Supplementary Material

A Proofs

A.1 Proof of Theorem 3.1

Proof. Define $\tilde{K}_A(x, x') := K(x, x') + \eta \tilde{k}_x (A + \eta \tilde{K})^{-1} \tilde{k}_{x'}$. Then since $\sup_{x \in \mathcal{X}} ||\tilde{k}_x||^2 = n\tilde{n}$ and since $\tilde{Q}$ is positive definite,

$$\sup_{x, x' \in \mathcal{X}} |\tilde{K}_A(x, x') - \tilde{K}(x, x')| = \eta \sup_{x, x' \in \mathcal{X}} |\tilde{k}_x (A + \eta \tilde{K})^{-1} (\tilde{Q}^{-1} + \eta \tilde{K})^{-1} \tilde{k}_{x'}|$$

$$\leq \eta ||(A + \eta \tilde{K})^{-1} - (\tilde{Q}^{-1} + \eta \tilde{K})^{-1}||_2 \sup_{x \in \mathcal{X}} ||\tilde{k}_x||^2$$

$$\leq \epsilon + \text{h.o.t.}$$

where h.o.t. denotes terms in $\epsilon^2$ or greater. Define $\tilde{K}_A(x, x') := K(x, x') + \eta \tilde{k}_x (A + \eta \tilde{K})^{-1} \tilde{k}_{x'}$. Then since $\sup_{x \in \mathcal{X}} ||\tilde{k}_x||^2 = n\tilde{n}$ and since $\tilde{Q}$ is positive definite,

The higher order terms involve terms in $\epsilon^2$ or higher order and potentially depend upon $n$. Thus by choosing $\epsilon$ asymptotically sufficiently small w.r.t. $n$ (for example if h.o.t. \(\in O(f(n))\) then choosing $\epsilon = \frac{\text{const}}{f(n)}$ suffices) then asymptotically $\epsilon$ dominates the higher order terms as both $n \to \infty$ and $\epsilon \to 0$ so that we can remove the higher order terms from the asymptotic analysis (for example by making a substitution $\epsilon' = \epsilon + \text{h.o.t.}$).

A.2 Proof of Theorem 4.3

Proof. Theorem 4.2 implies that in time $O(\tilde{n} s \log n (\log \log n)^2 \log \frac{\tilde{n} n\kappa}{\lambda_{\min}} + \tilde{n}^2 n)$, we can compute an $A$ such that,

$$|| (A + \eta K) - (\tilde{Q}^{-1} + \eta K) ||_2 < \frac{\epsilon \lambda_{\min}}{\eta \kappa}$$

which implies that (see for example Horn and Johnson, 1990, section 5.8),

$$|| (A + \eta K)^{-1} - (\tilde{Q}^{-1} + \eta K)^{-1} ||_2 < \frac{\epsilon}{\eta \kappa} + \text{h.o.t.}$$

A.3 Proof of Theorem 4.4

Proof. We begin by making $p\tilde{n}$ calls to the solver of Koutis et al. (2011) to iteratively solve the equations

$$R z^{(j)}_i = z^{(j-1)}_i$$

for each $i$ where $x_i \in \tilde{K}_S$ and all $1 \leq j \leq p$ and where $z^{(0)}_i = e_{s_i}$. This gives $z^{(j)}_i = R^+ z^{(j-1)}_i + r^{(j)}_i$ such that

$$||r^{(j)}_i||_R \leq \epsilon ||R^+ z^{(j-1)}_i||_R,$$

in total time $O(p\tilde{n} s \log n (\log \log n)^2 \log \frac{1}{\epsilon})$ by Lemma 4.1. Now note that,

$$z^{(j)}_i = R^+ z^{(j-1)}_i + r^{(j)}_i$$

$$= R^+ (R^+ z^{(j-2)}_i + r^{(j-1)}_i) + r^{(j)}_i$$

$$= (R^+)^j z^{(0)}_i + (R^+)^{j-1} r^{(1)}_i + \ldots + R^+ r^{(j-1)}_i + r^{(j)}_i.$$

Thus,

$$||z^{(j)}_i - (R^+)^j e_{s_i}||_R \leq ||(R^+)^{j-1} r^{(1)}_i||_R + \ldots + ||R^+ r^{(j-1)}_i||_R + ||r^{(j)}_i||_R$$

$$\leq ||R^+||_2^{j-1} ||r^{(1)}_i||_R + \ldots + ||R^+||_2 ||r^{(j-1)}_i||_R + ||r^{(j)}_i||_R$$

(16)
where h.o.t. denotes terms in $\epsilon^2$ or greater. And so plugging this into (16) gives,

$$
||z_i^{(j)} - (R^+)^j e_s||_R \leq j\epsilon ||R^+||_2^{(j-\frac{1}{2})} + \text{h.o.t.}
$$

$$
||z_i^{(j)} - (R^+)^j e_s||_R \leq j\epsilon ||R||_2^{(j-1)/2} ||R^+||_2^{(j-\frac{1}{2})} + \text{h.o.t.}
$$

$$
||z_i^{(p)} - Q^+ e_s||_Q \leq p\epsilon |||R||_2^{(p-1)/2} ||R^+||_2^{(p-\frac{1}{2})} + \text{h.o.t.}
$$

Now let $Z := \left( z_1^{(p)} \ldots z_n^{(p)} \right)$ and

$$
A := Z^T QZ = Z^T R^T Z,
$$

and note that $A$ can be computed with $O(psn + \hat{n}^2n)$ operations since $R$ has $s$ non-zero entries. Now note that,

$$
|\hat{Q}_{ij}^T - A_{ij}| = |Q_{s,ij}^+ - A_{ij}|
$$

$$
= |e_i^T Q^+ e_s - z_i^{(p)} Q z_j^{(p)}|
$$

$$
= |(Q^+ e_s - z_i^{(p)})^T Q Q^+ e_s + (Q^+ e_s - z_j^{(p)})^T Q Q^+ e_s - (Q^+ e_s - z_i^{(p)})^T Q (Q^+ e_s - z_j^{(p)})|
$$

$$
\leq ||Q^+ e_s - z_i^{(p)}||_Q \sqrt{e_s^T Q^+ e_s} + ||Q^+ e_s - z_i^{(p)}||_Q \sqrt{e_s^T Q^+ e_s}
$$

$$
+ ||Q^+ e_s - z_j^{(p)}||_Q ||Q^+ e_s - z_j^{(p)}||_Q
$$

$$
\leq 2p\epsilon ||R||_2^{(p-1)/2} ||R^+||_2^{\frac{3p-1}{2}} + \text{h.o.t.}
$$

which, after setting $\epsilon'$ such that $\epsilon = 2p\epsilon' |||R||_2^{(p-1)/2} ||R^+||_2^{\frac{3p-1}{2}}$, we have that in time complexity,

$$
O(p\hat{n}s \log n (\log \log n)^2 \log \frac{1}{\epsilon'} + p\hat{n}^2s)
$$

$$
= O(p\hat{n}s \log n (\log \log n)^2 (\log p + p \log ||R||_2 ||R^+||_2 + \log \frac{1}{\epsilon'}) + \hat{n}^2n)
$$

the guarantee,

$$
|\hat{Q}_{ij}^T - A_{ij}| \leq \epsilon + \text{h.o.t.}
$$

and the higher order terms can be removed as in the proof of Theorem 4.3. \qed