Controlling Selection Bias in Causal Inference

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Abstract

Selection bias, caused by preferential exclusion of samples from the data, is a major obstacle to valid causal and statistical inferences; it cannot be removed by randomized experiments and can hardly be detected in either experimental or observational studies. This paper highlights several graphical and algebraic methods capable of mitigating and sometimes eliminating this bias. These nonparametric methods generalize previously reported results, and identify the type of knowledge that is needed for reasoning in the presence of selection bias. Specifically, we derive a general condition together with a procedure for deciding recoverability of the odds ratio (OR) from s-biased data. We show that recoverability is feasible if and only if our condition holds. We further offer a new method of controlling selection bias using instrumental variables that permits the recovery of other effect measures besides OR.

1 Introduction

Selection bias is induced by preferential selection of units for data analysis, usually governed by unknown factors including treatment, outcome and their consequences. Case-control studies in Epidemiology are particularly susceptible to such bias, e.g., cases may be reported only when the outcome (disease or complication) is unusual, while non-cases remain unreported (see (Glymour and Greenland, 2008; Robins et al., 2000; Robins, 2001; Hernán et al., 2004)).

To illuminate the nature of this bias, consider the model of Fig. 1 (a) in which S is a variable affected by both X (treatment) and Y (outcome), indicating entry into the data pool. Such preferential selection to the pool amounts to conditioning on S, which creates spurious association between X and Y through two mechanisms. First conditioning on S induces spurious association between its parents, X and Y. Second, S is also a descendant of a "virtual collider" Y, whose parents are X and the error term U_Y (also called "omitted factors" or "hidden variable") which is always present, though often not shown in the diagram.[1]

A medical example of selection bias was reported in (Horwitz and Feinstein, 1978), and subsequently studied in (Hernán et al., 2004; Geneletti et al., 2009), in which it was noticed that the effect of Oestrogen (X) on Endometrial Cancer (Y) was overestimated in the data studied. One of the symptoms of the use of Oestrogen is vaginal bleeding (W) (Fig. 1(c)), and the hypothesis was that women noticing bleeding are more likely to visit their doctors, causing women using Oestrogen to be overrepresented in the study.

In causal inference studies, the two most common sources of bias are confounding (Fig. 1(b)) and selection (Fig. 1(a)). The former is a result of treatment X and outcome Y being affected by a common omitted variables $\mathbf{U}$, while the latter is due to treatment or outcome (or its descendants) affecting the inclusion of the subject in the sample (indexed by S). In both cases, we have unblocked extraneous "flow" of influence between treatment and outcome, which appear under the rubric of "spurious correlation." It is called spurious because it is not part of what we seek to estimate – the causal effect of X on Y in the target population. In the case of confounding, bias occurs because we cannot condition on the unmeasured confounders, while in selection, the distribution is always conditioned on S.

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[1] See (Pearl, 2009, pp. 339-341) for further explanation of this bias mechanism.

Formally, the distinction between these biases can be articulated thus: confounding bias is any $X - Y$ association that is attributable to selective choice of treatment, while selection bias is any association attributable to selective inclusion in the data pool. Operationally, confounding bias can be eliminated by randomization – selection bias cannot. Given this distinction, the two biases deserve different qualitative treatment and entail different properties, which we explore in this paper. Remarkably, there are special cases in which selection bias can be detected even from observations, as in the form of a non-chordal undirected component (Zhang, 2008).

As an interesting corollary of this distinction, it was shown (Pearl, 2010) that confounding bias, if such exists, can be amplified by conditioning on an instrumental variable Z (Fig. 1(d)). Selection bias, on the other hand, remains invariant under such conditioning.

We will use instrumental variables for the removal of selection bias in the presence of confounding bias, as shown in the scenario of Fig. 1(f). Whereas instrumental variables cannot ensure nonparametric identification of average causal effects, they can help provide reasonable bounds on those effects as well as point estimates in some special cases (Balke and Pearl, 1997). Since the bounding analysis assumed no selection bias, the question arises whether similar bounds can be derived in the presence of selection bias. We will show that selection bias can be removed entirely through the use of instrumental variables, therefore, the bounds on the causal effect will be narrowed to those obtained under the selection-free assumption.

This result is relevant in many areas because selection bias is pervasive in almost all empirical studies, including Machine Learning, Statistics, Social Sciences, Economics, Bioinformatics, Biostatistics, Epidemiology, Medicine, etc. For instance, one version of selection bias was studied in Economics, and led to the celebrated method developed by (Heckman, 1970). It removes the bias through a two-step process which assumes linearity, normality and, a probabilistic model of the selection mechanism.

Machine learning tasks suffer from a similar problem when training samples are selected preferentially, depending on feature-class combinations that differ from those encountered in the target environment (Zadrozny, 2004; Smith and Elkan, 2007; Storkey, 2009; Hein, 2009).

In Epidemiology, the prevailing approach is due to James Robins (Robins et al., 2000; Hernán et al., 2004), which assumes knowledge of the probability of selection given treatment. In some special cases, this probability can be estimated from data, requiring a record, for each treatment given, whether a follow up outcome (Y) is reported or not. We do not rely on such knowledge in this paper but assume, instead, that no data of treatment or outcome is available unless a case is reported (via S).

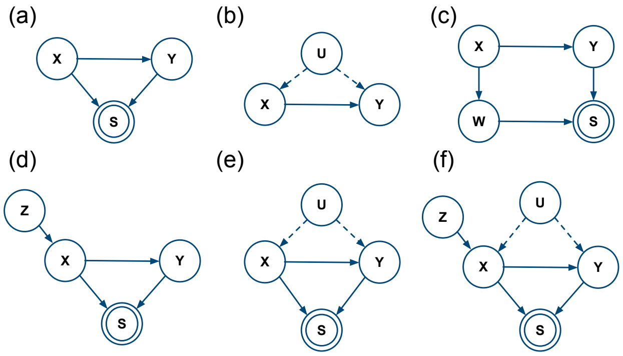


Figure 1: Different scenarios considered in this paper. (a,b) Simplest examples of selection and confounding bias, respectively. (c) Typical study with intermediary variable W between X and selection. (d) Instrumental variable with selection bias. (e) Selection combined with confounding. (f) Instrumental variable with confounding and selection bias simultaneously present.

Contributions

Our contributions are as follows. In Section 2, we give a complete graphical condition under which the population odds ratio (OR) and a covariate-specific causal odds ratio can be recovered from selection-biased data (Theorem 1). We then devise an effective procedure for testing this condition (Theorem 2, 3). These results, although motivated by causal considerations, are applicable to classification tasks as well, since the process of eliminating selection bias is separated from that of controlling for confounding bias.

In Section 3, we present universal curves that show the behavior of OR as the distribution $P(y \mid x)$ changes, and how the risk ratio (RR) and risk difference (RD) are related to OR. We further show that if one is interested in recovering RR and RD under selection bias, knowledge of $P(X)$ is sufficient for recovery.

In Section 4, we advance for other measures of effects besides odds ratio, and show that even when confounding and selection biases are simultaneously present (Fig. 1(e)), the latter can be entirely removed with the help of instrumental variables (Theorem 4). This result is surprising for two reasons: first, we generally do not expect selection bias to be removable; second, bias removal in the presence of confounding is generally expected to be a more challenging task. We finally show how this result is applicable to scenarios where other structural assumptions hold, for instance, when an instrument is not available but a certain back-door admissible set can be identified (Corollary 4).

2 Selection bias in a chain structure and its graphical generalizations

The chain structure of Figure 2(a) is the simplest structure exhibiting selection bias. The intuition gained from analyzing this example will serve as a basis for subsequently treating more complicated structures.

Consider a study of the effect of a training program (X) on earnings after 5 years of completion (Y), and assume that there is no confounding between treatment and outcome. Assume that subjects achieving higher income tend to report their status more frequently than those with lower income. The qualitative causal assumptions are depicted in Fig. 2(a). Given that all available data is obtained under selection bias, is the unbiased odds ratio recoverable?

To address this problem, we explicitly add a variable S to represent the selection mechanism, and assume that $S = 1$ represents presence in the sample, and zero otherwise. We will refer to samples selected by such mechanism as "s-biased". A similar representation was used in (Cooper, 1995; Lauritzen and Richardson, 2008; Geneletti et al., 2009; Didelez et al., 2010). In the chain structure of Fig. 2(a), X is d-separated from S by Y, which implies the conditional independence $(X \perp\!\!\!\perp S \mid Y)$, and encodes the assumption that entry to the data pool is determined by the outcome Y only, not by X. We define next some key concepts used along the paper and state some results that will support our analysis.

Definition 1 (Odds ratio)**.** *Consider two variables X and Y and a set $\mathbf{Z}$, the conditional odds ratio $OR(Y, X \mid \mathbf{Z} = \mathbf{z})$ is given by the ratio: $\big(Pr(y \mid \mathbf{z}, x')/Pr(y' \mid \mathbf{z}, x')\big)/\big(Pr(y \mid \mathbf{z}, x)/Pr(y' \mid \mathbf{z}, x)\big)$.*

$OR(Y, X \mid \mathbf{Z})$ measures the strength of association between X and Y conditioned on $\mathbf{Z}$ and it is symmetric, i.e., $OR(Y, X \mid \mathbf{Z}) = OR(X, Y \mid \mathbf{Z})$.

Definition 2 (G-Recoverability)**.** *Given a graph G, $OR(X, Y \mid \mathbf{Z})$ is said to be G-recoverable from s-biased data if the assumptions embedded in G renders it expressible in terms of the observable distribution $P(\mathbf{V_{xy}} \mid S = 1)$ where $\mathbf{V_{xy}} = \mathbf{V} \setminus \{S\}$. Formally, for every two probability distributions $P_1(.)$ and $P_2(.)$ compatible with G, $P_1(\mathbf{v_{xy}} =\mid S = 1) = P_2(\mathbf{v_{xy}} \mid S = 1)$ implies $OR_1(X, Y \mid \mathbf{Z}) = OR_2(X, Y \mid \mathbf{Z})$.*

Definition 3 (Collapsibility)**.** *Consider two variables X and Y and disjoint sets $\mathbf{Z}$ and $\mathbf{W}$. We say that the odds ratio $OR(X, Y \mid \mathbf{Z}, \mathbf{W})$ is collapsible over $\mathbf{W}$ if $OR(X, Y \mid \mathbf{Z} = \mathbf{z}, \mathbf{W} = \mathbf{w}) = OR(X, Y \mid \mathbf{Z} = \mathbf{z}, \mathbf{W} = \mathbf{w}') = OR(X, Y \mid \mathbf{Z} = \mathbf{z})$, for all $\mathbf{w} \neq \mathbf{w}'$.*

Definition 3 and the following Lemma are stated in (Didelez et al., 2010) and are based on long tradition in Epidemiology starting with (Cornfield, 1951) and followed by (Whittemore, 1978; Geng, 1992).[2]

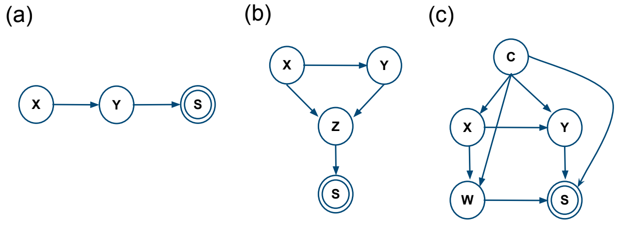


Figure 2: (a) Chain graph where X represents treatment, Y is the outcome, and S an indicator variable for the selection mechanism. (b) Scenario where there exists a blocking set from $\{X, Y\}$ to S yet the OR is not G-recoverable. (c) Example where the **c**-specific OR is G-recoverable.

Lemma 1. *For any two sets, $\mathbf{Z}$ and $\mathbf{W}$, the conditional odds ratio $OR(Y, X \mid \mathbf{Z}, \mathbf{W})$ is collapsible over $\mathbf{W}$ (that is, $OR(Y, X \mid \mathbf{Z}, \mathbf{W}) = OR(Y, X \mid \mathbf{Z})$), if either $(X \perp\!\!\!\perp \mathbf{W} \mid \{Y, \mathbf{Z}\})$ or $(Y \perp\!\!\!\perp \mathbf{W} \mid \{X, \mathbf{Z}\})$.*

The following Corollary provides a graphical test for G-recoverability (Def. 2) based on Lemma 1:

Corollary 1. *Given a graph G in which node S represents selection, the $OR(X, Y \mid \mathbf{Z})$ is G-recoverable from s-biased data if $\mathbf{Z}$ is such that $(X \perp\!\!\!\perp S \mid \{Y, \mathbf{Z}\})_G$ or $(Y \perp\!\!\!\perp S \mid \{X, \mathbf{Z}\})_G$.*

There is an important subtlety here. One might surmise that selection bias of $OR(X, Y)$ can be removed if the condition of Corollary 1 holds, i.e., there exists a separating set $\mathbf{Z}$ such that $(X \perp\!\!\!\perp S \mid \{Y, \mathbf{Z}\})_G$ or $(Y \perp\!\!\!\perp S \mid \{X, \mathbf{Z}\})_G$, but this is not the case. Consider Fig. 2(b) where the set $\mathbf{Z}$ d-separates $\{X, Y\}$ from S and therefore permits us to remove S by writing $OR(X, Y \mid \mathbf{Z}, S = 1)$ as $OR(X, Y \mid \mathbf{Z})$, yet the unconditional OR is not G-recoverable because we cannot re-apply the condition of Corollary 1 to eliminate $\mathbf{Z}$ from $OR(X, Y \mid \mathbf{Z})$. Moreover, the resulting quantity, $OR(X, Y \mid \mathbf{Z})$, though estimable for every level $\mathbf{Z} = \mathbf{z}$, does not represent a meaningful relation for decision making or interpretation, because it does not stand for a causal effect in a stable subset of individuals (see discussion about the causal OR at the end of this section). Since $\mathbf{Z}$ is X-dependent in G, the class of units for which $\mathbf{Z} = \mathbf{z}$ under $do(X = 1)$ is not the same as the class of units for which $\mathbf{Z} = \mathbf{z}$ under $do(X = 0)$. The conditional odds ratio $OR(X, Y \mid \mathbf{Z})$ would be meaningful only if $\mathbf{Z}$ is restricted to pre-treatment covariates, which are X-invariant, hence stable.

[2] Cornfield's result and some of its graphical ramifications were brought to our attention by Sander Greenland. See also (Greenland and Pearl, 2011).

We next introduce a criterion, followed by a procedure to decide whether it is legitimate to replace $\mathbf{Z}$ with a set $\mathbf{C}$ of pre-treatment covariates, for which $OR(Y, X \mid \mathbf{C})$ is a meaningful $\mathbf{c}$-specific causal effect. Typical examples of $\mathbf{c}$-specific effects would be $\mathbf{C} = \{age, sex\}$ or, when average behavior is desired, $\mathbf{C} = \{\}$.

Definition 4 (OR-admissibility)**.** *A set* $\mathbf{Z} = \{Z_1, ..., Z_n\}$ *is OR-admissible relative to an ordered triplet* $(X, Y, \mathbf{C})$ *whenever an ordering* $(Z_1, ..., Z_n)$ *exists such that for each* Z_k*, either* $(X \perp\!\!\!\perp Z_k \mid \mathbf{C}, Y, Z_1, ..., Z_{k-1})$ *or* $(Y \perp\!\!\!\perp Z_k \mid \mathbf{C}, X, Z_1, ..., Z_{k-1})$*.*

Corollary 2 (Didelez et al. (2010))**.** *OR-admissibility of* $\mathbf{Z}$ *implies* $OR(Y, X \mid \mathbf{C}, \mathbf{Z}) = OR(Y, X \mid \mathbf{C})$*.*

This Corollary follows by successive application of Lemma 1 to the elements $Z_1, ..., Z_n$ of $\mathbf{Z}$.

Theorem 1 (OR G-recoverability)**.** *Let graph G contain the arrow $X \to Y$ and a set $\mathbf{C}$ of measured X-independent covariates. The $\mathbf{c}$-specific odds ratio $OR(Y, X \mid \mathbf{C})$ is G-recoverable from s-biased data if and only if there exists an additional set $\mathbf{Z}$ of measured variables such that the following conditions hold in G:*

1. $(X \perp\!\!\!\perp S \mid \{Y, \mathbf{Z}, \mathbf{C}\})_G$ *or* $(Y \perp\!\!\!\perp S \mid \{X, \mathbf{Z}, \mathbf{C}\})_G$*.*
2. $\mathbf{Z}$ *is OR-admissible relative to* $(X, Y, \mathbf{C})$*.*

Moreover, $OR(Y, X \mid \mathbf{C}) = OR(Y, X \mid \mathbf{C}, \mathbf{Z}, S = 1)$*.* [3]

Proof. See the supplementary material. □

Note that unlike the control of confounding, which requires averaging over the adjusted covariates, a single instantiation of the variables in $\mathbf{Z}$ is all that is needed for removing selection bias.

Let us consider the causal story of section 1 concerning the effect of Oestrogen (X) on Endometrial Cancer (Y) as depicted in in Fig. 1(c). This problem is solvable by setting $\mathbf{Z} = \{W\}$ and applying Theorem 1 – we can readily verify that $\mathbf{Z}$ is OR-admissible relative to $(X, Y, \{\})$ (i.e., $(W \perp\!\!\!\perp Y \mid X)$), and $(X \perp\!\!\!\perp S \mid \{Y, W\})$ holds. Thus, we can write $OR(Y, X) = OR(Y, X \mid W) = OR(X, Y \mid W) = OR(X, Y \mid W, S = 1)$, which shows a mapping from the target (unbiased) quantity (without any S) to the s-biased data (conditioned on $S = 1$, which was measured). (In the sequel we will drop G finding no need to distinguish conditional independencies from d-separation statements.) [4]

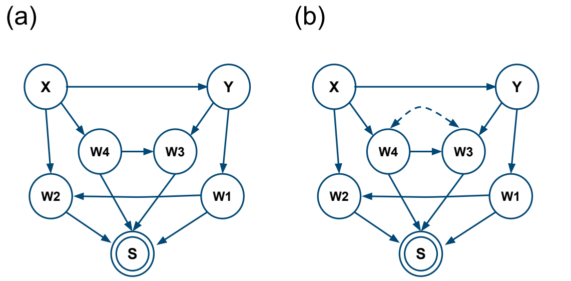


Figure 3: Scenario where OR is G-recoverable and $\mathbf{Z} = \{W_1, W_2, W_4\}$ (a), and it is not G-recoverable in (b).

Theorem 1 defines the boundary that distinguishes the class of graphs that permit G-recoverability of OR from those that do not. To show the power of Theorem 1, let us consider the more intricate scenario of Fig. 3(a), in which $\mathbf{Z} = \{W_1, W_2, W_4\}$ satisfies the conditions of Theorem 1. This can be seen through the following sequence of reductions verified by the graph: $(X \perp\!\!\!\perp S \mid \{Y, W_1, W_2, W_4\}) \to (Y \perp\!\!\!\perp W_2 \mid \{X, W_1, W_4\}) \to (X \perp\!\!\!\perp W_1 \mid \{Y, W_4\}) \to (Y \perp\!\!\!\perp W_4 \mid X)$. The final result is

$$OR(Y, X) = OR(Y, X \mid W_1, W_2, W_4, S = 1)$$

where the term on the left is our target quantity and the one on the right is estimable from the s-biased data. Fig. 3(b) shows an example where OR is not G-recoverable, because we must start with $\mathbf{Z} = \{W_1, W_2, W_3, W_4\}$ or $\mathbf{Z} = \{W_1, W_3, W_4\}$ to separate S from X or Y, respectively – these two sets are not OR-admissible since each set contains the variable W_3 which cannot be separated from X or Y by any set.

Theorem 1 relies on OR-admissibility, for which Definition 4 gives a declarative, non-procedural criterion. Taken literally, it requires that we first find a proper $\mathbf{Z}$ and then, out of the $n!$ orderings of the elements in Z, find one that will satisfy the d-separation tests specified in Definition 4. We will now supplement Theorem 1 with a simple graphical condition, followed by an effective procedure for finding such a sequence if one exists.

Theorem 2. *Let graph G contain the arrow $X \to Y$, a necessary condition for G to permit the G-recoverability of $OR(Y, X \mid \mathbf{C})$ for a given set $\mathbf{C}$ of pre-treatment covariates is that S and every ancestor A_i of S that is also a descendant of X have a separat-*

[3] This Theorem builds on and extends the results in (Didelez et al., 2010) which are summarized by Definition 4 and Corollary 2. First, it supplements the sufficient condition with its necessary counterpart. This is made possible by defining G-recoverability in terms of identifiability (Def. 2). Second, Theorem 1 explicitly avoid meaningless ORs (i.e., $OR(X, Y \mid \mathbf{Z})$, where $\mathbf{Z}$ is X-dependent). Finally, the proof of the sufficiency part prepares the ground for a procedure for finding an admissible sequence if such exists, to be shown next.

[4] Furthermore, the graph symmetric to Fig. 1(c) where the positions of X and Y are interchanged yields the same result. Similarly, another common variant of Fig. 1(c), with the edge $X \to W$ reversed, is solvable as well.

ing set $\mathbf{T_i}$ that either d-separates A_i from X given Y, or d-separates A_i from Y given X. [5]

Proof. See the supplementary material. □

Theorem 3. *Let G be a DAG containing the arrow $X \rightarrow Y$ and two sets of variables, measured $\mathbf{V}$ and unmeasured $\mathbf{U}$. A necessary and sufficient condition for G to permit the G-recoverability of $OR(Y, X \mid \mathbf{C})$ for a given set $\mathbf{C}$ of pre-treatment variables is when the sink-procedure below terminates. Moreover, $OR(Y, X \mid \mathbf{C}) = OR(Y, X \mid \mathbf{C}, \mathbf{Z}, \mathbf{T}, S = 1)$, where $\mathbf{Z} = \big(An(S) \setminus An(Y)\big) \cap \mathbf{V}$ and $\mathbf{T}$ is given by the sink-procedure.*

Procedure (Sink reduction)

1. Set $\mathbf{T} = \{\}$, and consider $\mathbf{Z}$ as previously defined. Remove $\mathbf{V} \setminus An(Y \cup S)$ from $\mathbf{G}$, and name the new graph $\mathbf{G}^*$. Consider an ordering compatible with $\mathbf{G}^*$ such that $Z_i < Z_j$ whenever Z_i is non-descendant of Z_j.

2. Test if sink Z_i of $\mathbf{G}^*$ satisfies the following condition: $(Z_i \perp\!\!\!\perp X \mid C, T, Y, Z_1, ..., Z_i - 1)$ or $(Z_i \perp\!\!\!\perp Y \mid C, T, X, Z_1, ..., Z_i - 1)$. If so, go to step 4. Otherwise, continue.

3. Test if there exists a minimal set $\mathbf{T_i}$ of non-descendants of $\mathbf{X}$ that, if added to $\mathbf{T}$ would render step 2 successful, if none exists, exit with failure.[5] Else, add $\mathbf{T_i}$ to $\mathbf{T}$ and continue with step 4.

4. Remove Z_i from $\mathbf{G}^*$ and $\mathbf{Z}$, and repeat step 2 recursively until $\mathbf{Z}$ is empty. If so, go to step 5.

5. Test if $(\mathbf{T} \perp\!\!\!\perp Y \mid \mathbf{C}, X)$, if so, the sequence $(Z_1, Z_2, ...Z_m)$ with $\mathbf{T}$ constitutes a witness for the OR-admissibility of $\mathbf{Z}$ relative to $(X, Y, \mathbf{C})$, for a set $\mathbf{C}$ of X-independent variables. Otherwise, exit with failure.

Proof. See the supplementary material. □

The algorithm exploits the graph structure to construct a mapping from the observed s-biased data and the desired target OR. Since the OR is symmetric, it is not necessary to separate S from X and Y simultaneously, but only from one of them (given the other.) For simplicity, denote the expression "X given Y or Y given X" by the symbol Φ_{xy}. A separating set from S to Φ_{xy} is first sought in step 2, starting with all observable ancestors of S that are non-ancestors of Y. If the test succeeds and this set is a separator, the algorithm iterates trying to separate Φ_{xy} from the deepest node in the remaining set. In case of failure, the algorithm attempts (step 3) to achieve separability using pre-treatment covariates $\mathbf{T_i}$. In case no separability can be found using these added covariates, the algorithm fails. Otherwise, at the end, the algorithm further requires that all $\mathbf{T_i}$ added along these iterations be separable from Y (step 5).

To illustrate, running the procedure on the graph of Fig. 3(b) with $\mathbf{C} = \{\}$, the graph remaining after the removal of S has two sink nodes, W_2 and W_3. Removing W_2 leaves two other sinks, W_3, and W_1. Removing W_1 leaves W_3 as the only remaining sink node which fails the test of Step 3. Since no non-descendant of X exists that yields separability, we must exit with failure. On the other hand, if we are able to measure U, the hidden variable responsible for the double arrow arc between W_3 and W_4, we would add this node to $\mathbf{T}$, W_3 will pass the test, followed by W_4, and we will end up with U as the only non-descendant of X remaining in $\mathbf{T}$. In step 5 we remove U from $\mathbf{T}$, yielding $OR(X, Y) = OR(X, Y \mid \mathbf{W}, U, S = 1)$.

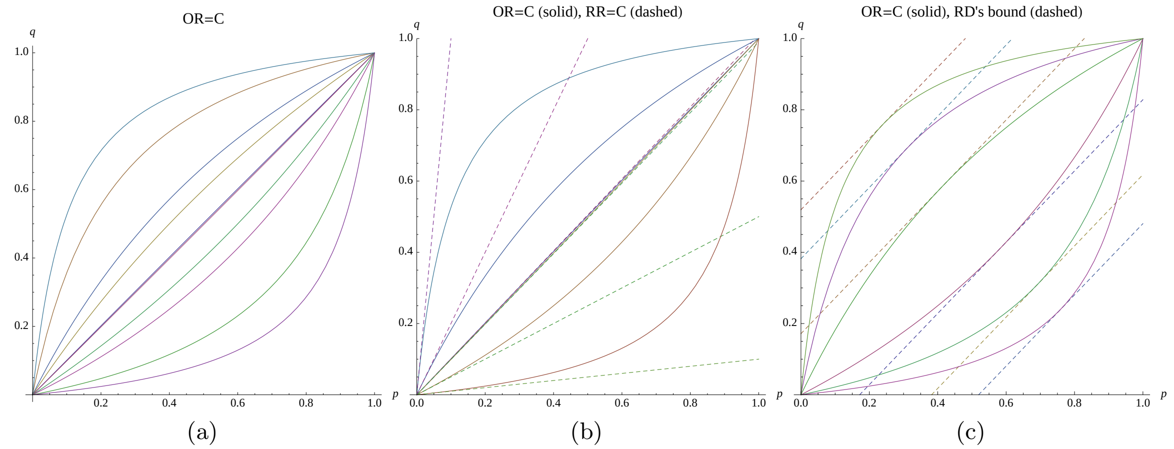
Thus far, we assumed that the treatment X is unconfounded, therefore the OR is identical to the causal OR defined as $COR(X, Y) = \frac{P(y|do(x))P(y'|do(x'))}{P(y|do(x'))P(y'|do(x))}$. In the presence of confounding, it is not enough to recover OR in s-biased data, we need to go further and assure that the recovered $OR(X, Y \mid \mathbf{C})$ is such that C satisfies the back-door criterion (2nd rule of do-calculus, observing and intervening are equivalent), in which case $OR(X, Y \mid \mathbf{C})$ will represent the **c**-specific causal OR. For example, in Fig. 2(c) the $COR(X, Y \mid \mathbf{C})$ will be G-recoverable because once we condition on $\mathbf{C}$ all conditional independencies will be identical to those of Fig. 1(c), and $P(Y \mid do(X), \mathbf{C}) = P(Y \mid X, \mathbf{C})$.

Note, however that although we can recover the **c**-specific causal OR, we cannot recover the population $COR(X, Y)$. For such measure to be recoverable we need to add assumptions which will make it possible to infer averageable measures of causal effects such as RD and RR, to be handle next.

3 OR and other measures of causal effects

Consider again the chain structure in Fig. 2(a) and define the causal effect as $COR(X, Y)$. The fact that X and Y are not confounded permits us to estimate the causal effect $COR(X, Y)$ by the odd ratio $OR(X, Y)$ which, by the results in the previous section, will re-

[5] A polynomial time algorithm for finding a minimal separating set in DAGs is given in (Tian et al., 1998). The *restricted minimal separation* version of that algorithm finds a minimal separator in a DAG with latent variables (equivalently, semi-Markovian models). A fast test for the non-separability of X and A_i is the existence of an inducing path between the two variables (Verma and Pearl, 1990). For example, the path $X \rightarrow W_4 \rightarrow W_3$ in Fig. 3(b).



(a) (b) (c)

Figure 4: (a) Constant odds ratio curves for $c = \{1.00, 1.01, 1.50, 2.00, 5.00, 10.00\}$ and their inverses; Superimposed constant odds ratio with constant risk ratio curves (b) and constant risk difference curves (c).

main invariant to conditioning on $S = 1$. However, if we define the causal effect as $ACE = Pr(y \mid do(x)) - Pr(y \mid do(x'))$ (also known as the causal risk difference), a bias will be introduced upon conditioning.

The invariance of OR can be represented in the following intuitive and pictorial way. We characterize the conditional distribution $P(Y \mid X)$ by two independent parameters $p = P(y \mid x)$ and $q = P(y \mid x')$, which define a point (p, q) in the unit square. The condition $OR(X, Y) = c$ describes a curve in the $(p, q)-$plane. For $c = 1$, the curve is the unit slope line. For $c > 1$, this curve separate points with $OR(.) > c$ from those with $OR(.) < c$ in the region below the unit slope line (symmetrically for the inverses ($c < 1$) in the region above $q = p$). See Fig. 4.

Now, by conditioning on $S = 1$, we obtain a new conditional probability, also characterized by two independent parameters $p_s = P(y \mid x, S = 1), q_s = P(y \mid x', S = 1)$. The fact that $OR(Y, X \mid S = 1) = OR(Y, X)$ means that conditioning on $S = 1$ must shift the initial (p, q) point along a constant OR curve, not anywhere else. We show these universal curves of constant OR for $c = \{1.00, 1.01, 1.50, 2.00, 5.00, 10.00\}$ and their respective inverses in Fig. 4(a). Fig. 4(b) shows curves for constant risk ratio (RR: $\frac{p}{q} = c$), which are variable slope lines going through the origin, and bounded by the slope $\frac{1}{c}$. Similarly, Fig. 4(c) shows curves for constant risk difference.

We see that even though RR does not remain constant (upon conditioning), the constancy of OR constrains the behavior of the RR. This follows by noting (after some algebra) that $RR = c + (1 - c)p$, i.e., RR has intercept c and slope $1 - c$. For instance, if OR is constant and $c = 1$, we have unit slope line for OR, but RR does not move and is equal to one. For constant OR and $\frac{1}{2} < c < 1$, the slope is positive but less than $\frac{1}{2}$, and the intercept is greater than $c = \frac{1}{2}$, which implies that RR lies inside the interval $[c, 1]$. Similar bounds can be obtained for other values of c.

Recovering RR and RD under selection bias

In this section we show that, in some situations, point estimates of RR and RD can be recoverable from s-biased data in studies where the prior probability $P(X)$ is available. [6] In other words, we refer back to the chain structure of Fig. 2(a) and ask whether $P(Y \mid X)$ can be recovered from $P(X)$ and $P(X, Y \mid S = 1)$.

The solution can be obtained algebraically, noting that Y d-separates X from S, which permits us to write:

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{\Big(P(Y \mid X)P(X) + P(Y \mid \neg X)P(\neg X)\Big)}$$

$$P(X \mid \neg Y) = \frac{P(\neg Y \mid X)P(X)}{\Big(P(\neg Y \mid X)P(X) + P(\neg Y \mid \neg X)P(\neg X)\Big)}$$

This can be turn into a two linear equations with two unknowns, $\{P(Y \mid X), P(Y \mid \neg X)\}$, which gives:

$$P(Y \mid X) = -\frac{P(X \mid Y)\Big(P(X \mid Y) - P(X)\Big)}{\Big(P(X \mid Y) - P(X \mid \neg Y)\Big)P(X)}$$

[6] Potentially, we are under a RCT setup or have an alternative way to access it through external studies as census' data.

$$P(Y \mid \neg X) = \frac{P(\neg X \mid Y)\Big(P(X \mid \neg Y) - P(X)\Big)}{\Big(P(X \mid \neg Y) - P(X \mid Y)\Big)P(\neg X)}, \tag{1}$$

where $P(X \mid Y) = P(X \mid Y, S=1), \forall X, Y.$[7]

This simple result exemplifies a general theme of correcting for selection bias (section 4); the bias induced by preferential selection can be removed if we have enough unconfounded variables that constraint the distribution of the remaining variables in a specific way.

Note that this case is different than as previously discussed in which we were just interested in the OR. Next we extend this result for more elaborated scenarios.

4 Randomization with non-compliance under selection bias

Let us consider the more general problem depicted in Fig. 5(a) in which confounding and selection biases are simultaneously present, and there are instrumental variables available.

Our goal is to infer the most accurate bounds for the causal effect of X on Y, knowing that there is no unbiased estimate for this quantity even when selection bias is not present. This scenario is usually presented under the rubric of "randomization with non-compliance", and it is pervasive in the Economics literature, we defer to (Pearl, 2009, Ch. 8) for a more comprehensive discussion of the relevance of this setup, we focus here on the technical aspects of the problem.

Generally, the bounding analysis assumes no selection bias, and the natural question that arises is whether selection bias can be treated and under which conditions bounds free from selection can be recovered.

We show next that this problem can be solved assuming the existence of two instrumental variables Z_1 and Z_2. [8] Noteworthy, the set of assumptions used in our analysis are commonplace in daily Econometrics practice, and its convoluted appearance is diluted when one observes them more vividly through the causal graph depicted in Fig. 5(a). In a nutshell, they are the same assumptions of randomization with non-compliance together with selection bias (such that treatment and outcome affect entry in the data pool).

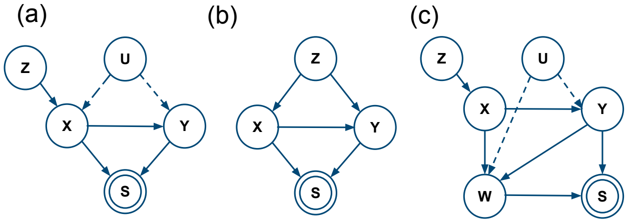


Figure 5: Different scenarios in which Theorem 4 can be applied. (a) Typical study with randomization and non-compliance (IV as incentive-mechanism) where selection and confounding are both present. (b) Selection bias in the back-door case. (c) More complex study with an intermediary variable W between treatment and selection. In this case, Y directly cause W and there is a common cause between them (extension of Fig. 1(c), see corollary 5.)

Theorem 4. *The joint distribution of $P(X, Y, \mathbf{Z})$ is recoverable from s-biased data whenever the following conditions hold: (i) the S node is affected by the set $\mathbf{Z}$ only through $\{X, Y\}$; (ii) the set $\mathbf{Z}$ is d-connected to $\{X, Y\}$ (and combinations); (iii) the dimensionality of $\mathbf{Z}$ matches the dimensionality of $\{X, Y\}$; (iv) the marginal probability of $\mathbf{Z}$ is known. In other words, the distribution $P(X, Y, \mathbf{Z})$ is recoverable from s-biased data whenever $(S \perp\!\!\!\perp \mathbf{Z} \mid X, Y)$, $(\mathbf{Z} \not\perp\!\!\!\perp \{X, Y\})$, $(\mathbf{Z} \not\perp\!\!\!\perp X \mid Y)$,$(\mathbf{Z} \not\perp\!\!\!\perp Y \mid X)$, the dimensionality of $\mathbf{Z}$ and $X \cup Y$ matches, and the marginal distribution of $P(\mathbf{Z})$ is given.*

Proof. See the supplementary material. □

Corollary 3. *The bounds for $P(y \mid do(x))$ in the scenario of randomization with non-compliance (Fig. 5(a)) are recoverable from s-biased data whenever the conditions of the Theorem 4 hold.*

Proof. It follows directly from Theorem 4 together with the bounds in (Balke and Pearl, 1997). □

Corollary 4. *The causal effect $P(y \mid do(x))$ in the back-door scenario (Fig. 5(b)) is recoverable from s-biased data whenever the conditions of the Theorem 4 hold.*

Proof. It follows directly from Theorem 4. □

Corollary 5. *The causal effect of Oestrogen (X) on Endometrial Cancer (Y) as studied in (Horwitz and Feinstein, 1978; Hernán et al., 2004) (Fig. 5(c)) is recoverable from s-biased data whenever there is an IV set $\mathbf{Z}$ pointing to X, and the conditions of the Theorem*

[7] In Epidemiology, there are many "longitudinal data settings" where selection bias is sequential, in which it can be possible easier to estimate the probability of selection instead of $P(X)$ – this observation was brought to our attention by Onyebuchi A. Arah.

[8] Call $\mathbf{Z} = Z_1 \cup Z_2$, or consider one IV with the same number of levels. Let us name both cases by instrumental variable set.

4 hold. Moreover, the same holds without relying on $\mathbf{Z}$ *whenever the following conditions hold: (i)* X *has the same dimensionality of* $\{W, Y\}$*; (ii) the marginal distribution of* $P(X)$ *is available.*

Proof. See the supplementary material. □

Some observations on the method

Methods that handle selection bias under different causal assumptions try to model the distribution of S, which is unobservable and usually hard to estimate; we take a different approach and avoid doing this explicit manipulation of the selection mechanism by exploiting the topology of the causal graph and the underlying data-generating process. We are not aware of other approaches trying to do so.

The main idea is to exploit the conditional independence of the IV set $\mathbf{Z}$ and the selection mechanism S given the distribution of the treatment and outcome – interestingly, the latter is what we seek to estimate. The method hinges on two properties about the induced system, that it is linearizable and full rank – both facts were not obvious nor expected a priori.

It is worth to make some additional remarks that follow the proof of Theorem 4. First note that the proposed method relies on a sample size approaching infinity, which is difficult to obtain in practice. As a possible improvement, the problem could be cast as an optimization problem. The formulation goes as follows. We associate error terms $\epsilon_{z_1 z_2, xy}$ to each $\gamma_{z_1 z_2, xy}$ term, and proceed the analysis minimizing the (square) mean error subject to constraints. The constraints emerge naturally from the induced system of equations together with the additional constraints of positivity and integrality. Our original goal was to show feasibility of removing selection bias (identifiability) but not the estimation per se, still, this should be an interesting exercise to pursue. Further investigation is needed to check the applicability of this suggestion.

We envision our method being used as a first step in a pre-processing stage, before the application of any bounding (Balke and Pearl, 1997) or estimation procedure. The method returns the same values of $P(X, Y, \mathbf{Z})$ whenever the collected data is not under selection bias, which means that its usage will not hurt and should be considered as a "good practice."

Finally, it is also important to mention that there are scenarios not solvable by our method or in which our assumptions are not applicable. For instance, we show in Fig. 6 one of this kind, in which selection and confounding biases are entangled in such way that it does not seem possible to detach one from another. We conjecture that this case is not solvable in general without further assumptions. Notice that even if we remove the edge $U \rightarrow X$, the example is still hard to resolve.

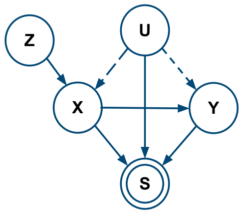


Figure 6: Scenario in which selection and confounding biases are present, entangled, and thus not recoverable.

5 Conclusion

We showed that qualitative knowledge of the selection mechanism and the use of instrumental variables can eliminate selection bias in many realistic problems. In particular, the paper provides a general graphical condition together with an algorithm that operates on a general DAG, with measured and unmeasured nodes, and decides whether and how a given **c**-specific odds ratio can be recovered from selection-biased data characterized by a selection node S. We further showed by algebraic methods that selection bias can be removed with the help of instrumental variables under a mild set of conditions.

This paper complements recent work on transportability (Pearl and Bareinboim, 2011) which deals with transferring causal information from one environment to another, in which only passive observations can be collected. The solution to the transportability problem assumes that disparities between the two environments are represented graphically in the form of unobserved factors capable of causing such disparities. The problem of selection bias also seeks extrapolation between two environments; from one in which samples are selected preferentially, to one in which no preferential sampling takes place. Both problems represent environmental differences in the form of auxiliary (selection) variables, the influence of which we seek to eliminate. However the semantics of those variables is different. In selection bias the auxiliary s-variables represent disparities in the data-gathering process, whereas in transportability problem they represent disparities in the structure of the data-generation process itself.

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References

Balke, A. and Pearl, J. (1997). Bounds on treatment effects from studies with imperfect compliance. *Journal of the American Statistical Association* **92** 1172–1176.

Cooper, G. (1995). Causal discovery from data in the presence of selection bias. *Artificial Intelligence and Statistics* 140–150.

Cornfield, J. (1951). A method of estimating comparative rates from clinical data; applications to cancer of the lung, breast, and cervix. *Journal of the National Cancer Institute* **11** 1269–1275.

Didelez, V., Kreiner, S. and Keiding, N. (2010). Graphical models for inference under outcome-dependent sampling. *Statistical Science* **25(3)** 368–387.

Geneletti, S., Richardson, S. and Best, N. (2009). Adjusting for selection bias in retrospective, case-control studies. *Biostatistics* **10(1)**.

Geng, Z. (1992). Collapsibility of relative risk in contingency tables with a response variable. *Journal Royal Statistical Society* **54** 585–593.

Glymour, M. and Greenland, S. (2008). Causal diagrams. In *Modern Epidemiology* (K. Rothman, S. Greenland and T. Lash, eds.), 3rd ed. Lippincott Williams & Wilkins, Philadelphia, PA, 183–209.

Greenland, S. and Pearl, J. (2011). Adjustments and their consequences – collapsibility analysis using graphical models. *International Statistical Review* **79** 401–426.

Heckman, J. (1970). Sample selection bias as a specification error. *Econometrica* **47** 153–161.

Hein, M. (2009). Binary classification under sample selection bias. In *Dataset Shift in Machine Learning* (J. Candela, M. Sugiyama, A. Schwaighofer and N. Lawrence, eds.). MIT Press, Cambridge, MA, 41–64.

Hernán, M., Hernández-Díaz, S. and Robins, J. (2004). A structural approach to selection bias. *Epidemiology* **15** 615–625.

Horwitz, R. and Feinstein, A. (1978). Alternative analytic methods for case-control studies of estrogens and endometrial cancer. *New England Journal of Medicine* **299** 368–387.

Lauritzen, S. L. and Richardson, T. S. (2008). Discussion of mccullagh: Sampling bias and logistic models. *J. Roy. Statist. Soc. Ser. B* **70** 140–150.

Pearl, J. (2009). *Causality: Models, Reasoning, and Inference*. 2nd ed. Cambridge University Press, New York.

Pearl, J. (2010). On a class of bias-amplifying variables that endanger effect estimates. In *Proceedings of the Twenty-Sixth Conference on Uncertainty in Artificial Intelligence*. AUAI, Corvallis, OR, 425–432.

Pearl, J. and Bareinboim, E. (2011). Transportability of causal and statistical relations: A formal approach. In *Proceedings of the Twenty-Fifth National Conference on Artificial Intelligence*. AAAI Press, Menlo Park, CA, 247–254.

Robins, J. (2001). Data, design, and background knowledge in etiologic inference. *Epidemiology* **12** 313–320.

Robins, J. M., Hernan, M. and Brumback, B. (2000). Marginal structural models and causal inference in epidemiology. *Epidemiology* **11** 550–560.

Smith, A. T. and Elkan, C. (2007). Making generative classifiers robust to selection bias. In *Proceedings of the 13th ACM SIGKDD international conference on Knowledge discovery and data mining*. KDD '07, ACM, New York, NY, USA.

Storkey, A. (2009). When training and test sets are different: characterising learning transfer. In *Dataset Shift in Machine Learning* (J. Candela, M. Sugiyama, A. Schwaighofer and N. Lawrence, eds.). MIT Press, Cambridge, MA, 3–28.

Tian, J., Paz, A. and Pearl, J. (1998). Finding minimal separating sets. Tech. Rep. R-254, <http://ftp.cs.ucla.edu/pub/stat_ser/r254.pdf>, Computer Science Department, University of California, Los Angeles, CA.

Verma, T. and Pearl, J. (1990). Equivalence and synthesis of causal models. In *Uncertainty in Artificial Intelligence, Proceedings of the Sixth Conference*. Cambridge, MA. Also in P. Bonissone, M. Henrion, L.N. Kanal and J.F. Lemmer (Eds.), *Uncertainty in Artificial Intelligence 6*, Elsevier Science Publishers, B.V., 255–268, 1991.

Whittemore, A. (1978). Collapsibility of multidimensional contingency tables. *Journal of the Royal Statistical Society, Series B* **40** 328–340.

Zadrozny, B. (2004). Learning and evaluating classifiers under sample selection bias. In *ICML* (C. E. Brodley, ed.), vol. 69 of *ACM International Conference Proceeding Series*. ACM.

Zhang, J. (2008). On the completeness of orientation rules for causal discovery in the presence of latent confounders and selection bias. *Artif. Intell.* **172** 1873–1896.