) - Q^*(s_t,a_t), \frac{3\epsilon}{4H}\right) = 0.$$

Combining Lemma 6, Lemma 8 and Lemma 9, and by the definition of $\tilde{\beta}_t$, we have that

Lemma 10. *With probability* $1-(2+6SAH(\check{J}+\bar{J}))p$, $\sum_{t\geqslant 1}\tilde{\beta}_t$ *is bounded by*

$$O\left(\frac{SABH^4\iota}{\epsilon}+\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon B}+SABH^3\ln(N_0)\right).$$

5.4. Putting Everything Together

Invoking Lemma 5 with $K=\frac{c_2}{3H^2\iota}\left(\frac{SABH^4\iota}{\epsilon}+\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon B}+SABH^3\ln(N_0)\right)\geqslant 1$ for some large enough universal constant c_2, we have that conditioned on the successful event E_1,

$$\begin{aligned}&\mathbb{P}\left[\sum_{t\in\mathcal{T}}\beta_t\geqslant 12KH^3\iota+24SAH^4B\ln(N_0)\right]\\&\leqslant\mathbb{P}\Bigg[\sum_{t\in\mathcal{T}}\beta_t\geqslant 12KH^3\iota+24SAH^4B\ln(N_0),\\&\qquad\qquad\sum_{t\geqslant 1}\tilde{\beta}_t<3KH^2\iota\Bigg]\\&\quad+\mathbb{P}\left[\sum_{t\geqslant 1}\tilde{\beta}_t\geqslant 3KH^2\iota\right] &(28)\\&\leqslant(4SAH(\check{J}+\bar{J})+H+2)p, &(29)\end{aligned}$$

where the second term in (28) bounded due to Lemma 10. Combining Proposition 4 with (29), we obtain that with probability $1-(8SA(\check{J}+\bar{J})+(H+3))p$, it holds that

$$\begin{aligned}\frac{|\mathcal{T}|\epsilon}{2}&\leqslant\sum_{t\in\mathcal{T}}\beta_t\\&\leqslant O\left(\frac{SABH^5\iota}{\epsilon}+\frac{SAH^6\ln(\frac{4H}{\epsilon})\iota}{\epsilon B}+SAH^4B\ln(N_0)\right).\end{aligned}\tag{30}$$

Noting that $B=\sqrt{H}$, we conclude that the number of ϵ-suboptimal steps is bounded by

$$\begin{aligned}&O\left(\frac{SAH^{5.5}\ln(\frac{4H}{\epsilon})\iota}{\epsilon^2}+\frac{SAH^{4.5}\ln(N_0)}{\epsilon}\right)\\&\leqslant O\left(\frac{SAH^{5.5}\ln(\frac{4H}{\epsilon})(\ln(N_0)+\iota)}{\epsilon^2}\right)\end{aligned}$$

for any $\epsilon\in(0,\frac{1}{1-\gamma}]$. Noting that $H=\tilde{O}(\frac{1}{1-\gamma})$, $\check{J}=O(SAH\ln(N_0))$ and $\bar{J}=O(SAHB\ln(N_0))$, we finish the proof of Theorem 2 by replacing p with $\frac{p}{8SA(\check{J}+\bar{J})+H+3}$.

6. Conclusion

We design a stage-based model-free Q-learning Algorithm UCB-MultiStage-Advantage, which achieves a near-optimal sample complexity of $\tilde{O}\left(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^3}\right)$ for discountted reinforcement leaning problem asymptotically. By adjusting the number of stages, we also show a non-asymptotic sample complexity of $\tilde{O}\left(\frac{SA\ln(1/p)}{\epsilon^2(1-\gamma)^{5.5}}\right)$, which outperforms all previous model-free and model-based algorithms with tight dependence on S. We introduce a multi-stage update rule for Q-learning algorithm, which may be useful for other RL settings such as RL with linear function approximation.

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Appendices

A. Comparison with Previous Works

Table 1. Comparisons of PAC-RL algorithms for discounted MDPs

	Algorithm	Sample complexity	Space complexity
Model-based	R-max (Kakade, 2003)	$\tilde{O}\left(\frac{S^2 A \ln(1/p)}{\epsilon^3(1-\gamma)^6}\right)$	$O(S^2 A)$
	MoRmax (Szita & Szepesvari, 2010)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^2(1-\gamma)^6}\right)$	
	UCRL-γ (Lattimore & Hutter, 2012)	$\tilde{O}\left(\frac{S^2 A \ln(1/p)}{\epsilon^2(1-\gamma)^3}\right)$	
Model-free	Delayed Q-learning (Strehl et al., 2006)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^4(1-\gamma)^8}\right)$	$O(SA)$
	Infinite Q-learning with UCB (Dong et al., 2019)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^2(1-\gamma)^7}\right)$	
	UCB-MULTISTAGE-ADVANTAGE (Theorem 1)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^2(1-\gamma)^3}\right)$ (for $\epsilon < \frac{(1-\gamma)^{14}}{S^2A^2}$)	
	UCB-MULTISTAGE (Theorem 2)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^2(1-\gamma)^{5.5}}\right)$	
	MEDIAN-PAC(Pazis et al., 2016)	$\tilde{O}\left(\frac{SA \ln(1/p)}{\epsilon^2(1-\gamma)^4}\right)$	$\tilde{O}\left(\frac{SAH^4}{\epsilon^2}\right)$
Lower bound		$\Omega\left(\frac{SA}{\epsilon^2(1-\gamma)^3}\right)$ (Lattimore & Hutter, 2012)	

B. Technical Lemmas

Lemma 11. *Let $M_1, M_2, ..., M_k, ...$ be a series of random variables which range in $[0,1]$ and $\{\mathcal{F}_k\}_{k\geqslant 0}$ be a filtration such that M_k is measurable with respect to $\mathcal{F}_k$ for $k \geqslant 1$. Define $\mu_k := \mathbb{E}[M_k|\mathcal{F}_{k-1}]$.*

For any $p \in (0,1)$ and $c \geqslant 1$, it holds that

$$\mathbb{P}\left[\exists n, \sum_{k=1}^{n} \mu_k \geqslant 4c\iota, \sum_{k=1}^{n} M_k \leqslant c\iota\right] \leqslant p.$$

Proof. Let $\lambda < 0$ be fixed. Let M be a random variable taking values in $[0,1]$ with mean μ. By convexity of $e^{\lambda x}$ in x, we have that $\mathbb{E}\left[e^{\lambda M}\right] \leqslant \mu e^{\lambda} + (1-\mu) = 1 + \mu(e^{\lambda}-1) \leqslant e^{\mu(e^{\lambda}-1)}$. Then we obtain that for any $k \geqslant 1$

$$\mathbb{E}\left[e^{\lambda M_k - (e^{\lambda}-1)\mu_k}|\mathcal{F}_{k-1}\right] \leqslant 1,$$

which means $\{Y_k := e^{\lambda \sum_{i=1}^{k} M_i - (e^{\lambda}-1)\sum_{i=1}^{k}\mu_i}\}_{k\geqslant 0}$ is a super-martingale with respect to $\{\mathcal{F}_k\}_{k\geqslant 0}$. Let τ be the least n with $\sum_{k=1}^{n} \mu_k \geqslant 4c\iota$. It is easy to verify that $|Y_{\min\{\tau,n\}}| \leqslant e^{(1-e^{\lambda})(4c\iota+1)}$ for any n. By the optional stopping theorem, we have that $\mathbb{E}\left[Y_\tau\right] \leqslant 1$. Then

$$\begin{aligned}
&\mathbb{P}\left[\exists n, \sum_{k=1}^{n} \mu_k \geqslant 4c\iota, \sum_{k=1}^{n} M_k \leqslant c\iota\right] \\
&\leqslant \mathbb{P}\left[\sum_{k=1}^{\tau} M_k \leqslant c\iota\right] \\
&\leqslant \frac{1}{e^{(1-e^{\lambda})4c\iota+\lambda c\iota}}. \qquad (31)
\end{aligned}$$

By setting $\lambda = -\frac{1}{2}$, we obtain that $\frac{1}{e^{(1-e^{\lambda})4c\iota+\lambda c\iota}} \leqslant \frac{1}{e^{c\iota}} = (\frac{p}{2})^c \leqslant p$. The proof is completed. □

Lemma 12 (Freedman's Inequality, Theorem 1.6 of (Freedman et al., 1975))**.** *Let $(M_n)_{n\geqslant 0}$ be a martingale such that $M_0 = 0$ and $|M_n - M_{n-1}| \leqslant c$. Let $\mathrm{Var}_n = \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]$ for $n \geqslant 0$, where $\mathcal{F}_k = \sigma(M_0, M_1, M_2, \ldots, M_k)$. Then, for any positive x and for any positive y,*

$$\mathbb{P}\left[\exists n : M_n \geqslant x \text{ and } \mathrm{Var}_n \leqslant y\right] \leqslant \exp\left(-\frac{x^2}{2(y+cx)}\right). \qquad (32)$$

Lemma 13. *Let $(M_n)_{n\geqslant 0}$ be a martingale such that $M_0 = 0$ and $|M_n - M_{n-1}| \leqslant c$ for some $c > 0$ and any $n \geqslant 1$. Let $\mathrm{Var}_n = \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]$ for $n \geqslant 0$, where $\mathcal{F}_k = \sigma(M_1, M_2, ..., M_k)$. Then for any positive integer n, and any $\epsilon, p > 0$, we have that*

$$\mathbb{P}\left[|M_n| \geqslant 2\sqrt{2}\sqrt{\mathrm{Var}_n \log(\frac{1}{p})} + 2\sqrt{\epsilon \log(\frac{1}{p})} + 2c\log(\frac{1}{p})\right] \leqslant 2\left(\log_2(\frac{nc^2}{\epsilon}) + 1\right)p. \qquad (33)$$

Proof. For any fixed n, we apply Lemma 12 with $y = 2^i\epsilon$ and $x = \pm(2\sqrt{y\log(\frac{1}{p})} + 2c\log(\frac{1}{p}))$. For each $i = 0, 1, 2, \ldots, \log_2(\frac{nc^2}{\epsilon})$, we get that

$$\begin{aligned}
&\mathbb{P}\left[|M_n| \geqslant 2\sqrt{2}\sqrt{2^{i-1}\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p}), \mathrm{Var}_n \leqslant 2^i\epsilon\right] \\
&= \mathbb{P}\left[|M_n| \geqslant 2\sqrt{2^i\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p}), \mathrm{Var}_n \leqslant 2^i\epsilon\right] \\
&\leqslant 2p. \qquad (34)
\end{aligned}$$

Then via a union bound, we have that

$$\begin{aligned}
&\mathbb{P}\left[|M_n| \geqslant 2\sqrt{2}\sqrt{\mathrm{Var}_n \log(\frac{1}{p})} + 2\sqrt{\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p})\right] \\
&\leqslant \sum_{i=1}^{\log_2(\frac{nc^2}{\epsilon})} \mathbb{P}\left[|M_n| \geqslant 2\sqrt{2}\sqrt{2^{i-1}\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p}), 2^{i-1}\epsilon \leqslant \mathrm{Var}_n \leqslant 2^i\epsilon\right] \\
&\quad + \mathbb{P}\left[|M_n| \geqslant 2\sqrt{\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p}), \mathrm{Var}_n \leqslant \epsilon\right] \qquad (35) \\
&\leqslant \sum_{i=1}^{\log_2(\frac{nc^2}{\epsilon})} \mathbb{P}\left[|M_n| \geqslant 2\sqrt{(i-1)\epsilon\log(\frac{1}{p})} + 2\sqrt{\epsilon\log(\frac{1}{p})} + 2c\log(\frac{1}{p}), \mathrm{Var}_n \leqslant i\epsilon\right] + 2p \\
&\leqslant 2\left(\log_2(\frac{nc^2}{\epsilon}) + 1\right)p. \qquad (36)
\end{aligned}$$

□

C. Missing Proofs in Section 5

C.1. Proof of Lemma 3

Proof. Recall that $\check{l}_i$ is the time when the i-th visit in the j-th type-I stage of (s,a) occurs and $\check{b}^{(j)} = \min\{2\sqrt{\frac{H^2\iota}{\check{e}_j}}, \frac{1}{1-\gamma}\}$ for $j \geqslant 2$. By Azuma's inequality, with probability $1-2p$, it holds that

$$\frac{1}{\check{e}_j}\sum_{i=1}^{\check{e}_j} V^*(s_{\check{l}_i+1}) + \check{b}^{(j)} \geqslant P_{s,a}V^*; \tag{37}$$

$$\left|\frac{1}{\check{e}_j}\sum_{i=1}^{\check{e}_j}\left(V_{\check{l}_i}(s_{\check{l}_i+1}) - P_{s,a}V_{\check{l}_i}\right)\right| \leqslant \check{b}^{(j)}, \tag{38}$$

which implies that $\mathbb{P}[\check{E}^j(s,a)] \geqslant 1-2p$.

Also recall that $\bar{l}_i$ is the time when the i-th visit in the j-th type-II stage of (s,a) occurs and $\bar{b}^{(j)} = \min\{2\sqrt{\frac{H^2\iota}{\bar{e}_j}}, \frac{1}{1-\gamma}\}$ for $j \geqslant 1$. By Azuma's inequality, for any $1 \leqslant j \leqslant \bar{J}$ and (s,a), with probability $1-2p$, it holds that

$$\frac{1}{\bar{e}_j}\sum_{i=1}^{\bar{e}_j} V^*(s_{\bar{l}_i+1}) + \bar{b}^{(j)} \geqslant P_{s,a}V^*; \tag{39}$$

$$\left|\frac{1}{\bar{e}_j}\sum_{i=1}^{\bar{e}_j}\left(V_{\bar{l}_i}(s_{\bar{l}_i+1}) - P_{s,a}V_{\bar{l}_i}\right)\right| \leqslant \bar{b}^{(j')}, \tag{40}$$

which implies that $\mathbb{P}[\bar{E}^j(s,a)] \geqslant 1-2p$. Finally, recall

$$E_1 = \Big(\bigcap_{s,a,1\leqslant j\leqslant \check{J}} \check{E}^{(j)}(s,a)\Big)\bigcap\Big(\bigcap_{s,a,1\leqslant j'\leqslant \bar{J}} \bar{E}^{(j')}(s,a)\Big). \tag{41}$$

Then $\mathbb{P}\left[E_1\right] \geqslant 1 - \left(\sum_{(s,a),1\leqslant j\leqslant \check{J}}(1-\mathbb{P}[\check{E}^{(j)}(s,a)])\right) - \left(\sum_{(s,a),1\leqslant j'\leqslant \bar{J}}(1-\mathbb{P}[\bar{E}^{(j')}(s,a)])\right) \geqslant 1 - 2SA(\check{J}+\bar{J})p$. The proof is completed. □

C.2. Proof of Proposition 4

Proof of Proposition 4. By the update rule, $Q_{t+1}(s,a) \leqslant Q_t(s,a)$ for any $t \geqslant 1$ and (s,a). We will prove $Q_t(s,a) \geqslant Q^*(s,a)$ for any $t \geqslant 1$ and (s,a) by induction conditioned on E_1.

For $t=1$, $Q_t(s,a) = \frac{1}{1-\gamma} \geqslant Q^*(s,a)$ for any (s,a). For $t \geqslant 2$, assume $Q_{t'}(s,a) \geqslant Q^*(s,a)$ for $1 \leqslant t' < t$ and all (s,a) pairs. With a slight abuse of notations, we use $\check{l}_i^{(j)}(s,a)$ to denote the time step of the i-th visit in the j-th type-I stage of (s,a). If there exists (j,s,a) such that the j-th type-I update of (s,a) happens at the $(t-1)$-th step, by (37) we have that

$$\begin{aligned}
Q_t(s,a) &= \min\{r(s,a) + \frac{\gamma}{\check{e}_j}\sum_{i=1}^{\check{e}_j} V_{\check{l}_i^{(j)}(s,a)}(s_{\check{l}_i^{(j)}(s,a)+1}) + \check{b}^{(j)}, Q_{t-1}(s,a)\} \\
&\geqslant \min\{r(s,a) + \frac{\gamma}{\check{e}_j}\sum_{i=1}^{\check{e}_j} V^*(s_{\check{l}_i^{(j)}(s,a)+1}) + \check{b}^{(j)}, Q_{t-1}(s,a)\} \\
&\geqslant \min\{r(s,a) + \gamma P_{s,a}V^*, Q_{t-1}(s,a)\} \\
&\geqslant Q^*(s,a).
\end{aligned}$$

In a similar way, if there exists (j,s,a) such that the j-th type-II update of (s,a) happens at the $(t-1)$-th step, by (39), it holds that $Q_t(s,a) \geqslant Q^*(s,a))$. Otherwise, $Q_t(s,a) = Q_{t-1}(s,a) \geqslant Q^*(s,a)$ for any (s,a). The proof is completed.

□

C.3. Proof of Lemma 5

We split $\mathcal{T}$ into H separate subsets by define $\mathcal{V}_k = \{t \in \mathcal{T} : t \mod H = k\}$ for $k = 0, 1, 2, \ldots, H-1$. We will prove Lemma 5 by showing that for each k, it holds that

$$\mathbb{P}\Big[\sum_{t\in\mathcal{V}_k} \beta_t \geqslant 12KH^2\iota + 24SAH^3B\ln(N_0), \quad \sum_{t\geqslant 1}\tilde{\beta}_t < 3KH^2\iota\Big] \leqslant p. \tag{42}$$

If (42) holds for each k, then we have

$$\begin{aligned}
&\mathbb{P}\Big[\sum_{t\in\mathcal{T}} \beta_t \geqslant 12KH^3\iota + 24SAH^4B\ln(N_0), \quad \sum_{t\geqslant 1}\tilde{\beta}_t < 3KH^2\iota\Big] \\
&\leqslant \sum_{k=0}^{H-1}\mathbb{P}\Big[\sum_{t\in\mathcal{V}_k} \beta_t \geqslant 12KH^2\iota + 24SAH^3B\ln(N_0), \quad \sum_{t\geqslant 1}\tilde{\beta}_t < 3KH^2\iota\Big] \\
&\leqslant Hp.
\end{aligned} \tag{43}$$

Let

$$U_t = \mathbb{I}\left[\exists t' \in \{t, t+1, ..., t+H-1\} \text{ and } (s,a) \text{ such that } Q_{t'+1}(s,a) \neq Q_{t'}(s,a)\right].$$

We define

$$\hat{\beta}_t := 3H^2U_t + (1-U_t)\sum_{i=0}^{H-1}\gamma^i\left(2\text{clip}(\check{b}_t(s_{t+i},a_{t+i}), \frac{\epsilon}{8H}) + \gamma P_{s_{t+i},a_{t+i}}\text{clip}(V_{\underline{\rho}_t(s_{t+i},a_{t+i})} - V_t, \frac{\epsilon}{8H})\right).$$

For fixed $k \in \{0, 1, 2, \ldots, H-1\}$, we let

$$\hat{\beta}_t^k := \frac{\hat{\beta}_{tH+k}\mathbb{I}\left[tH+k \in \mathcal{T}\right]}{3H^2}.$$

Noting that $\hat{\beta}_t^k \in [0,1]$ is measurable with respect to $\mathcal{F}_t^k := \mathcal{F}_{(t+1)H+k-1}$ and $\mathbb{E}\left[\hat{\beta}_t^k|\mathcal{F}_{t-1}^k\right] \geqslant \beta_t^k := \frac{\beta_{tH+k}\mathbb{I}[tH+k\in\mathcal{T}]}{3H^2}$, by Lemma 11 we obtain that for any $K \geqslant 1$,

$$\mathbb{P}\left[\exists n, \sum_{t=1}^{n}\beta_t^k \geqslant 4K\iota + 16SAHB\ln(N_0), \quad \sum_{t=1}^{n}\hat{\beta}_t^k \leqslant K\iota + 4SAHB\ln(N_0)\right] \leqslant p,$$

which is equivalent to

$$\begin{aligned}
\mathbb{P}\Big[\exists n, \sum_{t=1}^{n}\beta_t\mathbb{I}\left[t\in\mathcal{V}_k\right] &\geqslant 12KH^2\iota + 24SAH^3B\ln(N_0), \\
&\sum_{t=1}^{n}\hat{\beta}_t\mathbb{I}\left[t\in\mathcal{V}_k\right] \leqslant 3KH^2\iota + 6SAH^3B\ln(N_0)\Big] \leqslant p.
\end{aligned} \tag{44}$$

By definition of $\hat{\beta}_t$, and noting that if $U_t = 0$, $\check{b}_t(s_{t+i},a_{t+i}) = \check{b}_{t+i}(s_{t+i},a_{t+i})$ and $V_{\underline{\rho}_t(s_{t+i},a_{t+i})} = V_{\underline{\rho}_{t+i}(s_{t+i},a_{t+i})}$ for any $0 \leqslant i \leqslant H-1$, we have

$$\begin{aligned}
\hat{\beta}_t &= 3H^2U_t + (1-U_t)\sum_{i=0}^{H-1}\gamma^i\left(2\text{clip}(\check{b}_t(s_{t+i},a_{t+i}), \frac{\epsilon}{8H}) + \gamma P_{s_{t+i},a_{t+i}}\text{clip}(V_{\underline{\rho}_t(s_{t+i},a_{t+i})} - V_t, \frac{\epsilon}{8H})\right) \\
&\leqslant 3H^2U_t + (1-U_t)\sum_{i=0}^{H-1}\left(2\text{clip}(\check{b}_t(s_{t+i},a_{t+i}), \frac{\epsilon}{8H}) + \gamma P_{s_{t+i},a_{t+i}}\text{clip}(V_{\underline{\rho}_t(s_{t+i},a_{t+i})} - V_t, \frac{\epsilon}{8H})\right) \\
&\leqslant 3H^2U_t + \sum_{i=0}^{H-1}\left(2\text{clip}(\check{b}_{t+i}(s_{t+i},a_{t+i}), \frac{\epsilon}{8H}) + \gamma P_{s_{t+i},a_{t+i}}\text{clip}(V_{\underline{\rho}_{t+i}(s_{t+i},a_{t+i})} - V_t, \frac{\epsilon}{8H})\right).
\end{aligned}$$

Then it follows that

$$\begin{aligned}\sum_{t\in\mathcal{V}_k}\hat{\beta}_t &\leqslant \sum_{t\in\mathcal{V}_k}\sum_{i=0}^{H-1}\Big(2\mathrm{clip}(\check{b}_{t+i}(s_{t+i},a_{t+i}),\frac{\epsilon}{8H}) + P_{s_{t+i},a_{t+i}}\mathrm{clip}(V_{\underline{\rho}_{t+i}(s_{t+i},a_{t+i})} - V_t,\frac{\epsilon}{8H})\Big)\\ &\quad + 3H^2\sum_{t\in\mathcal{V}_k}U_t\\ &\leqslant \sum_{t\geqslant 1}\Big(2\mathrm{clip}(\check{b}_t(s_t,a_t),\frac{\epsilon}{8H}) + P_{s_t,a_t}\mathrm{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t,\frac{\epsilon}{8H})\Big) + 6SAH^3B\ln(N_0) \qquad (45)\\ &= \sum_{t\geqslant 1}\tilde{\beta}_t + 6SAH^3B\ln(N_0). \qquad (46)\end{aligned}$$

Here Inequality (45) holds because for each update, there is at most one element $t\in\mathcal{T}'$, such that $U_t = 1$ due to this update.

By (44) and (46), we have that

$$\begin{aligned}&\mathbb{P}\left[\sum_{t\in\mathcal{V}_k}\beta_t \geqslant 12CH^2\iota + 24SAH^3B\ln(N_0),\quad \sum_{t\geqslant 1}\tilde{\beta}_t < 3CH^2\iota\right]\\ &\leqslant \mathbb{P}\left[\sum_{t\in\mathcal{V}_k}\beta_t \geqslant 12CH^2\iota + 24SAH^3B\ln(N_0),\quad \sum_{t\geqslant 1}\hat{\beta}_t < 3CH^2\iota + 6SAH^3B\ln(N_0)\right]\\ &\leqslant p.\end{aligned}$$

The proof is completed.

C.4. Proof of Lemma 6

Proof of Lemma 6. Recall that $\check{b}_t(s_t,a_t) = 2\sqrt{\frac{H^2}{\check{n}_t(s_t,a_t)}\iota}$, so $\mathrm{clip}(\check{b}_t(s_t,a_t),\frac{\epsilon}{8H}) \leqslant 2\sqrt{\frac{H^2\iota}{\check{n}_t(s_t,a_t)}}\mathbb{I}[\check{n}_t < 256\frac{H^4\iota}{\epsilon^2}]$. Noting that $\check{n}_t \geqslant \frac{n_t}{2HB}$, we obtain that

$$\begin{aligned}\sum_{t\geqslant 1}\mathrm{clip}(\check{b}_t(s_t,a_t),\frac{\epsilon}{8H}) &\leqslant SAH^2 + \sum_{t\geqslant 1}2\sqrt{\frac{2H^3B\iota}{n_t(s_t,a_t)}}\mathbb{I}\left[n_t < 512\frac{H^5B\iota}{\epsilon^2}\right]\\ &\leqslant SAH^2 + 182\frac{SAH^4B\iota}{\epsilon}.\end{aligned}$$

□

C.5. Proof of Lemma 8

Proof of Lemma 8. We fix (s,a) and consider to bound $\alpha(s,a) := \sum_{t\geqslant 1}\alpha_t\mathbb{I}[(s_t,a_t) = (s,a)] = \sum_{t\geqslant 1}P_{s_t,a_t}\mathrm{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t,\frac{\epsilon}{8H})\cdot\mathbb{I}[(s_t,a_t) = (s,a,N_t(s_t,a_t) < N_0]$. Define $T(j,s,a)$ to be the set of indices of samples in the j-th type-I stage with respect to (s,a), i.e., $T(j,s,a) := \{t\geqslant 1|(s_t,a_t) = (s,a),\sum_{i=1}^{j-1}\check{e}_i \leqslant N_t(s,a) < \sum_{i=1}^{j}\check{e}_i\}$. It is then clear that for any $t\in T(j,s,a)$, $\underline{\rho}_t(s,a) = \rho(j-1,s,a)$ and $\overline{\rho}_t(s,a) = \rho(j+1,s,a)$. (The definitions of ρ, $\underline{\rho}_t$ and $\overline{\rho}_t$ are at the beginning of Section 5.)

For $j\geqslant 2$, by the definition of α_t and the fact V_t is non-increasing in t, we obtain that

$$\sum_{t\in T(j,s,a)}\alpha_t\mathbb{I}[(s_t,a_t) = (s,a)] \leqslant \check{e}_jP_{s,a}\left(\mathrm{clip}(V_{\rho(j-1,s,a)} - V_{\rho(j+1,s,a)},\frac{\epsilon}{8H})\right),$$

and therefore

$$\alpha(s,a) \leqslant H\sum_{i=1}^{HB}\check{e}_i + \sum_{HB+1\leqslant j\leqslant j_\infty(s,a)}\check{e}_jP_{s,a}\left(\mathrm{clip}(V_{\rho(j-1,s,a)} - V_{\rho(j+1,s,a)},\frac{\epsilon}{8H})\right). \qquad (47)$$

Here also recall that $j_t(s,a)$ is defined at the beginning of Section 5, and $j_\infty(s,a)$ is defined to be $\max_{t\geqslant 1} j_t(s,a) \leqslant \check{J}$.

We next define

$$j(s,a,s',\epsilon') := \max\{j \leqslant j_\infty(s,a) | V_{\rho(j,s,a)}(s') - V^*(s') > \epsilon'\}$$

and

$$\tilde{\tau}(s,a,s',\epsilon') := \sum_{i=1}^{j(s,a,s',\epsilon')} \check{e}_i$$

for $s' \in \mathcal{S}$ and $\epsilon' > 0$. Let $\epsilon_i = \frac{2^i \epsilon}{H}$ for $i = 0,1,2,\ldots,k$ where $k = \lceil \log_2(\frac{H}{(1-\gamma)\epsilon}) \rceil$. By (47), we have that

$$\begin{aligned}
\alpha(s,a) &\leqslant H\sum_{i=1}^{HB} \check{e}_i + \sum_{s'} \sum_{HB+1 \leqslant j < j(s,a,s',\frac{\epsilon}{8H})+1} \check{e}_j P_{s,a}(s') \left(V_{\rho(j-1,s,a)}(s') - V_{\rho(j+1,s,a)}(s')\right) \\
&\leqslant O(BH^2\check{e}_1) + \sum_{s'} \sum_{i=1}^{k} \sum_{\max\{j(s,a,s',\epsilon_i),HB\} < j \leqslant j(s,a,s',\epsilon_{i-1})} \check{e}_{j+1} P_{s,a}(s') \theta(s,a,s',j) \\
&\leqslant O(BH^2\check{e}_1) + \sum_{s'} \sum_{i=1}^{k} \frac{2\sum_{1\leqslant j \leqslant j(s,a,s',\epsilon_{i-1})} \check{e}_j}{HB} P_{s,a}(s') \sum_{j(s,a,s',\epsilon_i) < j \leqslant j(s,a,s',\epsilon_{i-1})} \theta(s,a,s',j) \qquad (48) \\
&= O(BH^2\check{e}_1) + \sum_{i=1}^{k} \frac{2\tilde{\tau}(s,a,s',\epsilon_{i-1})}{HB} P_{s,a}(s') \psi(s,a,s',i) \\
&\leqslant O(BH^2\check{e}_1) + \frac{4}{HB} \sum_{i=1}^{k} \tilde{\tau}(s,a,s',\epsilon_{i-1}) P_{s,a}(s') \epsilon_i, \qquad (49)
\end{aligned}$$

where

$$\begin{aligned}
&\theta(s,a,s',j) := V_{\rho(j,s,a)}(s') - V_{\rho(j+2,s,a)}(s'), \\
&\psi(s,a,s',i) := \sum_{j(s,a,s',\epsilon_i) < j \leqslant j(s,a,s',\epsilon_{i-1})} \theta(s,a,s',j) \leqslant 2\epsilon_i.
\end{aligned}$$

Here Inequality (48) is by the fact $\check{e}_{j+1} \leqslant \frac{2}{HB}\sum_{i=1}^{j} \check{e}_i$ for $j \geqslant HB$ and Inequality (49) is by the definition of $j(s,a,s',\epsilon_i)$.

In the next subsection, we will prove the following lemma.

Lemma 14. *For any $\epsilon > 0$, with probability $1-(1+SA(\check{J}+\bar{J}))p$ it holds that*

$$\sum_{s,a,s'} \tilde{\tau}(s,a,s',\epsilon) P_{s,a}(s') \leqslant O\left(\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + SAHB\ln(N_0) \right).$$

Now, by (49) and Lemma 14 we have that

$$\begin{aligned}
\sum_{t\geqslant 1} \alpha_t &= \sum_{s,a} \alpha(s,a) \\
&\leqslant \sum_{s,a} \left(BH^2\check{e}_1 + \frac{4}{HB} \sum_{s'} \sum_{i=1}^{k} \tilde{\tau}(s,a,s',\epsilon_{i-1}) P_{s,a}(s') \epsilon_i \right) \\
&\leqslant O(SABH^3) + O\left(\frac{4}{HB} \sum_{i=1}^{k} \left(\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon_{i-1}^2} + SAHB\ln(N_0) \right) \epsilon_i \right) \qquad (50) \\
&\leqslant O(SABH^3) + O\left(\frac{1}{HB} \cdot \frac{SAH^6 \ln(\frac{4H}{\epsilon})\iota}{\epsilon} + \frac{SA\ln(N_0)}{1-\gamma} \right) \\
&\leqslant O\left(\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon B} + SABH^3 + SAH\ln(N_0) \right).
\end{aligned}$$

The proof is completed.

□

C.6. Proof of Lemma 14

Recall that by Lemma 7, we have that Conditioned on the successful event E_1 defined in (41), for any $\epsilon_1 \in [\epsilon, \frac{1}{1-\gamma}]$ it holds that

$$\sum_{t=1}^{\infty} \mathbb{I}\left[V_t(s_t) - V^*(s_t)) \geqslant \epsilon_1\right] \leqslant \sum_{t=1}^{\infty} \mathbb{I}\left[Q_t(s_t,a_t) - Q^*(s_t,a_t)) \geqslant \epsilon_1\right] \leqslant O\left(\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon_1^2}\right) \tag{51}$$

With the help of Lemma 7, we prove Lemma 14 as follows.

Proof of Lemma 14. We start with defining

$$\tau(s,a,s',\epsilon) := \sum_{t\geqslant 1} \mathbb{I}\left[(s_t,a_t) = (s,a), V_t(s') - V^*(s') > \epsilon\right].$$

Recalling that $\tilde{\tau}(s,a,s',\epsilon) = \sum_{i=1}^{j(s,a,s',\epsilon)} \check{e}_i$, we have

$$\tilde{\tau}(s,a,s',\epsilon) = \sum_{i=1}^{j(s,a,s',\epsilon)} \check{e}_i \leqslant H + (1+\frac{2}{H}) \sum_{i=1}^{j(s,a,s',\epsilon)-1} \check{e}_i \leqslant H + (1+\frac{2}{H})\tau(s,a,s',\epsilon).$$

So it suffices to prove that

$$\sum_{s,a,s'} \tau(s,a,s',\epsilon)P_{s,a}(s') \leqslant O\left(\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + SAHB\ln(N_0)\right). \tag{52}$$

To prove (52), we define λ_t to be the vector such that $\lambda_t(s) = \mathbb{I}\left[V_t(s) - V^*(s) > \epsilon\right]$. Note that

$$\sum_{s,a,s'} \tau(s,a,s',\epsilon)P_{s,a}(s') = \sum_{t\geqslant 1} P_{s_t,a_t}\lambda_t$$

and due to the infrequent updates, we have that

$$\sum_{t\geqslant 1} \left(\lambda_t(s_{t+1}) - \lambda_{t+1}(s_{t+1})\right) \leqslant \sum_{t\geqslant 1} \mathbb{I}\left[V_t(s_{t+1}) \neq V_{t+1}(s_{t+1})\right] \leqslant 2SAHB\ln(N_0).$$

For C a large enough constant, we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\sum_{s,a,s'} \tau(s,a,s',\epsilon)P_{s,a}(s') \geqslant 4C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + 8SAHB\ln(N_0)\right] \\
&= \mathbb{P}\left[\sum_{t\geqslant 1} P_{s_t,a_t}\lambda_t \geqslant 4C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + 8SAHB\ln(N_0)\right] \\
&\leqslant \mathbb{P}\left[\sum_{t\geqslant 1} P_{s_t,a_t}\lambda_t \geqslant 4C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + 8SAHB\ln(N_0), \sum_{t\geqslant 1}\lambda_t(s_{t+1}) \leqslant C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + 2SAHB\ln(N_0)\right] \\
&\quad + \mathbb{P}\left[\sum_{t\geqslant 1}\lambda_t(s_{t+1}) > C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2} + 2SAHB\ln(N_0)\right] \\
&\leqslant p + \mathbb{P}\left[\sum_{t\geqslant 1}\lambda_t(s_t) \geqslant C\frac{SAH^5 \ln(\frac{4H}{\epsilon})\iota}{\epsilon^2}\right] && (53) \\
&\leqslant p + \mathbb{P}\left[E_1\right] && (54) \\
&\leqslant p + SA(\check{J} + \bar{J})p, && (55)
\end{aligned}$$

where Inequality (53) is by Lemma 11 with $M_k = \lambda_k(s_{k+1})$ and $\mathcal{F}_k = \sigma(s_1, a_1, ..., s_k, a_k, s_{k+1})$ for $k \geqslant 1$, Inequality (54) is by Lemma 7 and Inequality (55) is by Proposition 4. The proof is completed. □

C.7. Proof of Lemma 7

The proof of Lemma 7 uses similar techniques as presented in in Appendix.B of (Dong et al., 2019) and Appendix.B.2 of (Zhang et al., 2020). However, it requires more twists since the Q function is only updated by at most $SA(\check{J}+\bar{J})$ times for each state-action pair.

We first introduce a few simplified notations. Define $\delta^t := Q_t(s_t,a_t) - Q^*(s_t,a_t)$. Clearly $\delta^t \geqslant V_t(s_t) - V^*(s_t)$ and $\sum_{t\geqslant 1}\mathbb{I}[\delta^t \geqslant x] \geqslant \sum_{t\geqslant 1}\mathbb{I}[V_t(s_t) - V^*(s_t) \geqslant x]$ for any $x \geqslant 0$. Throughout this subsection, we use $\bar{n}^t$, $\bar{b}^t$ and $\bar{l}_i^t$ as short hands of $\bar{n}_t(s_t,a_t)$, $\bar{b}_t(s_t,a_t)$ and $\bar{l}_{t,i}(s_t,a_t)$ respectively.

Conditioned on E_1 defined in (41), we note that (37) and (39) hold for any $j \geqslant 1$ and $j' \geqslant 1$ respectively. We will use these inequalities without additional explanation.

Let $\mathcal{T}_1 := \{t \geqslant 1 | N_t(s_t,a_t) \geqslant N_0\}$. We then have the following lemma.

Lemma 15. *Conditioned on successful event E_1 defined in* (41)*, it holds that for any $t \in \mathcal{T}_1$ (if $\mathcal{T}_1$ is not empty)*

$$Q_t(s_t,a_t) - Q^*(s_t,a_t) \leqslant \frac{\epsilon}{2H}.$$

Proof. For each $i = 1,2,\ldots,S$, if there are at least i states with total visit number greater or equal to N_0, we let $s^{(i)}$ be the i-th such state (sorted in the order of time to reach N_0) and let T_i be the corresponding time (i.e., $n_{T_i}(s^{(i)}) = N_0$ and $s_{T_i} = s^{(i)}$). Otherwise we let $s^{(i)}$ be a random state in $\mathcal{S}\backslash\{s^{(1)},...,s^{(i-1)}\}$ and set $T_i = \infty$.

It suffices prove that $V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leqslant \frac{\epsilon}{2H}$ for $s^{(i)}$ with finite T_i. We prove this by applying induction on i to prove the stronger statement that $V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leqslant \frac{\epsilon i}{2HS}$.

<u>*Base case ($i=1$):*</u> Note that for any $t \notin \mathcal{T}_1$, we have following inequality by the update rule (3) and event E_1,

$$\begin{aligned}
\delta^t &= Q_t(s_t,a_t) - Q^*(s_t,a_t) \\
&\leqslant \frac{\mathbb{I}\left[\bar{n}^t = 0\right]}{1-\gamma} + \left(\bar{b}^t + \frac{\gamma}{\bar{n}^t}\sum_{i=1}^{\bar{n}^t} V_{\bar{l}_i^t}(s_{\bar{l}_i^t+1}) - P_{s_t,a_t}V^*\right) \\
&\leqslant \frac{\mathbb{I}\left[\bar{n}^t = 0\right]}{1-\gamma} + \left(2\bar{b}^t + \frac{\gamma}{\bar{n}^t}\sum_{i=1}^{\bar{n}^t}\left(V_{\bar{l}_i^t}(s_{\bar{l}_i^t+1}) - V^*(s_{\bar{l}_i^t+1})\right)\right) \\
&\leqslant \frac{\mathbb{I}\left[\bar{n}^t = 0\right]}{1-\gamma} + 2\bar{b}^t + \frac{\gamma}{\bar{n}^t}\sum_{i=1}^{\bar{n}^t}\left(V_{\bar{l}_i^t+1}(s_{\bar{l}_i^t+1}) - Q^*(s_{\bar{l}_i^t+1}, a_{\bar{l}_i^t+1}) + \theta^{\bar{l}_i^t+1}\right) \\
&= \frac{\mathbb{I}\left[\bar{n}^t = 0\right]}{1-\gamma} + 2\bar{b}^t + \frac{\gamma}{\bar{n}^t}\sum_{i=1}^{\bar{n}^t}(\delta^{\bar{l}_i^t+1} + \theta^{\bar{l}_i^t+1}),
\end{aligned} \tag{56}$$

where we define $\theta^{\bar{l}_i^t+1} := V_{\bar{l}_i^t}(s_{\bar{l}_i^t+1}) - V_{\bar{l}_i^t+1}(s_{\bar{l}_i^t+1})$.

It is obvious that $t \notin \mathcal{T}_1$ if $t < T_1$. Then for any non-negative weights $\{w_t\}_{t\geqslant 1}$, we have that

$$\sum_{t<T_1} w_t\delta^t \leqslant \sum_{t<T_1}\frac{w_t\mathbb{I}[\bar{n}^t = 0]}{1-\gamma} + 2\sum_{t<T_1} w_t\bar{b}^t + \sum_{t<T_1} w_t'(\delta^t + \theta^t), \tag{57}$$

where

$$w_t' = \gamma\sum_{u<T_1}\frac{1}{\bar{n}^u}\sum_{i=1}^{\bar{n}^t}\mathbb{I}\left[t = \bar{l}_i^u + 1\right]. \tag{58}$$

If we choose a sequence of non-negative weights $\{w_t\}_{t\geqslant 1}$ such that $\sup_{t<T_1} w_t \leqslant C$ and $\sum_{t<T_1} w_t \leqslant W$ for two positive constant C and W, then for all $t \geqslant 1$, we have that

$$w_t' \leqslant \gamma(1+\frac{1}{H})C \leqslant (1-\frac{1}{2H})C, \tag{59}$$

and

$$\sum_{t<T_1} w'_t \leqslant \gamma(1+\frac{1}{H})W \leqslant (1-\frac{1}{2H})W. \tag{60}$$

Lemma 16. *Let $\{w_t\}_{t\geqslant 1}$ be a sequence of non-negative weights such that $0 \leqslant w_t \leqslant C$ for any $t \notin \mathcal{T}_1$ and $\sum_{t\notin\mathcal{T}_1} w_t \leqslant W$, then it holds that*

$$\sum_{t\notin\mathcal{T}_1} \frac{w_t\mathbb{I}[\bar{n}^t=0]}{1-\gamma} \leqslant \frac{CSAH}{1-\gamma} \leqslant CSAH^2, \tag{61}$$

$$2\sum_{t\notin\mathcal{T}_1} w_t\bar{b}^t \leqslant 40(1+\frac{1}{H})\sqrt{SAH^3WC\iota} \leqslant 60\sqrt{SAH^3WC\iota}, \tag{62}$$

$$\sum_{t\notin\mathcal{T}_1} w_t\theta^t \leqslant \frac{SAC}{1-\gamma} \leqslant SCH. \tag{63}$$

Proof. The first inequality holds because $\sum_{t\geqslant 1}\mathbb{I}[\bar{n}^t = 0] \leqslant SAH$, and the third inequality holds because $\sum_{t\geqslant 1}\mathbb{I}[s_t=s]\theta^t \leqslant 1/(1-\gamma)$. For the second inequality, we note that $\bar{b}^t \leqslant 2\sqrt{H^2\iota/\bar{n}^t}$, it then follows that

$$\begin{aligned}\sum_{t\notin\mathcal{T}_1} w_t\bar{b}^t &\leqslant 2\sqrt{H^2\iota}\sum_{t\notin\mathcal{T}_1} w_t\sqrt{1/\bar{n}^t}\\ &= 2\sqrt{H^2\iota}\sum_{s,a}\sum_{t\notin\mathcal{T}_1}\mathbb{I}\left[(s_t,a_t)=(s,a)\right]w_t\sqrt{1/\bar{n}^t}.\end{aligned}$$

Let $\tilde{w}(s,a) = \sum_{t\notin\mathcal{T}_1} w_t\mathbb{I}\left[(s_t,a_t)=(s,a)\right]$. We fix $\tilde{w}(s,a)$ and consider to maximize

$$\sum_{t\notin\mathcal{T}_1}\mathbb{I}\left[(s_t,a_t)=(s,a)\right]w_t\sqrt{1/\bar{n}^t}.$$

Define $\bar{T}(j,s,a) := \{t \geqslant 1 | (s_t,a_t) = (s,a), \sum_{i=1}^{j-1}\bar{e}_j \leqslant N_t(s,a) < \sum_{i=1}^{j}\bar{e}_j\}$. Note that for each $j \geqslant 2$, $\sum_{t\notin\mathcal{T}_1, t\in\bar{T}(j,s,a)} w_t \leqslant (1+\frac{1}{H})C\bar{e}_{j-1}$. By rearrangement inequality we have that,

$$\begin{aligned}\sum_{t\notin\mathcal{T}_1}\mathbb{I}\left[(s_t,a_t)=(s,a)\right]w_t\sqrt{1/\bar{n}^t} &= \sum_{j\geqslant 2}\left(\sum_{t\notin\mathcal{T}_1, t\in\bar{T}(j,s,a)} w_t\right)\sqrt{1/\bar{e}_{j-1}}\\ &\leqslant C(1+\frac{1}{H})\sum_{j\geqslant 1}\sqrt{e_j}\mathbb{I}\left[\sum_{i=1}^{j-1}Ce_i \leqslant \tilde{w}(s,a)\right]\\ &\leqslant 10(1+\frac{1}{H})\sqrt{HC\tilde{w}(s,a)}.\end{aligned}$$

By Cauchy-Schwartz inequality, we obtain that

$$\sum_{t\notin\mathcal{T}_1} w_t\bar{b}^t \leqslant 20(1+\frac{1}{H})\sqrt{H^3C\iota}\sum_{s,a}\sqrt{\tilde{w}(s,a)} \leqslant 20(1+\frac{1}{H})\sqrt{SAH^3WC\iota}.$$

The proof is completed.

□

By Lemma 16 we derive that

$$\sum_{t<T_1} w_t\delta^t \leqslant \sum_{t<T_1} w'_t\delta^t + 2SACH^2 + 60\sqrt{SAH^3WC\iota}. \tag{64}$$

By iteratively unrolling (64) for $2H\ln(\frac{4H^2S}{\epsilon})$ times and setting the initial weights by $w_t = \mathbb{I}\left[s_t = s^{(1)}\right]$ so that $C = 1$ and $W = N_0$, we have

$$\sum_{t<T_1} \mathbb{I}\left[s_t = s^{(1)}\right]\delta^t \leqslant 2H\ln(\frac{4H^2S}{\epsilon})\left(2SAH^2 + 60\sqrt{SAH^3N_0\iota}\right) + \frac{\epsilon\sum_{t<T_1}\mathbb{I}\left[s_t = s^{(1)}\right]}{4HS}. \tag{65}$$

If $V_{T_1}(s^{(1)}) - V^*(s^{(1)}) > \frac{\epsilon}{2HS}$, then $\mathbb{I}\left[s_t = s^{(1)}\right]\delta^t > \frac{\epsilon}{2HS}$ for $t < T_1$ due to the fact that V_t is non-increasing in t, which implies that

$$\frac{\epsilon N_0}{4HS} \leqslant 2H\ln(\frac{4H^2S}{\epsilon})(2SAH^2 + 60\sqrt{SAH^3N_0\iota}), \tag{66}$$

which contradicts to the definition of N_0 ($N_0 = c_1\frac{SAH^5S^2\ln(\frac{4H^2S}{\epsilon})\iota}{\epsilon^2}$) . As a result, we have that $V_{T_1}(s^{(1)}) \leqslant V^*(s^{(1)}) + \frac{\epsilon}{2HS}$.

Induction step: Now suppose that $V_{T_i}(s^{(i)}) - V^*(s^{(i)}) \leqslant \frac{k\epsilon}{2HS}$ holds for all $1 \leqslant i \leqslant k$ for some $k \geqslant 1$. We will prove that $V_{T_{k+1}}(s^{(k+1)}) - V^*(s^{(k+1)}) \leqslant \frac{(k+1)\epsilon}{2HS}$ assuming that $T_{k+1} \neq \infty$.

Note that if $t < T_{k+1}$ and $T \in \mathcal{T}_1$, $\delta^t \leqslant \frac{k\epsilon}{2HS}$. It then follows that for non-negative weights $\{w_t\}_{t\geqslant 1}$ such that $\sup_{t<T_{k+1}} w_t \leqslant C$ and $\sum_{t<T_{k+1}} w_t \leqslant W$,

$$\begin{aligned}
\sum_{t<T_{k+1}} w_t\delta^t &\leqslant \sum_{t<T_{k+1}, t\notin\mathcal{T}_1} w_t\delta^t + \sum_{t<T_{k+1}, t\in\mathcal{T}_1}\frac{w_tk\epsilon}{2HS} \\
&\leqslant \sum_{t<T_{k+1}, t\notin\mathcal{T}_1}\left(\frac{w_t\mathbb{I}\left[\bar{n}^t = 0\right]}{1-\gamma} + 2w_t\bar{b}^t\right) + \sum_{t<T_{k+1}} w'_t(\delta^t + \theta^t) + \sum_{t<T_{k+1}, t\in\mathcal{T}_1}\frac{w_tk\epsilon}{2HS} && (67)\\
&\leqslant 2SACH^2 + 60\sqrt{SAH^3W_1} + \sum_{t<T_{k+1}} w'_t\delta^t + \sum_{t<T_{k+1}, t\in\mathcal{T}_1}\frac{w_tk\epsilon}{2HS} && (68)\\
&\leqslant 2SACH^2 + 60\sqrt{SAH^3W_1} + \sum_{t<T_{k+1}} w'_t\delta^t + \frac{(W-W_1)k\epsilon}{2HS}, && (69)
\end{aligned}$$

where $W_1 = \sum_{t<T_{k+1}, t\notin\mathcal{T}_1} w_t$ and $w'_t = \gamma\sum_{u<T_{k+1}, u\notin\mathcal{T}_1}\frac{1}{\bar{n}^u}\sum_{i=1}^{\bar{n}^t}\mathbb{I}\left[t = \bar{l}_i^u + 1\right]$. Here, Inequality (68) is by Lemma 16. Because $w'_t \leqslant (1 - \frac{1}{2H})C, \forall t \geqslant 1$ and $\sum_{t<T_{k+1}, t\notin\mathcal{T}_1} w'_t \leqslant (1 - \frac{1}{2H})W_1$, by iteratively applying (69) for $2H\ln(\frac{3H^2S}{\epsilon})$ times, we have that

$$\sum_{t<T_{k+1}} w_t\delta^t \leqslant 2H\ln(\frac{4H^2S}{\epsilon})\left(2SAH^2 + 60\sqrt{SAH^3N_0\iota}\right) + \frac{Wk\epsilon}{2HS} + \frac{W\epsilon}{4HS}. \tag{70}$$

If $V_{T_{k+1}}(s^{(k+1)}) - V^*(s^{(k+1)}) > \frac{(k+1)\epsilon}{2HS}$, choosing $w_t = \mathbb{I}\left[s_t = s^{(k+1)}, t < T_{k+1}\right]$ so that $C = 1$ and $W = N_0$ in (70), we obtain that

$$\frac{N_0(k+1)\epsilon}{2HS} \leqslant 2H\ln(\frac{4H^2S}{\epsilon})\left(2SAH^2 + 60\sqrt{SAH^3N_0\iota}\right) + \frac{N_0k\epsilon}{2HS} + \frac{N_0\epsilon}{4HS},$$

which again contradicts to the definition of N_0. Therefore we have proved that $V_{T_{k+1}}(s^{(k+1)}) - V^*(s^{(k+1)}) \leqslant \frac{(k+1)\epsilon}{2HS}$. □

Proof of Lemma 7. Let $\epsilon_1 \in [\epsilon, \frac{1}{1-\gamma}]$ be fixed. Let $\{w_t\}_{t\geqslant 1}$ be a non-negative sequence such that $\sup_{t\geqslant 1} w_t \leqslant C$ and $\sum_{t\geqslant 1} w_t \leqslant W$. Following the derivation of (64) we have that

$$\sum_{t\geqslant 1} w_t\delta^t = \sum_{t\geqslant 1, t\notin \mathcal{T}_1} w_t\delta^t + \sum_{t\geqslant 1, t\in\mathcal{T}_1} w_t\delta^t$$
$$\leqslant \sum_{t\geqslant 1, t\notin \mathcal{T}_1} w_t\delta^t + \frac{W_1\epsilon}{2H} \tag{71}$$
$$\leqslant \sum_{t\geqslant 1} w'_t\delta^t + 2SACH^2 + 60\sqrt{SAH^3WC\iota} + \frac{W_1\epsilon}{2H}. \tag{72}$$

where $\{w'_t\}_{t\geqslant 1} = \gamma\sum_{u\geqslant 1, u\notin\mathcal{T}_1}\frac{1}{\bar{n}^u}\sum_{i=1}^{\bar{n}^t}\mathbb{I}\left[t=\bar{l}_i^u+1\right]$ and $W_1=\sum_{t\in\mathcal{T}_1} w_t$. Similarly, it holds that $w'_t \leqslant (1-\frac{1}{2H})C, \forall t\geqslant 1$ and $\sum_{t\geqslant 1} w'_t \leqslant (1-\frac{1}{2H})(W-W_1)$. Here Inequality (71) holds by Lemma 15 and Inequality (72) holds by Lemma 16. Again by applying (72) iteratively for $2H\ln(\frac{4H}{\epsilon})$ times, we have that

$$\sum_{t\geqslant 1} w_t\delta^t \leqslant 2H\ln(\frac{4H}{\epsilon})\left(2SACH^2 + 60\sqrt{SAH^3WC\iota}\right) + \frac{W\epsilon}{2H} + \frac{W\epsilon}{4}. \tag{73}$$

By choosing $w_t = \mathbb{I}\left[\delta^t > \epsilon_1\right]$ so that $C=1$ and $W = N(\epsilon_1) := \sum_{t\geqslant 1}\mathbb{I}\left[\delta^t>\epsilon_1\right]$ into (73), we obtain that

$$\frac{N(\epsilon_1)\epsilon_1}{2} \leqslant 2H\ln(\frac{4H}{\epsilon})\left(2SAH^2 + 60\sqrt{SAH^3N(\epsilon_1)\iota}\right), \tag{74}$$

which means that $N(\epsilon_1)\leqslant O(\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon_1^2})$. The proof is completed. □

C.8. Proof of Lemma 9

Proof of Lemma 9. By Lemma 15, conditioned on the successful event E_1, for any t such that $N_t(s_t,a_t)\geqslant N_0$, it holds that $Q_t(s_t,a_t)-Q^*(s_t,a_t)\leqslant\frac{\epsilon}{2H}<\frac{3\epsilon}{4H}$, which implies that $\mathrm{clip}(Q_t(s_t,a_t)-Q^*(s_t,a_t),\frac{3\epsilon}{4H})=0$. □

D. Achieving Asymptotically Near-Optimal Sample Complexity

As mentioned in Section 4, in the UCB-MultiStage-Advantage algorithm, we set B to be a much larger value (indeed, $B=H^3$), an employ the reference-advantage decomposition variance reduction technique (Zhang et al., 2020), and re-design the exploration bonus $\check{b}$ to incorporate the Bernstein-type variance estimation. To prove Theorem 1 (the sample complexity bound for UCB-MultiStage-Advantage), in the analysis we split the error incurred due to the exploration bonus into two parts: the *bandit loss* $b_t^*(s_t,a_t)$ (defined in (79)) and the rest part that is due to the estimation variance of the real bandit loss. While the second part can be dealt with the variance reduction technique (Lemma 22), the bandit loss contributes the main $\tilde{O}(SAH^3\iota/\epsilon^2)$ term in the sample complexity (Lemma 21).

The rest of this section is organized as follows. In Appendix D.1, we present the details of the UCB-MultiStage-Advantage algorithm. In Appendix D.2, we prove Theorem 1, while the proofs of all technical lemmas are deferred to Appendix D.3.

D.1. The UCB-MultiStage-Advantage Algorithm

The UCB-MultiStage-Advantage algorithm (Algorithm 2) has almost the same updating structure as UCB-MultiStage. More specifically, the stopping condition and update triggers of UCB-MultiStage-Advantage are the same as that of UCB-MultiStage. The main difference between these two algorithms is 1) that UCB-MultiStage-Advantage utilized a more delicate exploration bonus with the help of a reference value function in the type-I updates; 2) we set $B=H^3$ in UCB-MultiStage-Advantage. Recall $\check{\mathcal{L}}=\{\sum_{i=1}^{j}\check{e}_i|1\leqslant j\leqslant\check{J}\}$ and $\bar{\mathcal{L}}=\{\sum_{i=1}^{j}\bar{e}_i|1\leqslant j\leqslant\bar{J}\}$.

The Statistics. Besides the statistics maintained in UCB-MultiStage, we let μ^{ref} and σ^{ref} be the accumulators of the reference value function and square of the reference value function respectively. Different from UCB-MultiStage, in UCB-MultiStage-Advantage we use $\check{\mu}$ and $\check{\sigma}$ denote respectively the accumulator of the advantage function and square of the advantage function in the current type-I stage.

Algorithm 2 UCB-MULTISTAGE-ADVANTAGE

Initialize: $\forall(s,a)\in\mathcal{S}\times\mathcal{A}$: $Q(s,a), Q^{\text{ref}}(s,a)\leftarrow\frac{1}{1-\gamma}$, $N(s,a), \check{N}(s,a), \bar{N}(s,a), \check{\mu}(s,a), \bar{\mu}(s,a)\leftarrow 0$;
for $t=1,2,3,\ldots$ **do**
Observe s_t;
Take action $a_t=\arg\max_a Q(s_t,a)$ and observe s_{t+1};
\\ *Maintain the statistics*
$(s,a,s')\leftarrow(s_t,a_t,s_{t+1})$;
$n:=N(s,a)\xleftarrow{+}1$; $\check{n}:=\check{N}(s,a)\xleftarrow{+}1$; $\bar{n}:=\bar{N}(s,a)\xleftarrow{+}1$;
$\check{\mu}:=\check{\mu}(s,a)\xleftarrow{+}V(s')-V^{\text{ref}}(s')$; $\mu^{\text{ref}}:=\mu^{\text{ref}}(s,a)\xleftarrow{+}V^{\text{ref}}(s')$; $\bar{\mu}:=\bar{\mu}(s,a)\xleftarrow{+}V(s')$;
$\check{\sigma}:=\check{\sigma}(s,a)\xleftarrow{+}(V(s')-V^{\text{ref}}(s'))^2$; $\sigma^{\text{ref}}:=\sigma^{\text{ref}}(s,a)\xleftarrow{+}(V^{\text{ref}}(s'))^2$;
\\ *Update triggered by a type-I stage*
if $n\in\check{\mathcal{L}}$ **then**

$$\check{b}\leftarrow\min\{2\sqrt{2}\left(\sqrt{\frac{\check{\sigma}/\check{n}-(\check{\mu}/\check{n})^2}{\check{n}}\iota}+\sqrt{\frac{\sigma^{\text{ref}}/n-(\mu^{\text{ref}}/n)^2}{n}\iota}\right)+7\left(\frac{H\iota^{3/4}}{n^{3/4}}+\frac{H\iota^{3/4}}{\check{n}^{3/4}}\right)+4\left(\frac{H\iota}{n}+\frac{H\iota}{\check{n}}\right),\frac{1}{1-\gamma}\}; \quad (75)$$

$$Q(s,a)\leftarrow\min\{r(s,a)+\gamma\big(\check{\mu}/\check{n}+\mu^{\text{ref}}/n+\check{b}\big),Q(s,a)\} \quad (76)$$

$$\check{N}(s,a)\leftarrow 0;\quad \check{\mu}(s,a)\leftarrow 0;\quad V(s)\leftarrow\max_a Q(s,a);$$

end if
\\ *Update triggered by a type-II stage*
if $n\in\bar{\mathcal{L}}$ **then**

$$\begin{aligned}&\bar{b}\leftarrow\min\{2\sqrt{H^2\iota/\bar{n}},1/(1-\gamma)\};\\&Q(s,a)\leftarrow\min\{r(s,a)+\gamma\big(\bar{\mu}/\bar{n}+\bar{b}\big),Q(s,a)\};\\&\bar{N}(s,a)\leftarrow 0;\quad\bar{\mu}(s,a)\leftarrow 0;\quad V(s)\leftarrow\max_a Q(s,a);\end{aligned} \quad (77)$$

end if
if $\sum_{a'}N(s,a')=N_1$ **then** $V^{\text{ref}}(s)\leftarrow V(s)$; {*Learn the reference value function*}
end for

D.2. Proof of Theorem 1

We start from showing that the Q function is optimistic and non-increasing.

Proposition 17. *With probability $\left(1-SA\left(4\check{J}(2\log_2(N_0H)+1)+\bar{J}\right)p\right)$, it holds that $Q_t(s,a)\geqslant Q^*(s,a)$ and $Q_{t+1}(s,a)\leqslant Q_t(s,a)$ for any $t\geqslant 1$ and $(s,a)\in\mathcal{S}\times\mathcal{A}$.*

In the proof of Proposition 17 in Appendix D.3.1, we introduce the desired event E_2 by (89). Moreover, we use $\overline{E}_2$ to denote the complement event of E_2. As will be shown later in (92), we have

$$\mathbb{P}\left[E_2\right]\geqslant\left(1-SA\left(4\check{J}(2\log_2(N_0H)+1)+\bar{J}\right)p\right),$$

and thus

$$\mathbb{P}\left[\overline{E}_2\right]\leqslant SA\left(4\check{J}(2\log_2(N_0H)+1)+\bar{J}\right)p.$$

The analysis will be done assuming the successful event E_2 throughout the rest of this section.

Since the type-II stages in UCB-MULTISTAGE-ADVANTAGE are exactly the same as that in UCB-MULTISTAGE, using the the same way as in the proof of Lemma 7, we can prove the following lemma (and the proof is omitted).

Lemma 18. *Conditioned on E_2, for any $\epsilon_1\in[\epsilon,\frac{1}{1-\gamma}]$, it holds that*

$$\sum_{t=1}^{\infty}\mathbb{I}\left[V_t(s_t)-V^*(s_t)\geqslant\epsilon_1\right]\leqslant\sum_{t=1}^{\infty}\mathbb{I}\left[Q_t(s_t,a_t)-Q^*(s_t,a_t)\geqslant\epsilon_1\right]\leqslant O\left(\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon_1^2}\right).$$

Recall that $\mathcal{T}_1=\{t|N_t(s_t,a_t)>N_0\}$. Similar as Lemma 15, we have that (the proof is omitted)

Lemma 19. *Conditioned on successful event E_2 , it holds that for any $t \in \mathcal{T}_1$ (if $\mathcal{T}_1$ is not empty)*

$$Q_t(s_t, a_t) - Q^*(s_t, a_t) \leqslant \frac{\epsilon}{2H}.$$

Define λ_t to be the vector such that $\lambda_t(s) = \mathbb{I}\left[\sum_a N_t(s,a) < N_1\right]$ where $N_1 := c_{10} SAH^5 B \ln(\frac{4H}{\epsilon})\iota$ for some large enough constant c_{10}. By Lemma 18, $\lambda_t(s) = 0$ implies that $V_t^{\mathrm{ref}}(s) = V^{\mathrm{REF}}(s)$.

We then show that the Bellman error of the Q-function is properly bounded.

Lemma 20. *Define $l_i(s,a)$ to be the time the i-th visit of (s,a) occurs and $\bar{N}_t(s,a)$ to be the visit count of (s,a) before the current stage of (s,a). Conditioned on E_2, it holds that*

$$Q_t(s,a) - r(s,a) - P_{s,a} V_t \leqslant P_{s,a}(V_{\underline{\rho}_t}(s,a) - V_t) + P_{s,a}\tilde{\lambda}_t(s,a) \tag{78}$$

for any $t \geqslant 1$ and any $(s,a) \in \mathcal{S} \times \mathcal{A}$, where

$$\tilde{\lambda}_t(s,a) := \frac{1}{1-\gamma}\left(\frac{1}{\bar{N}_t(s,a)} \sum_{i=1}^{\bar{N}_t(s,a)} \lambda_{l_i(s,a)}\right).$$

The proof of Lemma 20 is given in Section D.3.2. We now define the bandit loss

$$b_t^*(s,a) := \min\{2\sqrt{2}\sqrt{\frac{\mathbb{V}(P_{s,a}, V^*)\iota}{n_t(s,a)}}, \frac{1}{1-\gamma}\}. \tag{79}$$

By (78), with the definition that $\tilde{w}_t(s,a) := w_t(s,a) \cdot \mathbb{I}[N_t(s,a) < N_0]$ we can show that

$$\begin{aligned}
&V_t(s) - V^{\pi_t}(s) \\
&\leqslant \sum_{s,a} \tilde{w}_t(s,a)\left(2\check{b}_t(s,a) + P_{s,a}\tilde{\lambda}_t(s,a) + \gamma P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t)\right) \\
&\qquad + \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) \geqslant N_0] \cdot (Q_t(s,a) - Q^*(s,a)) + \frac{\epsilon}{8} \\
&= 2\sum_{s,a} \tilde{w}_t(s,a) b_t^*(s,a) + 2\sum_{s,a}\tilde{w}_t(s,a)(\check{b}_t(s,a) - b_t^*(s,a)) + \gamma\sum_{s,a}\tilde{w}_t(s,a)P_{s,a}(V_{\underline{\rho}_t(s,a)} - V_t) \\
&\qquad + \sum_{s,a}\tilde{w}_t(s,a)P_{s,a}\tilde{\lambda}_t(s,a) \\
&\qquad + \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) \geqslant N_0] \cdot (Q_t(s,a) - Q^*(s,a)) + \frac{\epsilon}{8} \\
&\leqslant 2\sum_{s,a}\tilde{w}_t(s,a)b_t^*(s,a) + 2\sum_{s,a}\tilde{w}_t(s,a)\mathrm{clip}(\check{b}_t(s,a) - b_t^*(s,a), \frac{\epsilon}{16H}) \\
&\qquad + \gamma\sum_{s,a}\tilde{w}_t(s,a)P_{s,a}\mathrm{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{16H}) + \sum_{s,a}\tilde{w}_t(s,a)P_{s,a}\mathrm{clip}(\tilde{\lambda}_t(s,a), \frac{\epsilon}{16H}) \\
&\qquad + \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) \geqslant N_0] \cdot \mathrm{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}) + \frac{7\epsilon}{8} &(80)\\
&= 2\sum_{s,a}\tilde{w}_t(s,a)b_t^*(s,a) + \beta_t + \frac{7\epsilon}{8}. &(81)
\end{aligned}$$

where we re-define β_t as follows.

$$\begin{aligned}
\beta_t := \sum_{s,a}\tilde{w}_t(s,a)&\left(2\mathrm{clip}(\check{b}_t(s,a) - b_t^*(s,a), \frac{\epsilon}{16H}) + \gamma P_{s,a}\mathrm{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{16H})\right. \\
&\qquad\qquad \left. + P_{s,a}\mathrm{clip}(\tilde{\lambda}_t(s,a), \frac{\epsilon}{16H})\right) \\
&+ \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) \geqslant N_0] \cdot \mathrm{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}).
\end{aligned}$$

Plugging in the definition of $\tilde{w}_t$, we get that

$$\beta_t = \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) < N_0]\Big(2\text{clip}(\check{b}_t(s,a) - b_t^*(s,a), \frac{\epsilon}{16H}) + \gamma P_{s,a}\text{clip}(V_{\underline{\rho}_t(s,a)} - V_t, \frac{\epsilon}{16H}) + P_{s,a}\text{clip}(\tilde{\lambda}_t(s,a), \frac{\epsilon}{16H})\Big) + \sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) \geqslant N_0] \cdot \text{clip}(Q_t(s,a) - Q^*(s,a), \frac{3\epsilon}{4H}). \tag{82}$$

We also re-define the following notations,

$$\begin{aligned}
\alpha_t &:= \mathbb{I}[N_t(s_t,a_t) < N_0] P_{s_t,a_t}\text{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t, \frac{\epsilon}{16H}), \\
\upsilon_t &:= \mathbb{I}[N_t(s_t,a_t) < N_0] P_{s,a}\text{clip}(\tilde{\lambda}_t(s,a), \frac{\epsilon}{16H}), \\
\tilde{\beta}_t &:= \mathbb{I}[N_t(s_t,a_t) < N_0] \cdot \Big(2\text{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) + P_{s_t,a_t}\text{clip}(V_{\underline{\rho}_t(s_t,a_t)} - V_t, \frac{\epsilon}{16H}) \\
&\qquad + P_{s,a}\text{clip}(\tilde{\lambda}_t(s,a), \frac{\epsilon}{16H})\Big) \\
&\qquad + \mathbb{I}[N_t(s_t,a_t) \geqslant N_0] \cdot \text{clip}(Q_t(s_t,a_t) - Q^*(s_t,a_t), \frac{3\epsilon}{4H}).
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\tilde{\beta}_t = {} & \mathbb{I}[N_t(s_t,a_t) < N_0] \cdot 2\text{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) + \alpha_t + \upsilon_t \\
& + \mathbb{I}[N_t(s_t,a_t) \geqslant N_0] \cdot \text{clip}(Q_t(s_t,a_t) - Q^*(s_t,a_t), \frac{3\epsilon}{4H}).
\end{aligned}$$

To handle the first term in **RHS** of (80), we prove that

Lemma 21. *Define* $\Lambda = \left\lceil \log_2(\frac{256H^4}{\epsilon^2}) \right\rceil$. *With probability* $(1 - 2H\Lambda p)$*, it holds that*

$$\sum_{t \geqslant 1} \mathbb{I}\left[\sum_{s,a} w_t(s,a)\mathbb{I}[N_t(s,a) < N_0] b_t^*(s,a) \geqslant \frac{\epsilon}{16}\right] \leqslant O\left(\frac{SAH^3\Lambda^3\iota}{\epsilon^2} + \frac{SAH^4B\Lambda^2\ln(N_0)}{\epsilon}\right).$$

We remark that our proof of Lemma 21 is quite similar to the method of *knowness* in (Lattimore & Hutter, 2012), in the sense that both methods rely on an argument based on the partition of the states. However, our way of partitioning seems to be simpler as we divide the states into different subsets only according to their numbers. The detailed proof is presented in Appendix D.3.3.

For the second term, in Appendix D.3.4, we prove the pseudo-regret bounds as below.

Lemma 22. *If we choose* $B = H^3$*, with probability* $1 - SA\check{J}(2\mathbb{P}[\overline{E}_2] + 4p)$ *it holds that*

$$\begin{aligned}
&\sum_{t \geqslant 1} \text{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) \\
&\leqslant O\left(\frac{SAH^2\iota}{\epsilon}\right) + \tilde{O}\left(\frac{S^{3/2}A^{3/2}H^{17/4}\iota}{\epsilon^{1/2}} + \frac{SAH^{59/12}\iota}{\epsilon^{1/3}} + \frac{S^{5/4}A^{5/4}H^3\iota}{\epsilon^{1/4}} + S^2A^2H^9\iota\right).
\end{aligned}$$

Following the same arguments as the proof of Lemma 8, for the third term we show the following lemma (the proof of which is omitted).

Lemma 23. *With probability* $1 - (\mathbb{P}\left[\overline{E}_2\right] + p)$ *it holds that*

$$\sum_{t \geqslant 1} \alpha_t \leqslant O\left(\frac{SAH^5\ln(\frac{4H}{\epsilon})\iota}{\epsilon B} + SABH^3 + SAH\ln(N_0)\right).$$

Finally, in Appendix D.3.5, we show the following lemma.

Lemma 24. *With probability* $1-(\mathbb{P}\left[\overline{E}_2\right]+p)$*, it holds that*

$$\sum_{t\geqslant 1}\upsilon_t \leqslant O\left(\frac{H^2S(N_1+1)}{\epsilon}\right).$$

Similarly to the proof of Lemma 19, we also have the following lemma.

Lemma 25. *With probability* $1-(\mathbb{P}\left[\overline{E}_2\right]+p)$*, for any* t *it holds that*

$$\mathbb{I}[N_t(s_t,a_t)\geqslant N_0]\cdot \mathrm{clip}(Q_t(s_t,a_t)-Q^*(s_t,a_t),\frac{3\epsilon}{4H})=0.$$

By Lemmas 22, 23, 24 and 25, we obtain that

Lemma 26. *With probability* $1-\left(SA\check{J}(2\mathbb{P}[\overline{E}_2]+4p)+3\mathbb{P}\left[\overline{E}_2\right]+3p\right)$*, it holds that*

$$\sum_{t\geqslant 1}\tilde{\beta}_t \leqslant O\left(\frac{SAH^2\ln(\frac{4H}{\epsilon})\iota}{\epsilon}\right)+\tilde{O}\left(\frac{S^2A^2H^{10}\iota}{\epsilon^{1/2}}\right). \tag{83}$$

Following the same arguments in Section 5.4, we obtain that with probability

$$1-\left(SA\check{J}(2\mathbb{P}[\overline{E}_2]+4p)+3\mathbb{P}\left[\overline{E}_2\right]+34p\right),$$

it holds that

$$\sum_{t\geqslant 1}\mathbb{I}\left[\beta_t>\frac{\epsilon}{8}\right]\leqslant O(\frac{SAH^2\ln(\frac{4H}{\epsilon})\iota}{\epsilon^2})+\tilde{O}(\frac{S^2A^2H^{10}\iota}{\epsilon^{3/2}}). \tag{84}$$

By Proposition 17,(81) and (84), we conclude that with probability $1-\left(SA\check{J}(2\mathbb{P}[\overline{E}_2]+4p)+3\mathbb{P}\left[\overline{E}_2\right]+2H\Lambda p+3p\right)$, it holds that

$$\begin{aligned}
&\sum_{t\geqslant 1}\mathbb{I}\left[V^*(s_t)-V^{\pi_t}(s_t)>\epsilon\right]\\
&\leqslant \sum_{t\geqslant 1}\mathbb{I}\left[\sum_{s,a}w_t(s,a)b_t^*(s,a)>\frac{\epsilon}{8}\right]+\sum_{t\geqslant 1}\mathbb{I}\left[\beta_t>\frac{\epsilon}{4}\right]\\
&\leqslant O\left(\frac{SAH^3\Lambda^2\ln(\frac{4H}{\epsilon})\iota}{\epsilon^2}\right)+O\left(\frac{SAH^7\Lambda^2\ln(N_0)}{\epsilon}\right)+\tilde{O}\left(\frac{S^2A^2H^{10}\iota}{\epsilon^{3/2}}\right).
\end{aligned}$$

The proof is finished by replacing p with $\frac{p}{34S^2A^2\check{J}^2\log_2(N_0H)+4H\Lambda}$.

D.3. Missing Proofs in Appendix D.2

D.3.1. Proof of Proposition 17

Proposition 17 (restated). *With probability* $\left(1-SA\left(4\check{J}(2\log_2(N_0H)+1)+\bar{J}\right)p\right)$*, it holds that* $Q_t(s,a)\geqslant Q^*(s,a)$ *and* $Q_{t+1}(s,a)\leqslant Q_t(s,a)$ *for any* $t\geqslant 1$ *and* $(s,a)\in\mathcal{S}\times\mathcal{A}$. The rest of this subsection is devoted to the proof of Proposition 17.

Let (s,a,j) be fixed. Let $\underline{\mu}^{\mathrm{ref}}$, $\underline{\check{\mu}}$, $\underline{\sigma}^{\mathrm{ref}}$, $\underline{\check{\sigma}}$ and $\underline{\check{b}}$ be the values of μ^{ref}, $\check{\mu}$, σ^{ref}, $\check{\sigma}$ and $\check{b}$ in (76) in the j-th type-I update. Define $\check{l}_i$ to be the time when the i-th visit in the j-th type-I stage of (s,a) occurs and l_i to be the time the i-th visit of (s,a) occurs respectively. Let $\check{n}$ and n be the shorthands of $\check{e}_j$ and $\sum_{i=1}^{j}\check{e}_i$ respectively.

Define

$$\chi_1^{(j)}(s,a) := \frac{1}{n}\sum_{i=1}^{n}\left(V_{l_i}^{\mathrm{ref}}(s_{l_i+1}) - P_{s,a}V_{l_i}^{\mathrm{ref}}\right);$$

$$\chi_2^{(j)}(s,a) := \frac{1}{\check{n}}\sum_{i=1}^{\check{n}}\left(W_{\check{l}_i}(s_{\check{l}_i+1}) - P_{s,a}W_{\check{l}_i}\right).$$

We consider the events:

$$\check{E}_1^{(j)}(s,a) := \left\{|\chi_1^{(j)}(s,a)| \leqslant 2\sqrt{2}\sqrt{\frac{\underline{\sigma}^{\mathrm{ref}}/n - (\underline{\mu}^{\mathrm{ref}}/n)^2}{n}\iota} + \frac{7H\iota^{3/4}}{n^{3/4}} + \frac{4H\iota}{n}\right\}$$

and

$$\check{E}_2^{(j)}(s,a) := \left\{|\chi_2^{(j)}(s,a)| \leqslant 2\sqrt{2}\sqrt{\frac{\underline{\check{\sigma}}/\check{n} - (\underline{\check{\mu}}/\check{n})^2}{\check{n}}\iota} + \frac{7H\iota^{3/4}}{\check{n}^{3/4}} + \frac{4H\iota}{\check{n}}\right\},$$

where $W_t = V_t - V_t^{\mathrm{ref}}$. If both $\check{E}_1^{(j)}(s,a)$ and $\check{E}_2^{(j)}(s,a)$ occurs, then we have that

$$\begin{aligned}
& r(s,a) + \frac{\underline{\mu}^{\mathrm{ref}}}{n} + \frac{\underline{\check{\mu}}}{\check{n}} + \underline{\check{b}} \\
&= r(s,a) + P_{s,a}\left(\frac{1}{n}\sum_{i=1}^{n} V_{l_i}^{\mathrm{ref}}\right) + P_{s,a}\left(\frac{1}{\check{n}}\sum_{i=1}^{\check{l}_i}(V_{\check{l}_i} - V_{\check{l}_i}^{\mathrm{ref}})\right) + \chi_1^{(j)}(s,a) + \chi_2^{(j)}(s,a) + \underline{\check{b}} \\
&\geqslant r_h(s,a) + P_{s,a}\left(\frac{1}{\check{n}}\sum_{i=1}^{\check{l}_i} V_{\check{l}_i}\right) + \chi_1^{(j)}(s,a) + \chi_2^{(j)}(s,a) + \underline{\check{b}} \qquad (85) \\
&\geqslant r_h(s,a) + P_{s,a}\left(\frac{1}{\check{n}}\sum_{i=1}^{\check{l}_i} V_{\check{l}_i}\right), \qquad (86)
\end{aligned}$$

where Inequality (85) holds by the fact V_t^{ref} is non-increasing in t and Inequality (86) follows by the definition of $\underline{\check{b}}$.

On the other hand, for the j'-th type-II update, we consider the following same events as in the proof of Proposition 4,

$$\bar{E}^{(j')}(s,a) = \left\{\frac{1}{\bar{e}_{j'}}\sum_{i=1}^{\bar{e}_{j'}} V^*(s_{\bar{l}_i+1}) + \bar{b}^{(j)} \geqslant P_{s,a}V^*\right\}. \qquad (87)$$

Assuming $\bar{E}^{(j')}(s,a)$ holds, we then have

$$\begin{aligned}
& r(s,a) + \frac{\gamma}{\bar{e}_{j'}}\sum_{i=1}^{\bar{e}_{j'}} V_{\bar{l}_i}(s_{\bar{l}_i+1}) + \bar{b}^{(j)} \\
&\geqslant r(s,a) + \gamma P_{s,a}V^* + \gamma\left(\frac{1}{\bar{e}_{j'}}\sum_{i=1}^{\bar{e}_{j'}}(V_{\bar{l}_i}(s_{\bar{l}_i+1}) - V^*(s_{\bar{l}_i+1}))\right). \qquad (88)
\end{aligned}$$

Let

$$E_2 = (\cap_{s,a,j}\check{E}_1^{(j)}(s,a)) \cap (\cap_{s,a,j}\check{E}_2^{(j)}(s,a)) \cap (\cap_{s,a,j'}\bar{E}^{(j')}(s,a)). \qquad (89)$$

Assuming E_2 holds, by the update rule (76) and (77) and noting that V_t is non-increasing , for any $t \geqslant 2$ and (s,a), it holds either $Q_t(s,a) = Q_{t-1}(s,a)$ or

$$Q_t(s,a) \geqslant r_{s,a} + \gamma P_{s,a}V^* + \sum_{t'<t} v_{t'}(V_{t'} - V^*)$$

for some non-negative S-dimensional vectors $v_1, v_2, \ldots, v_{t-1}$. Noting that $Q_1(s,a) = \frac{1}{1-\gamma} \geqslant Q^*(s,a)$ for any (s,a), the conclusion follows easily by induction.

Therefore, it suffices to bound $\mathbb{P}\left[E_2\right]$.

Lemma 27. *For any* (s,a,j), $\mathbb{P}\left[\check{E}_1^{(j)}(s,a)\right] \geqslant 1 - 2(\log_2(N_0 H)+1)p$.

Proof. Define $\mathbb{V}(x,y) = xy^2 - (xy)^2$ for two vectors with the same dimension. Noticing that s_{l_i+1} is independent of $V^{\mathrm{ref}}_{l_i}$ conditioned on $\mathcal{F}_{l_i-1}$, by Lemma 13 with $\epsilon = H$, we have that with probability $(1 - 2\log_2(nH)p)$, it holds that

$$
\begin{aligned}
|\chi_1^{(j)}(s,a)| &= \left| \frac{1}{n}\sum_{i=1}^{n} \left(V^{\mathrm{ref}}_{l_i}(s_{l_i+1}) - P_{s,a}V^{\mathrm{ref}}_{l_i}\right) \right| \\
&\leqslant 2\sqrt{2}\sqrt{\frac{(\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V^{\mathrm{ref}}_{l_i}))\iota}{n^2}} + \frac{\sqrt{2H\iota}}{n} + \frac{2H\iota}{n} \\
&\leqslant 2\sqrt{2}\sqrt{\frac{(\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V^{\mathrm{ref}}_{l_i}))\iota}{n^2}} + \frac{4H\iota}{n}. \qquad (90)
\end{aligned}
$$

By definition of $\underline{\sigma}^{\mathrm{ref}}$ and $\underline{\mu}^{\mathrm{ref}}$, we have that

$$
\begin{aligned}
\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V^{\mathrm{ref}}_{l_i}) &= \sum_{i=1}^{n}\left(P_{s,a}(V^{\mathrm{ref}}_{l_i})^2 - (P_{s,a}V^{\mathrm{ref}}_{l_i})^2\right) \\
&= \sum_{i=1}^{n}(V^{\mathrm{ref}}_{l_i}(s_{l_i+1}))^2 - \frac{1}{n}\left(\sum_{i=1}^{n}V^{\mathrm{ref}}_{l_i}(s_{l_i+1})\right)^2 + \chi_3 + \chi_4 + \chi_5 \\
&= \underline{\sigma}^{\mathrm{ref}} - \frac{1}{n}(\underline{\mu}^{\mathrm{ref}})^2 + \chi_3 + \chi_4 + \chi_5,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_3 &:= \sum_{i=1}^{n}\left(P_{s,a}(V^{\mathrm{ref}}_{l_i})^2 - (V^{\mathrm{ref}}_{l_i}(s_{l_i+1}))^2\right) \\
\chi_4 &:= \frac{1}{n}\left(\sum_{i=1}^{n}V^{\mathrm{ref}}_{l_i}(s_{l_i+1})\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{n}P_{s,a}V^{\mathrm{ref}}_{l_i}\right) \\
\chi_5 &= \frac{1}{n}\left(\sum_{i=1}^{n}P_{s,a}V^{\mathrm{ref}}_{l_i}\right)^2 - \sum_{i=1}^{n}(P_{s,a}V^{\mathrm{ref}}_{l_i})^2.
\end{aligned}
$$

By Azuma's inequality, we have that

$$
\mathbb{P}\left[|\chi_3| > H^2\sqrt{2n\iota}\right] \leqslant p
$$

and

$$
\mathbb{P}\left[|\chi_4| > 2H^2\sqrt{2n\iota}\right] \leqslant \mathbb{P}\left[2H\cdot\left|\sum_{i=1}^{n}\left(V^{\mathrm{ref}}_{l_i}(s_{l_i+1}) - P_{s,a}V^{\mathrm{ref}}_{l_i}\right)\right| > 2H^2\sqrt{2n\iota}\right] \leqslant p.
$$

On the other hand, by Cauchy-Schwartz inequality, we have $\chi_5 \leqslant 0$. It then follow that

$$
\mathbb{P}\left[\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V^{\mathrm{ref}}_{l_i}) > \underline{\sigma}^{\mathrm{ref}} - \frac{1}{n}(\underline{\mu}^{\mathrm{ref}})^2 + 5H^2\sqrt{n\iota}\right] \leqslant 2p. \qquad (91)
$$

Combining (90) and (91), we have that

$$\mathbb{P}\left[\check{E}_1^{(j)}(s,a)\right] \geqslant 1-\mathbb{P}\left[|\chi_1^{(j)}(s,a)| > 2\sqrt{2}\sqrt{\frac{(\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V_{l_i}^{\mathrm{ref}}))\iota}{n^2}}+\frac{4H\iota}{n}\right]$$
$$-\mathbb{P}\left[\sum_{i=1}^{n}\mathbb{V}(P_{s,a},V_{l_i}^{\mathrm{ref}}) > \underline{\sigma}^{\mathrm{ref}}-\frac{1}{n}(\underline{\mu}^{\mathrm{ref}})^2+5H^2\sqrt{n\iota}\right]$$
$$\geqslant 1-2(\log_2(nH)+1)p$$
$$\geqslant 1-2(\log_2(N_0H)+1)p.$$

□

Following similar arguments as above, we can prove that $\mathbb{P}\left[\check{E}_2^{(j)}(s,a)\right] \geqslant 1-2(\log_2(N_0H)+1)p$ for any $1 \leqslant j \leqslant \check{J}$. At last, by Azuma's inequality, $\mathbb{P}\left[\bar{E}^{(j')}(s,a)\right] \geqslant 1-p$ for any j' and (s,a). Via a union bound over $1 \leqslant j \leqslant \check{J}$ and $1 \leqslant j' \leqslant \bar{J}$, we obtain that

$$\mathbb{P}\left[E_2\right] \geqslant 1-4SA\check{J}(\log_2(N_0H)+1)p-SA\bar{J}p. \tag{92}$$

The proof is completed.

D.3.2. PROOF OF LEMMA 20

Lemma 20 (restated). Define $l_i(s,a)$ to be the time the i-th visit of (s,a) occurs and $\bar{N}_t(s,a)$ to be the visit count of (s,a) before the current stage of (s,a). Conditioned on E_2, it holds that

$$Q_t(s,a)-r(s,a)-P_{s,a}V_t \leqslant P_{s,a}(V_{\underline{\rho}_t}(s,a)-V_t)+\frac{1}{1-\gamma}P_{s,a}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_{l_i}\right).$$

for any $t \geqslant 1$ and any $(s,a) \in \mathcal{S}\times\mathcal{A}$.

Let (s,a,j) be fixed. We use the same notations as that of Section D.3.1. For any t in the $j+1$-th type-I stage, by the arguments to derive (86), we have that

$$Q_t(s,a) = r(s,a)+\frac{\underline{\mu}^{\mathrm{ref}}}{n}+\frac{\check{\underline{u}}}{\check{n}}+\check{\underline{b}}$$
$$\leqslant r(s,a)+P_{s,a}\left(\frac{1}{n}\sum_{i=1}^{n}V_{l_i}^{\mathrm{ref}}\right)+P_{s,a}\left(\frac{1}{\check{n}}\sum_{i=1}^{\check{l}_i}(V_{\check{l}_i}-V_{\check{l}_i}^{\mathrm{ref}})\right)$$
$$\leqslant r(s,a)+P_{s,a}V_t+P_{s,a}(V_{\underline{\rho}_t}(s,a)-V_t)+P_{s,a}\left(\frac{1}{n}\sum_{i=1}^{n}V_{l_i}^{\mathrm{ref}}-V^{\mathrm{REF}}\right)$$
$$\leqslant r(s,a)+P_{s,a}V_t+P_{s,a}(V_{\underline{\rho}_t}(s,a)-V_t)+\frac{1}{1-\gamma}P_{s,a}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_{l_i}\right). \tag{93}$$

The proof is completed.

D.3.3. PROOF OF LEMMA 21

Lemma 21 (restated). *Define* $\Lambda = \left\lceil \log_2(\frac{256H^4}{\epsilon^2})\right\rceil$. *With probability* $(1-2H\Lambda p)$, *it holds that*

$$\sum_{t\geqslant 1}\mathbb{I}\left[\sum_{s,a}w_t(s,a)\mathbb{I}[N_t(s,a)<N_0]b_t^*(s,a) > \frac{\epsilon}{8}\right] \leqslant O\left(\frac{SAH^3\Lambda^3\iota}{\epsilon^2}+\frac{SAH^4B\Lambda^2\ln(N_0)}{\epsilon}\right).$$

The rest of this subsection is devoted to the proof of Lemma 21.

Define $\mathcal{S}_{t,0} := \{(s,a)|n_t(s,a) < \iota\}$, $\mathcal{S}_{t,u} := \{(s,a)|2^{u-1}\iota \leqslant n_t(s,a) < 2^u\iota\}$ for $u = 1, 2, \ldots, \Lambda = \lceil \log_2(\frac{256H^4}{\epsilon^2})\rceil$ and $\overline{\mathcal{S}}_t := \{(s,a)|n_t(s,a) > \frac{H^4}{\epsilon^2}\}$. Furthermore, we define

$$\beta^*_{t,u} := \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a) b^*_t(s,a)$$

and

$$\beta^*_t := \sum_u \beta^*_{t,u} = \sum_{s,a} w_t(s,a) b^*_t(s.a).$$

By the definition of $b^*_t(s,a)$, we obtain that for $1 \leqslant u \leqslant \Lambda$,

$$\begin{aligned}
\beta^*_{t,i} &= \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a) b^*_t(s,a) \\
&\leqslant 2\sqrt{2\iota} \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a) \sqrt{\frac{\mathbb{V}(P_{s,a}, V^*)}{n_t(s,a)}} \\
&\leqslant 2\sqrt{\frac{2}{2^{u-1}}} \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a) \sqrt{\mathbb{V}(P_{s,a}, V^*)} \\
&\leqslant 2\sqrt{\frac{2}{2^{u-1}}} \cdot \sqrt{\sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a)} \cdot \sqrt{\sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a)\mathbb{V}(P_{s,a}, V^*)}, \qquad (94)
\end{aligned}$$

and for $0 \leqslant u \leqslant \Lambda$,

$$\beta^*_{t,u} \leqslant \frac{1}{1-\gamma} \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a).$$

Define $w_{t,u} := \sum_{(s,a)\in\mathcal{S}_{t,u}} w_t(s,a)$ and $\nu_t = \sum_{s,a} w_t(s,a)\mathbb{V}(P_{s,a}, V^*)$. Note that

$$\begin{aligned}
\nu_t &= \sum_{s,a} w_t(s,a)(P_{s,a}(V^*)^2 - (P_{s,a}V^*)^2) \\
&= \sum_{s,a} w_t(s,a) P_{s,a}(V^*)^2 - \frac{1}{\gamma^2}\sum_{s,a} w_t(s,a)(Q^*(s,a) - r(s,a))^2 \\
&\leqslant \sum_{s,a} w_t(s,a) P_{s,a}(V^*)^2 - \sum_{s,a} w_t(s,a)(Q^*(s,a) - r(s,a))^2 \\
&\leqslant \sum_{s,a} w_t(s,a)(P_{s,a}(V^*)^2 - (Q^*(s,a))^2) + \frac{2H}{1-\gamma} \\
&= \sum_{s,a} w_t(s,a)(P_{s,a}(V^*)^2 - (V^*(s))^2) + \sum_{s,a} w_t(s,a)((V^*(s))^2 - (Q^*(s,a))^2)) + \frac{2H}{1-\gamma} \\
&\leqslant \sum_{s,a} w_t(s,a)(P_{s,a}(V^*)^2 - (V^*(s))^2) + \frac{2}{1-\gamma}\sum_{s,a} w_t(s,a)(V^*(s) - Q^*(s,a)) + \frac{2H}{1-\gamma} \\
&\leqslant \frac{1}{(1-\gamma)^2} + \frac{2}{1-\gamma}\sum_{s,a} w_t(s,a)(V^*(s) - Q^*(s,a)) + \frac{2H}{1-\gamma} \qquad (95) \\
&\leqslant \frac{1}{(1-\gamma)^2} + \frac{2}{(1-\gamma)}(V^*(s_t) - V^{\pi_t}(s_t)) + \frac{2H}{1-\gamma} \qquad (96) \\
&\leqslant 5H^2. \qquad (97)
\end{aligned}$$

Here Inequality (95) holds by the fact that

$$\sum_{s,a} w_t(s,a)(P_{s,a}-\mathbf{1}_s)(V^*)^2 = \sum_{s,a}\left(\mathbb{I}\left[a=\pi_t(s)\right]\sum_{i=0}^{H-1}\mathbf{1}_{s_t}^\top(\gamma P_{\pi_t})^i\mathbf{1}_s\right)\cdot(P_{s,a}-\mathbf{1}_s)(V^*)^2$$
$$= \sum_{s,a}\mathbb{I}\left[a=\pi_t(s)\right]\left(\mathbf{1}_{s_t}^\top(\gamma P_{\pi_t})^H\mathbf{1}_s - \mathbb{I}\left[s=s_t\right]\right)(V^*(s))^2$$
$$\leqslant \frac{1}{(1-\gamma)^2},$$

and Inequality (96) is due to the bound on the following telescoping sum,

$$V^*(s_t)-V^{\pi_t}(s_t) = \sum_{s,a}\left(\mathbb{I}\left[a=\pi_t(s)\right]\sum_{i=0}^{\infty}\mathbf{1}_{s_t}^\top(\gamma P_{\pi_t})^i\mathbf{1}_s\right)\cdot(V^*(s)-Q^*(s,a))$$
$$\geqslant \sum_{s,a} w_t(s,a)(V^*(s)-Q^*(s,a)).$$

Combining (97) with the fact that $\sum_{(s,a)\in\overline{\mathcal{S}}_t} w_t(s,a)b_t^*(s,a)\leqslant\frac{\epsilon}{16}$, we obtain that , if $\beta_t^*>\frac{\epsilon}{8}$, there exists u such that $\beta_{t,u}^*>\frac{\epsilon}{16\Lambda}$, which implies that $w_{t,u}>\max\{\frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2},\frac{\epsilon(1-\gamma)}{16\Lambda}\}$.

We will bound the number of steps in which there exists u satisfying $w_{t,u}>\max\{\frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2},\frac{\epsilon(1-\gamma)}{16\Lambda}\}$ by following lemma.

Lemma 28. *For any $k\in\{1,2,\ldots,H\}$ and $u\in\{1,2,\ldots,\Lambda\}$, with probability $1-p$,*

$$\sum_{t\geqslant 0}\mathbb{I}\left[w_{tH+k,u}>\frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2}\right]\leqslant O\left(\frac{SABH^4\Lambda^2\ln(N_0)}{2^{u-1}\iota\epsilon^2}+\frac{SAH^2\Lambda^2\iota}{\epsilon^2}\right). \tag{98}$$

Moreover, for any $u\geqslant 0$, with probability $1-p$,

$$\sum_{t\geqslant 0}\mathbb{I}\left[w_{tH+k,u}>\frac{\epsilon(1-\gamma)}{16\Lambda}\right]\leqslant O\left(\frac{H\Lambda}{\epsilon}(SAH^2B\ln(N_0)+SAH+2^{u+2}SA\iota)\right). \tag{99}$$

Proof. Define

$$\tilde{U}_{t,u}=\mathbb{I}\left[\exists(s,a),i\in\{1,2,\ldots,H-1\},\ \text{such that } \mathcal{S}_{t+i,u}\neq\mathcal{S}_{t,u} \text{ or } Q_{t+i}(s,a)\neq Q_t(s,a)\right],$$

and

$$\hat{w}_t(s,a)=(1-\tilde{U}_{t,u})\sum_{i=0}^{H-1}\mathbb{I}\left[(s_{t+i},a_{t+i})\in\mathcal{S}_{t+i,u}\right]+H\tilde{U}_{t,u}.$$

Note that $\hat{w}_{tH+k}$ is measurable with respect to $\mathcal{F}_t^k=\mathcal{F}_{(t+1)H+k-1}$ and $\mathbb{E}\left[\hat{w}_{tH+k}|\mathcal{F}_k^{t-1}\right]\geqslant w_{tH+k}$, we then have that by Lemma 14,

$$\mathbb{P}\Big[\sum_{t\geqslant 0}w_{tH+k}>8SAH^2B\ln(N_0)+8SAH+2^{u+2}SA\iota,$$
$$\sum_{t\geqslant 0}\hat{w}_{tH+k}\leqslant 2SAH^2B\ln(N_0)+2SAH+2^uSA\iota\Big]\leqslant p. \tag{100}$$

On the other hand, we have that

$$\sum_{t\geqslant 0}\hat{w}_{tH+k}\leqslant H\sum_{t\geqslant 0}\hat{U}_{tH+k}+\sum_{t\geqslant 1}\mathbb{I}\left[(s_t,a_t)\in\mathcal{S}_{t,u}\right]$$
$$\leqslant 2SAH^2B\ln(N_0)+2SAH+\sum_{t\geqslant 1}\mathbb{I}\left[(s_t,a_t)\in\mathcal{S}_{t,u}\right] \tag{101}$$
$$\leqslant 2SAH^2B\ln(N_0)+2SAH+2^uSA\iota, \tag{102}$$

where Inequality (101) is because $\mathcal{S}_{t,u}$ changes at most $2SA$ times in t, and Inequality (102) is by the fact that $2^{u-1}\iota \leqslant n_t(s,a) < 2^u\iota$ implies that $2^u\iota \leqslant N_t(s,a) < 2^{u+1}\iota$. It then follows that

$$\mathbb{P}\left[\sum_{t\geqslant 0} w_{tH+k} > 8SAH^2B\ln(N_0) + 8SAH + 2^{u+2}SA\iota\right] \leqslant p,$$

which means

$$\mathbb{P}\left[\sum_{t\geqslant 0}\mathbb{I}\left[w_{tH+k,u} > \frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2}\right] > 10240\left(\frac{16SABH^4\Lambda^2\ln(N_0)}{2^{u-1}\iota\epsilon^2} + \frac{8SAH^2\Lambda^2\iota}{\epsilon^2}\right)\right] \leqslant p$$

and

$$\mathbb{P}\left[\sum_{t\geqslant 0}\mathbb{I}\left[w_{tH+k,u} > \frac{\epsilon(1-\gamma)}{16\Lambda}\right] > \frac{16H\Lambda}{\epsilon}(8SAH^2B\ln(N_0) + 8SAH + 2^{u+2}SA\iota)\right] \leqslant p.$$

The proof is completed. □

For u such that $2^u \leqslant \frac{BH^2\ln(N_0)}{\iota}$ or $u=0$, we plug u and $k = 1, 2, \ldots, H$ into (99) and obtain that with probability $1 - Hp$,

$$\sum_{t\geqslant 1}\mathbb{I}\left[w_{t,u} > \frac{\epsilon(1-\gamma)}{16\Lambda}\right] \leqslant O\left(\frac{SAH^4B\Lambda\ln(N_0)}{\epsilon}\right). \tag{103}$$

For u such that $2^u > \frac{BH^2\ln(N_0)}{\iota}$, we plug u and $k = 1, 2, \ldots, H$ into (98) and obtain that with probability $1 - Hp$,

$$\sum_{t\geqslant 1}\mathbb{I}\left[w_{t,u} > \frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2}\right] \leqslant O\left(\frac{SAH^3\Lambda^2\iota}{\epsilon^2}\right). \tag{104}$$

Via a union bound over u, we have that with probability $1 - 2H\Lambda p$, it holds that

$$\begin{aligned}\sum_{t\geqslant 1}\mathbb{I}\left[\beta_t^* > \frac{\epsilon}{8}\right] &\leqslant \sum_{t\geqslant 1}\mathbb{I}\left[\exists u, w_{t,u} > \max\{\frac{1}{10240}\cdot\frac{2^{u-1}\epsilon^2}{H^2\Lambda^2}, \frac{\epsilon(1-\gamma)}{8\Lambda}\} \text{ and } w_{t,0} > \frac{\epsilon(1-\gamma)}{8\Lambda}\right] \\ &\leqslant O\left(\frac{SAH^3\Lambda^3\iota}{\epsilon^2} + \frac{SAH^4B\Lambda^2\ln(N_0)}{\epsilon}\right). \end{aligned} \tag{105}$$

D.3.4. PROOF OF LEMMA 22

Lemma 22 (restated). *With probability* $1 - SA\check{J}(2\mathbb{P}[\overline{E}_2] + 4p)$, *it holds that*

$$\begin{aligned}&\sum_{t\geqslant 1}\text{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) \\ &\leqslant O\left(\frac{SAH^2\iota}{\epsilon}\right) + \tilde{O}\left(\frac{S^{3/2}A^{3/2}H^{17/4}\iota}{\epsilon^{1/2}} + \frac{SAH^{59/12}\iota}{\epsilon^{1/3}} + \frac{S^{5/4}A^{5/4}H^3\iota}{\epsilon^{1/4}} + S^2A^2H^9\iota\right).\end{aligned} \tag{106}$$

The rest of this subsection is devoted to the proof of Lemma 22.

Let s, a, j be fixed. We follow the notations in Appendix D.3.1. For t in the $(j+1)$-th type-I stage of (s,a), recalling the definition

$$\begin{aligned}\check{b}_t(s_t,a_t) = \min\{2\sqrt{2}&\left(\sqrt{\frac{\check{\underline{\sigma}}/\check{n} - (\check{\underline{\mu}}/\check{n})^2}{\check{n}}\iota} + \sqrt{\frac{\underline{\sigma}^{\text{ref}}/n - (\underline{\mu}^{\text{ref}}/n)^2}{n}\iota}\right) \\ &+ 7\left(\frac{H\iota^{3/4}}{n^{3/4}} + \frac{H\iota^{3/4}}{\check{n}^{3/4}}\right) + 5\left(\frac{H\iota}{n} + \frac{H\iota}{\check{n}}\right),\ \frac{1}{1-\gamma}\},\end{aligned}$$

we have that

$$\begin{aligned}
&\mathrm{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) \\
&\leqslant \underbrace{4\mathrm{clip}(2\sqrt{2}\left(\sqrt{\frac{\underline{\sigma}^{\mathrm{ref}}/n - (\underline{\mu}^{\mathrm{ref}}/n)^2}{n}\iota} - \sqrt{\frac{\mathbb{V}(P_{s,a},V^*)}{n}\iota}\right), \frac{\epsilon}{64H})}_{①} + \underbrace{4\mathrm{clip}(2\sqrt{2}\sqrt{\frac{\check{\underline{\sigma}}/\check{n} - (\check{\underline{\mu}}/\check{n})^2}{\check{n}}\iota}, \frac{\epsilon}{64H})}_{②} \\
&\quad + \underbrace{4\mathrm{clip}(7\left(\frac{H\iota^{3/4}}{n^{3/4}} + \frac{H\iota^{3/4}}{\check{n}^{3/4}}\right), \frac{\epsilon}{64H})}_{③} + \underbrace{4\mathrm{clip}(5\left(\frac{H\iota}{n} + \frac{H\iota}{\check{n}}\right), \frac{\epsilon}{64H})}_{④}, \qquad (107)
\end{aligned}$$

and the trivial bound

$$\mathrm{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H}) \leqslant \frac{1}{1-\gamma}. \qquad (108)$$

Here, (107) is because $\mathrm{clip}(a+b, 2\epsilon) \leqslant 2\mathrm{clip}(a,\epsilon) + 2\mathrm{clip}(b,\epsilon)$ for any non-negative a, b, ϵ.

Let V_t^{ref} be the value of V^{ref} immediately before the beginning of the t-th step and $V^{\mathrm{REF}} := \lim_{t\to\infty} V_t^{\mathrm{ref}}$ (by the update rule of Algorithm 2, this limit exists). Recall that λ_t is defined as the vector such that $\lambda_t(s) = \mathbb{I}\left[\sum_a N_t(s,a) < N_1\right]$. By Lemma 18 with $\epsilon_1 = \omega := \frac{1}{\sqrt{B}}$ (assuming $\epsilon \leqslant \frac{1}{\sqrt{B}}$), we have that

$$\mathbb{P}\left[\forall t \geqslant 1, V_t^{\mathrm{ref}}(s_t) - V^*(s_t) \leqslant H\lambda_t(s_t) + \omega\right] \geqslant \mathbb{P}\left[E_2\right]. \qquad (109)$$

We will deal with the four terms in **RHS** of (107) separately.

The ① term To handle this term, we introduce a lemma to bound $\frac{\underline{\sigma}^{\mathrm{ref}}}{n} - (\frac{\underline{\mu}^{\mathrm{ref}}}{n})^2 - \mathbb{V}(P_{s,a}, V^*)$.

Lemma 29. *With probability* $1 - (\mathbb{P}[\overline{E}_2] + 4p)$*, it holds that*

$$\frac{\underline{\sigma}^{\mathrm{ref}}}{n} - (\frac{\underline{\mu}^{\mathrm{ref}}}{n})^2 - \mathbb{V}(P_{s,a},V^*) \leqslant 9\sqrt{2}H^3\sqrt{\frac{\iota}{n}} + \frac{1}{n}\left(2H^2SA(\check{J}+\bar{J}) + 10H^2SN_1\right) + 4H\omega.$$

Proof. Note that

$$\frac{\underline{\sigma}^{\mathrm{ref}}}{n} - (\frac{\underline{\mu}^{\mathrm{ref}}}{n})^2 - \mathbb{V}(P_{s,a},V^*) = \frac{1}{n}(\chi_6 + \chi_7 + \chi_8 + \chi_9), \qquad (110)$$

where

$$\begin{aligned}
\chi_6 &:= \sum_{i=1}^n \left((V_{l_i}^{\mathrm{ref}}(s_{l_i+1}))^2 - P_{s,a}(V_{l_i}^{\mathrm{ref}})^2\right), \\
\chi_7 &:= \frac{1}{n}\left(\sum_{i=1}^n P_{s,a}V_{l_i}^{\mathrm{ref}}\right)^2 - \frac{1}{n}\left(\sum_{i=1}^n V_{l_i}^{\mathrm{ref}}(s_{l_i+1})\right)^2 \\
\chi_8 &:= \sum_{i=1}^n (P_{s,a}V_{l_i}^{\mathrm{ref}})^2 - \frac{1}{n}\left(\sum_{i=1}^n P_{s,a}V_{l_i}^{\mathrm{ref}}\right)^2, \\
\chi_9 &:= \sum_{i=1}^n \mathbb{V}(P_{s,a}, V_{l_i}^{\mathrm{ref}}) - n\mathbb{V}(P_{s,a},V^*).
\end{aligned}$$

According to Azuma's inequality, with probability $(1-2p)$ it holds that

$$|\chi_6| \leqslant H^2\sqrt{2n\iota}, \qquad (111)$$

$$|\chi_7| \leqslant 2H\left|\sum_{i=1}^n \left(V_{l_i}^{\mathrm{ref}}(s_{l_i+1}) - P_{s,a}V_{l_i}^{\mathrm{ref}}\right)\right| \leqslant 2H^2\sqrt{2n\iota}. \qquad (112)$$

On the other hand, by direct computation, we have that

$$\begin{aligned}
\chi_8 &= \sum_{i=1}^{n}(P_{s,a}V_{l_i}^{\mathrm{ref}})^2 - \frac{1}{n}\left(\sum_{i=1}^{n}P_{s,a}V_{l_i}^{\mathrm{ref}}\right)^2 \\
&\leqslant \sum_{i=1}^{n}(P_{s,a}V_{l_i}^{\mathrm{ref}})^2 - \frac{1}{n}\left(\sum_{i=1}^{n}P_{s,a}V^{\mathrm{REF}}\right)^2 \qquad (113)\\
&= \sum_{i=1}^{n}\left((P_{s,a}V_{l_i}^{\mathrm{ref}})^2 - (P_{s,a}V^{\mathrm{REF}})^2\right) \\
&\leqslant 2H^2\sum_{i=1}^{n}P_{s,a}\lambda_{l_i} \qquad (114)\\
&= 2H^2\left(\sum_{i=1}^{n}\lambda_{l_i}(s_{l_i+1}) + \sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})\lambda_{l_i}\right) \\
&= 2H^2\sum_{i=1}^{n}(\lambda_{l_i}(s_{l_i+1}) - \lambda_{l_i+1}(s_{l_i+1})) + 2H^2\sum_{i=1}^{n}\lambda_{l_i+1}(s_{l_i+1}) + 2H^2\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})\lambda_{l_i} \\
&\leqslant 2H^2SA(\check{J}+\bar{J}) + 2H^2SN_1 + 2H^2\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})\lambda_{l_i}, \qquad (115)
\end{aligned}$$

where Inequality (113) is by the fact that $V_t^{\mathrm{ref}} \geqslant V^{\mathrm{REF}}$ for any $t \geqslant 1$, Inequality (114) is by the definition of λ_t and Inequality (115) holds because $\lambda_t \neq \lambda_{t+1}$ implies an update occurs at the t-th step and $\sum_{t\geqslant 1}\lambda_t(s_t) \leqslant SN_1$. Therefore, by Azuma's inequality it holds that

$$\mathbb{P}\left[\chi_8 > 2H^2SA(\check{J}+\bar{J}) + 2H^2SN_1 + 2H^3\sqrt{2n\iota}\right] \leqslant \mathbb{P}\left[2H^2\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})\lambda_{l_i} > 2H^3\sqrt{2n\iota}\right] \leqslant p. \qquad (116)$$

At last, the term χ_9 could be bounded by

$$\begin{aligned}
\chi_9 &= \sum_{i=1}^{n}\mathbb{V}(P_{s,a}, V_{l_i}^{\mathrm{ref}}) - n\mathbb{V}(P_{s,a}, V^*) \\
&\leqslant \frac{4H}{n}\sum_{i=1}^{n}P_{s,a}(V_{l_i}^{\mathrm{ref}} - V^*) \\
&= 4H\sum_{i=1}^{n}(V_{l_i}^{\mathrm{ref}}(s_{l_i+1}) - V_{l_i+1}^{\mathrm{ref}}(s_{l_i+1}) + V_{l_i+1}^{\mathrm{ref}}(s_{l_i+1}) - V^*(s_{l_i+1})) + 4H\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})(V_{l_i}^{\mathrm{ref}} - V^*) \\
&\leqslant 4H^2S + 4H\sum_{i=1}^{n}(V_{l_i+1}^{\mathrm{ref}}(s_{l_i+1}) - V^*(s_{l_i+1})) + 4H\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})(V_{l_i}^{\mathrm{ref}} - V^*), \qquad (117)
\end{aligned}$$

where Inequality (117) is by the fact that the number of updates of V^{ref} is at most S. Similarly, we have that

$$\begin{aligned}
&\mathbb{P}\left[\chi_9 > 4H^2S + 4H^2SN_1 + 4Hn\omega + 4H^2\sqrt{2n\iota}\right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{n}(V_{l_i+1}^{\mathrm{ref}}(s_{l_i+1}) - V^*(s_{l_i+1})) > \sum_{i=1}^{n}(H\lambda_{l_i+1}(s_{l_i+1}) + \omega)\right] \\
&\qquad + \mathbb{P}\left[\sum_{i=1}^{n}(P_{s,a}-\mathbf{1}_{s_{l_i+1}})(V_{l_i}^{\mathrm{ref}} - V^*) > H\sqrt{2n\iota}\right] \\
&\leqslant \mathbb{P}[\overline{E}_2] + p, \qquad (118)
\end{aligned}$$

where (118) holds by (109).

Combining (110), (111), (112), (116) and (118), with probability $1-(\mathbb{P}[\overline{E}_2]+5p)$ it holds that

$$\begin{aligned}
&\frac{\underline{\sigma}^{\mathrm{ref}}}{n}-(\frac{\underline{\mu}^{\mathrm{ref}}}{n})^2-\mathbb{V}(P_{s,a},V^*)\\
&\leqslant \frac{1}{n}\left(3H^2\sqrt{2n\iota}+2H^2SA(\check{J}+\bar{J})+2H^2SN_1+2H^3\sqrt{2n\iota}+4H\left(S+SN_1\right)+4H^2\sqrt{2n\iota}\right)+4Hw\\
&\leqslant 9\sqrt{2}H^3\sqrt{\frac{\iota}{n}}+\frac{1}{n}\left(2H^2SA(\check{J}+\bar{J})+10H^2SN_1\right)+4Hw.
\end{aligned}$$

□

By Lemma 29, with probability $1-(\mathbb{P}\left[\overline{E}_2\right]+4p)$ it holds that

$$\begin{aligned}
&\left(\sqrt{\frac{\underline{\sigma}^{\mathrm{ref}}/n-(\underline{\mu}^{\mathrm{ref}}/n)^2}{n}\iota}-\sqrt{\frac{\mathbb{V}(P_{s,a},V^*)}{n}\iota}\right)\\
&\qquad\leqslant\sqrt{\frac{9\sqrt{2}H^3\iota^{3/2}}{n^{3/2}}+\frac{(2H^2SA(\check{J}+\bar{J})+10H^2SN_1)\iota}{n^2}+\frac{4H\omega\iota}{n}}. \qquad (119)
\end{aligned}$$

As a result, for $n>N_2:=c_3\frac{H^3\omega\iota}{\epsilon^2}+c_4\frac{H^{10/3}\iota}{\epsilon^{4/3}}+c_5\frac{H\sqrt{(H^2SA(\check{J}+\bar{J})+H^2SN_1)\iota}}{\epsilon}$ with sufficient large constants c_4 and c_5, it holds that

$$2\sqrt{2}\left(\sqrt{\frac{\underline{\sigma}^{\mathrm{ref}}/n-(\underline{\mu}^{\mathrm{ref}}/n)^2}{n}\iota}-\sqrt{\frac{\mathbb{V}(P_{s,a},V^*)}{n}\iota}\right)<\frac{\epsilon}{64H}. \qquad (120)$$

The ② term Direct computation gives that

$$\frac{\check{\underline{\sigma}}/\check{n}-(\check{\underline{\mu}}/\check{n})^2}{\check{n}}\leqslant\frac{\check{\underline{\sigma}}}{\check{n}^2}=\frac{1}{\check{n}^2}\sum_{i=1}^{\check{n}}\left(V_{\check{l}_i}(s_{\check{l}_i+1})-V^{\mathrm{ref}}_{\check{l}_i}(s_{\check{l}_i+1})\right)^2\leqslant\frac{1}{\check{n}^2}\sum_{i=1}^{\check{n}}\left(V^{\mathrm{ref}}_{\check{l}_i}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2. \qquad (121)$$

Also note that

$$\begin{aligned}
&\left|\sum_{i=1}^{\check{n}}\left(\left(V^{\mathrm{ref}}_{\check{l}_i}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2-\left(V^{\mathrm{ref}}_{\check{l}_i+1}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2\right)\right|\\
&\qquad\leqslant 2H\cdot\left|\sum_{i=1}^{\check{n}}\left(V^{\mathrm{ref}}_{\check{l}_i}(s_{\check{l}_i+1})-V^{\mathrm{ref}}_{\check{l}_i+1}(s_{\check{l}_i+1})\right)\right|\leqslant 2H^2(SA(\check{J}+\bar{J})). \qquad (122)
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\mathbb{P}\left[\frac{\check{\underline{\sigma}}/\check{n}-(\check{\underline{\mu}}/\check{n})^2}{\check{n}}>\frac{H^2(2SN_1+2SA(\check{J}+\bar{J}))}{\check{n}^2}+\frac{2\omega^2}{\check{n}}\right]\\
\leqslant\ &\mathbb{P}\left[\sum_{i=1}^{\check{n}}\left(V^{\mathrm{ref}}_{\check{l}_i}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2>H^2(2SN_1+2SA(\check{J}+\bar{J}))+2\omega^2\check{n}\right]\\
\leqslant\ &\mathbb{P}\left[\sum_{i=1}^{\check{n}}\left(V^{\mathrm{ref}}_{\check{l}_i+1}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2>2H^2SN_1+2\omega^2\check{n}\right]\\
\leqslant\ &\mathbb{P}\left[\sum_{i=1}^{\check{n}}\left(V^{\mathrm{ref}}_{\check{l}_i+1}(s_{\check{l}_i+1})-V^*(s_{\check{l}_i+1})\right)^2>\sum_{i=1}^{\check{n}}\left(H\lambda_{\check{l}_i+1}(s_{\check{l}_i+1})+\omega\right)^2\right]\\
\leqslant\ &\mathbb{P}[\overline{E}_2],
\end{aligned}$$

where the last inequality is due to (109). Therefore, we have that

$$\mathbb{P}\left[\sqrt{\frac{\check{\sigma}/\check{n} - (\check{\underline{\mu}}/\check{n})^2}{\check{n}}} > \sqrt{\frac{H^2(2SN_1 + 2SA(\check{J} + \bar{J}))}{\check{n}^2} + \frac{2\omega^2}{\check{n}}}\right] \leqslant \mathbb{P}[\overline{E}_2]. \tag{123}$$

Note that $\check{n} \geqslant \frac{n}{2HB}$. For $n > N_3 = c_6 \frac{\omega^2 H^3 B\iota}{\epsilon^2} + c_7 \frac{\sqrt{H^4 BSN_1\iota}}{\epsilon}$ with large enough constants c_6 and c_7, we have that the following inequality holds with probability at least $1 - \mathbb{P}[\overline{E}_2]$,

$$2\sqrt{2}\sqrt{\frac{\check{\sigma}/\check{n} - (\check{\underline{\mu}}/\check{n})^2}{\check{n}}\iota} < \frac{\epsilon}{64H}. \tag{124}$$

The ③ term For $n > N_4 := c_8 \frac{H^{11/3}B\iota}{\epsilon^{4/3}}$ with large enough constant c_8, we have

$$7\left(\frac{H\iota^{3/4}}{n^{3/4}} + \frac{H\iota^{3/4}}{\check{n}^{3/4}}\right) < \frac{\epsilon}{64H}. \tag{125}$$

The ④ term For $n > N_5 := c_9 \frac{H^3 B\iota}{\epsilon}$ with large enough constant c_8, we have

$$5\left(\frac{H\iota}{n} + \frac{H\iota}{\check{n}}\right) < \frac{\epsilon}{64H}. \tag{126}$$

Combining (107) with the bounds (119), (120), (123), (124), (125) and (126), using the trivial bound $\mathrm{clip}(\check{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{16H}) \leqslant 1/(1-\gamma)$ for early stages, and summing over all possible s, a, j with a union bound, we obtain that with probability $1 - SA\check{J}(2\mathbb{P}[\overline{E}_2] + 4p)$,

$$\sum_{t\geqslant 1} \mathrm{clip}(\check{b}_t(s_t, a_t) - b_t^*(s_t, a_t), \frac{\epsilon}{16H}) \leqslant O(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4), \tag{127}$$

where (noting that $\check{n} \geqslant n/(2HB)$ in (123), (125) and (126))

$$\mathcal{M}_1 = \sum_{s,a}\left(H\iota + \sum_{n=\max\{\lfloor\iota\rfloor,1\}}^{N_2} \sqrt{\frac{9\sqrt{2}H^3\iota^{3/2}}{n^{3/2}} + \frac{(2H^2SA(\check{J} + \bar{J}) + 10H^2SN_1)\iota}{n^2} + \frac{4H\omega\iota}{n}}\right), \tag{128}$$

$$\mathcal{M}_2 = \sum_{s,a}\left(H\iota + \sum_{n=\max\{\lfloor\iota\rfloor,1\}}^{N_3} \sqrt{\frac{H^4B^2(2SN_1 + 2SA(\check{J} + \bar{J}))}{n^2} + \frac{2HB\omega^2}{n}}\right), \tag{129}$$

$$\mathcal{M}_3 = \sum_{s,a}\left(H\iota + \sum_{n=\max\{\lfloor\iota\rfloor,1\}}^{N_4} \left(\frac{H\iota^{3/4}}{n^{3/4}} + \frac{H^{7/4}B^{3/4}\iota^{3/4}}{n^{3/4}}\right)\right), \tag{130}$$

$$\mathcal{M}_4 = \sum_{s,a}\left(H\iota + \sum_{n=\max\{\lfloor\iota\rfloor,1\}}^{N_5} \left(\frac{H\iota}{n} + \frac{H^2B\iota}{n}\right)\right). \tag{131}$$

Straightforward calculation shows that

$$\mathcal{M}_1 \leqslant SA\cdot O\left(H\iota + N_2^{1/4}H^{3/2}\iota^{3/4} + \ln(\frac{N_2}{\iota})\sqrt{H^2SA\check{J}+H^2SN_1} + \sqrt{N_2H\omega\iota}\right)$$
$$\leqslant O\left(\frac{SAH^{5/4}\iota}{\epsilon}\right) + \tilde{O}\Big(\frac{SAH^{17/12}\iota}{\epsilon^{2/3}} + \frac{(S^{3/2}A^{3/2}H^{7/4}+S^{3/2}A^{5/4}H^{7/2}+SAH^{15/8})\iota}{\epsilon^{1/2}}$$
$$+\frac{SAH^{7/3}\iota}{\epsilon^{1/3}} + \frac{(S^{5/4}A^{5/4}H^{5/2}+S^{5/4}A^{9/8}H^{3})\iota}{\epsilon^{1/4}} + S^2A^2H^3\iota + S^2A^{3/2}H^{7/2}\iota\Big), \tag{132}$$

$$\mathcal{M}_2 \leqslant SA\cdot O\left(H\iota + \ln(\frac{N_3}{\iota})\sqrt{H^2B^2(H^2SN_1+H^2SA\check{J})} + \sqrt{N_3HB\omega^2\iota}\right)$$
$$\leqslant O\left(\frac{SAH^2\iota}{\epsilon}\right) + \tilde{O}\Big(\frac{S^{3/2}A^{5/4}H^{17/4}\iota}{\epsilon^{1/2}} + S^2A^{3/2}H^9\iota + S^2A^2H^7\iota\Big), \tag{133}$$

$$\mathcal{M}_3 \leqslant SA\cdot O\left(H\iota + N_4^{1/4}H^{7/4}B^{3/4}\iota^{3/4}\right) \leqslant O\left(\frac{SAH^{59/12}\iota}{\epsilon^{1/3}} + SAH\iota\right) \tag{134}$$

$$\mathcal{M}_4 \leqslant SA\cdot O\left(H\iota + \ln(\frac{N_5}{\iota})H^2B\iota\right) \leqslant \tilde{O}\left(SAH^5\iota\right). \tag{135}$$

Finally, together with (127), we conclude that

$$\sum_{t\geqslant 1}\mathrm{clip}(\check{b}_t(s_t,a_t) - b_t^*(s_t,a_t), \frac{\epsilon}{16H})$$
$$\leqslant O\left(\frac{SAH^2\iota}{\epsilon}\right) + \tilde{O}\left(\frac{S^{3/2}A^{3/2}H^{17/4}\iota}{\epsilon^{1/2}} + \frac{SAH^{59/12}\iota}{\epsilon^{1/3}} + \frac{S^{5/4}A^{5/4}H^3\iota}{\epsilon^{1/4}} + S^2A^2H^9\iota\right). \tag{136}$$

D.3.5. PROOF OF LEMMA 24

Lemma 22 (restated). With probability $1-(\mathbb{P}\left[\overline{E}_2\right]+p)$, it holds that

$$\sum_{t\geqslant 1}\upsilon_t \leqslant 64\log(\frac{16N_0H^2}{\epsilon})N_1.$$

By definition, we have that

$$\sum_{t\geqslant 1}\upsilon_t = \sum_{t\geqslant 1}\sum_{s}P_{s_t,a_t,s}\mathrm{clip}\left(\frac{1}{1-\gamma}\left(\frac{1}{\bar{N}_t(s,a)}\sum_{i=1}^{\bar{N}_t(s,a)}\lambda_{l_i(s_t,a_t)}(s)\right), \frac{\epsilon}{16H}\right)$$
$$\leqslant H\sum_{s}\sum_{t\geqslant 1}P_{s_t,a_t,s}\mathrm{clip}\left(\left(\frac{1}{\bar{N}_t(s,a)}\sum_{i=1}^{\bar{N}_t(s,a)}\lambda_{l_i(s_t,a_t)}(s)\right), \frac{\epsilon}{8H^2}\right). \tag{137}$$

Let $\tilde{T}(s,a,s')$ be the visit count of (s,a) before the smallest time t such that $\lambda_t(s')=0$. Then we have that

$$\frac{1}{\bar{N}_t(s,a)}\sum_{i=1}^{\bar{N}_t(s,a)}\lambda_{l_i(s_t,a_t)}(s) \leqslant \mathbb{I}\left[\bar{N}_t(s,a)\leqslant (1+\frac{1}{H})\tilde{T}(s,a,s')\right] + \frac{\tilde{T}(s,a,s')}{\bar{N}_t(s,a)}.$$

Noting that $\bar{N}_t(s,a)\leqslant N_t(s,a)\leqslant(1+\frac{1}{H})\bar{N}_t(s,a)$, we obtain that

$$\mathrm{clip}\left(\left(\frac{1}{\bar{N}_t(s,a)}\sum_{i=1}^{\bar{N}_t(s,a)}\lambda_{l_i(s_t,a_t)}(s)\right), \frac{\epsilon}{8H^2}\right) \leqslant \mathbb{I}\left[N_t(s,a)\leqslant 4\tilde{T}(s,a,s')\right] + \mathrm{clip}(\frac{2\tilde{T}(s,a,s')}{N_t(s,a)}, \frac{\epsilon}{8H^2}).$$

Combining this with (137), with probability $1-p$ it holds that

$$\sum_{t\geqslant 1} \upsilon_t \leqslant H\sum_{s}\sum_{t\geqslant 1} P_{s_t,a_t,s'}\mathbb{I}\left[N_t(s_t,a_t)\leqslant 4\tilde{T}(s_t,a_t,s')\right] + H\sum_{s'}\sum_{t\geqslant 1} P_{s_t,a_t,s'}\mathrm{clip}(\frac{2\tilde{T}(s_t,a_t,s')}{N_t(s_t,a_t)},\frac{\epsilon}{8H^2})$$

$$\leqslant 4H\sum_{s,a,s'} P_{s,a,s'}\tilde{T}(s,a,s') + 4H\sum_{s,a,s'} P_{s,a,s'}\tilde{T}(s,a,s')\log(\frac{16\tilde{T}(s,a,s')H^2}{\epsilon})$$

$$\leqslant 8\log(\frac{16N_0H^2}{\epsilon})\sum_{s,a,s'} P_{s,a,s'}\tilde{T}(s,a,s')$$

$$= 8\log(\frac{16N_0H^2}{\epsilon})\sum_{s'}\sum_{t\geqslant 1} P_{s_t,a_t,s'}\lambda_t(s')$$

$$\leqslant 32\log(\frac{16N_0H^2}{\epsilon})(\sum_{t\geqslant 1}\lambda_t(s_{t+1}) \tag{138}$$

$$\leqslant 64\log(\frac{16N_0H^2}{\epsilon})N_1. \tag{139}$$

The second last inequality holds with probability $1-p$ by Lemma 11, and the last inequality is by the facts $\sum_{t\geqslant 1}\lambda_t(s_t)\leqslant SN_1$ and $\sum_{t\geqslant 1}(\lambda_t(s_{t+1}-\lambda_{t+1}(s_{t+1}))\leqslant S$. The proof is completed.