Appendix

Proof of Lemma 4.1. Throughout the proof, we drop the subscript i on τ_i to ease the notation. Note that $q_i^{\tau(s+1)} = q_i^{\tau(s)+1}$ since the distribution is not updated when algorithm $\mathcal{A}_i$ is not invoked. Hence, conditioned on $\mathcal{F}_{\tau(s)}$, the variable $(q_i^{\tau(s+1)} - e_u)$ can be taken out of the expectation. We therefore need to show that

$$(q_i^{\tau(s+1)} - e_u) \cdot \mathbb{E}\left\{ Le_{j_{\tau(s+1)}} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \tag{6}$$

$$= (q_i^{\tau(s+1)} - e_u) \cdot \mathbb{E}\left\{ \tilde{f}_i^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \tag{7}$$

First, we can write $\mathbb{E}\left\{ h_{(i,j)}^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}$ as

$$\begin{aligned}
&\mathbb{E}\left\{ \sum_{t=\tau(s)+1}^{\tau(s+1)} b_{(i,j)}^t \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbb{E}\left\{ \sum_{t=\tau(s)+1}^{\infty} b_{(i,j)}^t \mathbf{I}\left\{ t \le \tau(s+1) \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ \mathbb{E}\left[b_{(i,j)}^t \mathbf{I}\left\{ t \le \tau(s+1) \right\} \;\middle|\; \mathcal{F}_{t-1} \right] \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} \mathbb{E}\left[b_{(i,j)}^t \;\middle|\; \mathcal{F}_{t-1} \right] \;\middle|\; \mathcal{F}_{\tau(s)} \right\}
\end{aligned}$$

The last step follows because the event $\{t \le \tau(s+1)\}$ is $\mathcal{F}_{t-1}$-measurable (that is, variables $k_1, \ldots, k_{t-1}$ determine the value of the indicator). By Eq. (2), we conclude

$$\begin{aligned}
&\mathbb{E}\left\{ h_{(i,j)}^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \qquad (8) \\
&= \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} p_i^t (e_j - e_i)^\intercal Le_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}
\end{aligned}$$

Since $\mathbf{I}\{t = \tau(s+1)\} = \mathbf{I}\{k_t = i\} \mathbf{I}\{t \le \tau(s+1)\}$, we have

$$\begin{aligned}
&\mathbb{E}\left\{ \mathbf{I}\left\{ t = \tau(s+1) \right\} e_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbb{E}\left\{ \mathbb{E}\left\{ \mathbf{I}\left\{ k_t = i \right\} \mathbf{I}\left\{ t \le \tau(s+1) \right\} e_{j_t} \mid \mathcal{F}_{t-1} \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} e_{j_t} \mathbb{E}\left\{ \mathbf{I}\left\{ k_t = i \right\} \mid \mathcal{F}_{t-1} \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} \mathbb{P}(k_t = i \mid \mathcal{F}_{t-1}) e_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} p_i^t e_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}.
\end{aligned}$$

Combining with Eq. (8),

$$\begin{aligned}
&\mathbb{E}\left\{ h_{(i,j)}^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ \mathbf{I}\left\{ t \le \tau(s+1) \right\} p_i^t (e_j - e_i)^\intercal Le_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ \mathbf{I}\left\{ t = \tau(s+1) \right\} (e_j - e_i)^\intercal Le_{j_t} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}
\end{aligned}$$

Observe that coordinates of $\tilde{f}_i^{s+1}$, $q_i^{\tau(s+1)}$, and e_u are zero outside of N_i. We then have that the jth coordinate (for $j \in [N]$) of the vector $\mathbb{E}\left\{ \tilde{f}_i^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}$ is equal to

$$\begin{aligned}
&\mathbf{I}\left\{ j \in N_i \right\} \mathbb{E}\left\{ h_{(i,j)}^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbf{I}\left\{ j \in N_i \right\} \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ (e_j - e_i)^\intercal Le_{j_t} \mathbf{I}\left\{ t = \tau(s+1) \right\} \middle| \mathcal{F}_{\tau(s)} \right\} \\
&= \mathbf{I}\left\{ j \in N_i \right\} \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ e_j Le_{j_t} \mathbf{I}\left\{ t = \tau(s+1) \right\} \middle| \mathcal{F}_{\tau(s)} \right\} - c 1_{N_i}
\end{aligned}$$

where

$$c = \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ e_i Le_{j_t} \mathbf{I}\left\{ t = \tau(s+1) \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}$$

is a scalar. When multiplying the above expression by $q_i^{\tau(s+1)} - e_u$, the term $c \cdot 1_{N_i}$ vanishes. Thus, minimizing regret with relative costs (with respect to the ith action) is the same as minimizing regret with the absolute costs. We conclude that

$$\begin{aligned}
&(q_i^{\tau(s+1)} - e_u) \mathbb{E}\left\{ \tilde{f}_i^{s+1} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= (q_i^{\tau(s+1)} - e_u) \\
&\quad \times \left[\sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ e_j Le_{j_t} \mathbf{I}\left\{ t = \tau(s+1) \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \right]_{j \in N_i} \\
&= (q_i^{\tau(s+1)} - e_u) \sum_{t=\tau(s)+1}^{\infty} \mathbb{E}\left\{ Le_{j_t} \mathbf{I}\left\{ t = \tau(s+1) \right\} \;\middle|\; \mathcal{F}_{\tau(s)} \right\} \\
&= (q_i^{\tau(s+1)} - e_u) \cdot \mathbb{E}\left\{ Le_{j_{\tau(s+1)}} \;\middle|\; \mathcal{F}_{\tau(s)} \right\}
\end{aligned}$$

□