APPENDIX—SUPPLEMENTARY MATERIAL

Using the facts that $P(Z_{m,n}=k|\, Y_m=t,\chi)=\chi_{t,k}$, $P(Y_m=t|\, \pi)=\pi_t$, $q(Y_m=t|\, \gamma_m)=\gamma_{m,t}$, and also $q(Z_{m,n}=k|\, \phi_{m,n})=\phi_{m,n,k}$, $P\left(X_{m,n}|\, Z_{m,n}=k,\, \beta\right)=P\left(X_{m,n}|\, \beta_k\right)$, we can easily see that L_m can be rewritten as

$$\begin{aligned}&L_m\left(\gamma_m,\phi_m;\pi,\chi,\beta\right)=\\&\sum_{t=1}^{T}\gamma_{m,t}\log\pi_t+\sum_{n=1}^{N_m}\sum_{t=1}^{T}\sum_{k=1}^{K}\gamma_{m,t}\phi_{m,n,k}\log\chi_{t,k}\\&+\sum_{n=1}^{N_m}\sum_{k=1}^{K}\phi_{m,n,k}\log P\left(X_{m,n}|\beta_k\right)-\sum_{t=1}^{T}\gamma_{m,t}\log\gamma_{m,t}\\&-\sum_{n=1}^{N_m}\sum_{k=1}^{K}\phi_{m,n,k}\log\phi_{m,n,k}.\end{aligned}$$

Let us start with calculating first $\phi^*_{m,n,k}=\arg\max_{\phi_{m,n,k}} L_m$. By introducing the λ Lagrange multiplier, we have to solve the following equation.

$$\begin{aligned}0&=\frac{\partial}{\partial\phi_{m,n,k}}\left[L_m+\lambda\left(\sum_{k=1}^{K}\phi_{m,n,k}-1\right)\right]\\&=\sum_{t=1}^{T}\gamma_{m,t}\log\chi_{t,k}+\log P\left(X_{m,n}|\beta_k\right)-\log\phi_{m,n,k}\\&\quad-1+\lambda\end{aligned}$$

Thus,

$$\phi^*_{m,n,k}=\frac{\exp\left(\sum_{t=1}^{T}\gamma_{m,t}\log\chi_{t,k}+\log P\left(X_{m,n}|\beta_k\right)\right)}{\sum_{j=1}^{K}\exp\left(\sum_{t=1}^{T}\gamma_{m,t}\log\chi_{t,j}+\log P\left(X_{m,n}|\beta_j\right)\right)}.$$

The derivation of the optimal $\gamma^*_{m,t}$ is similar, we just have to find $\gamma^*_{m,t}=\arg\max_{\gamma_{m,t}} L_m$.

$$\begin{aligned}0&=\frac{\partial}{\partial\gamma_{m,t}}\left[L_m+\lambda\left(\sum_{t=1}^{T}\gamma_{m,t}-1\right)\right]\\&=\log\pi_t+\sum_{n=1}^{N_m}\sum_{k=1}^{K}\phi_{m,n,k}\log\chi_{t,k}-\log\gamma_{m,t}\\&\quad-1+\lambda.\end{aligned}$$

Hence,

$$\gamma^*_{m,t}=\frac{\exp\left(\log\pi_t+\sum_{n=1}^{N}\sum_{k=1}^{K}\phi_{m,n,k}\log\chi_{t,k}\right)}{\sum_{\tau=1}^{T}\exp\left(\log\pi_\tau+\sum_{n=1}^{N}\sum_{k=1}^{K}\phi_{m,n,k}\log\chi_{\tau,k}\right)}.$$

We can use similar techniques to calculate the optimal $\pi^*\in\mathbb{S}^T$, as well.

$$\begin{aligned}0&=\frac{\partial}{\partial\pi_t}\left[\sum_{m=1}^{M}L_m+\lambda\left(\sum_{t=1}^{T}\pi_t-1\right)\right]\\&=\frac{1}{\pi_t}\sum_{m=1}^{M}\gamma_{m,t}+\lambda.\end{aligned}$$

Thus, we have that $\lambda=-\sum_{t=1}^{T}\sum_{m=1}^{M}\gamma_{m,t}$, and

$$\pi^*_t=\frac{\sum_{m=1}^{M}\gamma_{m,t}}{\sum_{\tau=1}^{T}\sum_{m=1}^{M}\gamma_{m,\tau}}.$$

To calculate the optimal $\chi^*_{t,k}$ we have to solve the following equation.

$$\begin{aligned}0&=\frac{\partial}{\partial\chi_{t,k}}\left[\sum_{m=1}^{M}L_m+\lambda\left(\sum_{k=1}^{K}\chi_{tk}-1\right)\right]\\&=\frac{1}{\chi_{t,k}}\sum_{m=1}^{M}\gamma_{m,t}\sum_{n=1}^{N_m}\phi_{m,n,k}+\lambda.\end{aligned}$$

And hence,

$$\chi^*_{t,k}=\frac{\sum_{m=1}^{M}\gamma_{m,t}\sum_{n=1}^{N_m}\phi_{m,n,k}}{\sum_{j=1}^{K}\sum_{m=1}^{M}\gamma_{m,t}\sum_{n=1}^{N_m}\phi_{m,n,j}}.$$