Appendix

Proof of Lemma 4.1. Throughout the proof, we drop the subscript $i$ on $\tau$ to ease the notation. Note that $q_{i}^{\tau(s+1)} = q_{i}^{\tau(s)+1}$ since the distribution is not updated when algorithm $A_{i}$ is not invoked. Hence, conditioned on $F_{\tau(s)}$, the variable $(q_{i}^{\tau(s+1)} - e_{u})$ can be taken out of the expectation. We therefore need to show that

$$
(q_{i}^{\tau(s+1)} - e_{u}) \cdot E \left\{ L_{e_{j},t_{s+1}} \right\} \bigg| F_{\tau(s)} \tag{6}
$$

$$
(q_{i}^{\tau(s+1)} - e_{u}) \cdot E \left\{ \tilde{f}_{i}^{s+1} \right\} \bigg| F_{\tau(s)} \tag{7}
$$

First, we can write $E \left\{ h_{(i,j)}^{s+1} \bigg| F_{\tau(s)} \right\}$ as

$$
E \left\{ \sum_{t = \tau(s)+1}^{\tau(s+1)} b_{(i,j)}^{t} \bigg| F_{\tau(s)} \right\}
$$

$$
= E \left\{ \sum_{t = \tau(s)+1}^{\infty} b_{(i,j)}^{t} I \{ t \leq \tau(s+1) \} \bigg| F_{\tau(s)} \right\}
$$

$$
= \sum_{t = \tau(s)+1}^{\infty} E \left\{ E \left[ b_{(i,j)}^{t} I \{ t \leq \tau(s+1) \} \bigg| F_{t-1} \right] \bigg| F_{\tau(s)} \right\}
$$

$$
= \sum_{t = \tau(s)+1}^{\infty} E \left\{ I \{ t \leq \tau(s+1) \} \cdot E \left[ b_{(i,j)}^{t} \bigg| F_{t-1} \right] \bigg| F_{\tau(s)} \right\}
$$

The last step follows because the event $\{ t \leq \tau(s+1) \}$ is $F_{t-1}$-measurable (that is, variables $k_{1}, \ldots, k_{t-1}$ determine the value of the indicator). By Eq. (2), we conclude

$$
E \left\{ h_{(i,j)}^{s+1} \bigg| F_{\tau(s)} \right\} = \sum_{t = \tau(s)+1}^{\infty} E \left\{ I \{ t \leq \tau(s+1) \} \sum_{i} p_{i}^{t}(e_{j} - e_{i})^{T}L_{e_{j}} \bigg| F_{\tau(s)} \right\}
$$

Since $I \{ t = \tau(s+1) \} = I \{ k_{t} = i \} I \{ t \leq \tau(s+1) \}$, we have

$$
E \left\{ I \{ t = \tau(s+1) \} e_{j_{t}} \bigg| F_{\tau(s)} \right\} = E \left\{ E \left[ I \{ k_{t} = i \} I \{ t \leq \tau(s+1) \} e_{j_{t}} \bigg| F_{t-1} \right] \bigg| F_{\tau(s)} \right\}
$$

$$
= E \left\{ I \{ t \leq \tau(s+1) \} e_{j_{t}} \cdot E \left[ I \{ k_{t} = i \} \bigg| F_{t-1} \right] \bigg| F_{\tau(s)} \right\}
$$

$$
= E \left\{ I \{ t \leq \tau(s+1) \} F(k_{t} = i) \bigg| F_{t-1} \right\} e_{j_{t}} \bigg| F_{\tau(s)} \right\}
$$

$$
= E \left\{ I \{ t \leq \tau(s+1) \} \sum_{i} p_{i}^{t}e_{j_{t}} \bigg| F_{\tau(s)} \right\}
$$

Combining with Eq. (8),

$$
E \left\{ h_{(i,j)}^{s+1} \bigg| F_{\tau(s)} \right\} = \sum_{t = \tau(s)+1}^{\infty} E \left\{ I \{ t \leq \tau(s+1) \} \sum_{i} p_{i}^{t}(e_{j} - e_{i})^{T}L_{e_{j}} \bigg| F_{\tau(s)} \right\}
$$

$$
= \sum_{t = \tau(s)+1}^{\infty} E \left\{ I \{ t = \tau(s+1) \} (e_{j} - e_{i})^{T}L_{e_{j}} \bigg| F_{\tau(s)} \right\}
$$

Observe that coordinates of $\tilde{f}_{i}^{s+1}$, $q_{i}^{\tau(s+1)}$, and $e_{u}$ are zero outside of $N_{i}$. We then have that the $j$th coordinate (for $j \in [N]$) of the vector $E \left\{ \tilde{f}_{i}^{s+1} \bigg| F_{\tau(s)} \right\}$ is equal to

$$
I \{ j \in N_{i} \} E \left\{ h_{(i,j)}^{s+1} \bigg| F_{\tau(s)} \right\}
$$

$$
= I \{ j \in N_{i} \} \sum_{t = \tau(s)+1}^{\infty} E \left\{ (e_{j} - e_{i})^{T}L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\}
$$

$$
= I \{ j \in N_{i} \} \sum_{t = \tau(s)+1}^{\infty} E \left\{ e_{j}^{T} L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\} - c1_{N_{i}}
$$

where

$$
c = \sum_{t = \tau(s)+1}^{\infty} E \left\{ e_{j}^{T} L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\}
$$

is a scalar. When multiplying the above expression by $q_{i}^{\tau(s+1)} - e_{u}$, the term $c \cdot 1_{N_{i}}$ vanishes. Thus, minimizing regret with relative costs (with respect to the $i$th action) is the same as minimizing regret with the absolute costs. We conclude that

$$
(q_{i}^{\tau(s+1)} - e_{u}) E \left\{ \tilde{f}_{i}^{s+1} \bigg| F_{\tau(s)} \right\}
$$

$$
= (q_{i}^{\tau(s+1)} - e_{u}) \times \left[ \sum_{t = \tau(s)+1}^{\infty} E \left\{ e_{j}^{T} L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\} \right]_{j \in N_{i}}
$$

$$
= (q_{i}^{\tau(s+1)} - e_{u}) \sum_{t = \tau(s)+1}^{\infty} E \left\{ L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\}
$$

$$
= (q_{i}^{\tau(s+1)} - e_{u}) \cdot E \left\{ L_{e_{j}} I \{ t = \tau(s+1) \} \bigg| F_{\tau(s)} \right\}
$$

$\square$