Supplementary Material

A Proofs

A.1 Proof of Theorem3.1

Proof. We just need to check the reproducing property of $K = \boldsymbol{R}^{+}$ for all $v \in \mathcal{V}$ and $\boldsymbol{h} \in \mathrm{im}(\boldsymbol{R})$: $\langle \boldsymbol{h}, K(v_i, \cdot)\rangle_{\mathcal{H}} = \langle \boldsymbol{h}, \boldsymbol{R}^{+}\boldsymbol{e}_i\rangle_{\mathcal{H}} = \boldsymbol{h}^{\top}\boldsymbol{R}\boldsymbol{R}^{+}\boldsymbol{e}_i = \boldsymbol{h}^{\top}\boldsymbol{e}_i = h_i = h(v_i)$. □

A.2 Proof of Theorem 4.3

Proof. Theorem 4.2 implies that in time $\mathcal{O}(\widehat{n}s\log n(\log\log n)^2\log\frac{q\widehat{n}\eta\kappa}{\epsilon\lambda^2_{\min}} + \widehat{n}^2 n)$, we can compute an $\boldsymbol{A}$ such that,

$$||(\boldsymbol{A} + \eta\boldsymbol{K}) - (\widehat{\boldsymbol{Q}}^{-1} + \eta\boldsymbol{K})||_2 < \frac{\epsilon\lambda^2_{\min}}{\eta\widehat{n}\kappa} \tag{13}$$

which implies that (see for example Horn and Johnson, 1990, section 5.8),

$$||(\boldsymbol{A} + \eta\boldsymbol{K})^{-1} - (\widehat{\boldsymbol{Q}}^{-1} + \eta\boldsymbol{K})^{-1}||_2 < \frac{\epsilon}{\eta\widehat{n}\kappa} + \text{h.o.t} \tag{14}$$

where h.o.t. denotes terms in ϵ^2 or greater. Define $\breve{K}_{\boldsymbol{A}}(x, x') := K(x, x') + \eta\widehat{\boldsymbol{k}}_x^{\top}(\boldsymbol{A} + \eta\widehat{\boldsymbol{K}})^{-1}\widehat{\boldsymbol{k}}_{x'}$. Then since $\sup_{x\in\mathcal{X}} ||\widehat{\boldsymbol{k}}_x||^2 = \kappa\widehat{n}$ and since $\widehat{\boldsymbol{Q}}$ is positive definite,

$$\begin{aligned}
\sup_{x,x'\in\mathcal{X}} |\breve{K}_{\boldsymbol{A}}(x, x') - \breve{K}(x, x')| &= \eta \sup_{x,x'\in\mathcal{X}} |\widehat{\boldsymbol{k}}_x^{\top}\left((\boldsymbol{A} + \eta\boldsymbol{K})^{-1} - (\widehat{\boldsymbol{Q}}^{-1} + \eta\boldsymbol{K})^{-1}\right)\widehat{\boldsymbol{k}}_{x'}| \\
&\le \eta||(\boldsymbol{A} + \eta\boldsymbol{K})^{-1} - (\widehat{\boldsymbol{Q}}^{-1} + \eta\boldsymbol{K})^{-1}||_2 \sup_{x\in\mathcal{X}} ||\widehat{\boldsymbol{k}}_x||^2 \\
&\le \epsilon + \text{h.o.t.}
\end{aligned}$$

The higher order terms involve terms in ϵ^2 or higher order and potentially depend upon n. Thus by choosing ϵ asymptotically sufficiently small w.r.t. n (for example if h.o.t $\in O(f(n))$ then choosing $\epsilon = \frac{\text{const.}}{f(n)}$ suffices) then asymptotically ϵ dominates the higher order terms as both $n \to \infty$ and $\epsilon \to 0$ so that we can remove the higher order terms from the asymptotic analysis (for example by making a substitution $\epsilon' = \epsilon + \text{h.o.t.}$). □

A.3 Proof of Theroem 4.4

Proof. We begin by making $p\widehat{n}$ calls to the solver of Koutis et al. (2011) to iteratively solve the equations

$$\boldsymbol{R}\boldsymbol{z}_i^{(j)} = \boldsymbol{z}_i^{(j-1)}$$

for each i where $\boldsymbol{x}_{s_i} \in \widehat{\mathcal{X}}_{\mathcal{S}}$ and all $1 \le j \le p$ and where $\boldsymbol{z}_i^{(0)} = \boldsymbol{e}_{s_i}$. This gives $\boldsymbol{z}_i^{(j)} = \boldsymbol{R}^{+}\boldsymbol{z}_i^{(j-1)} + \boldsymbol{r}_i^{(j)}$ such that

$$||\boldsymbol{r}_i^{(j)}||_{\boldsymbol{R}} \le \epsilon||\boldsymbol{R}^{+}\boldsymbol{z}_i^{(j-1)}||_{\boldsymbol{R}}, \tag{15}$$

in total time $\mathcal{O}(p\widehat{n}s\log n(\log\log n)^2\log\frac{1}{\epsilon})$ by Lemma 4.1. Now note that,

$$\begin{aligned}
\boldsymbol{z}_i^{(j)} &= \boldsymbol{R}^{+}\boldsymbol{z}_i^{(j-1)} + \boldsymbol{r}_i^{(j)} \\
&= \boldsymbol{R}^{+}(\boldsymbol{R}^{+}\boldsymbol{z}_i^{(j-2)} + \boldsymbol{r}_i^{(j-1)}) + \boldsymbol{r}_i^{(j)} \\
&= (\boldsymbol{R}^{+})^j\boldsymbol{z}_i^{(0)} + (\boldsymbol{R}^{+})^{j-1}\boldsymbol{r}_i^{(1)} + \ldots + \boldsymbol{R}^{+}\boldsymbol{r}_i^{(j-1)} + \boldsymbol{r}_i^{(j)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
||\boldsymbol{z}_i^{(j)} - (\boldsymbol{R}^{+})^j\boldsymbol{e}_{s_i}||_{\boldsymbol{R}} &\le ||(\boldsymbol{R}^{+})^{j-1}\boldsymbol{r}_i^{(1)}||_{\boldsymbol{R}} + \ldots + ||\boldsymbol{R}^{+}\boldsymbol{r}_i^{(j-1)}||_{\boldsymbol{R}} + ||\boldsymbol{r}_i^{(j)}||_{\boldsymbol{R}} \\
&\le ||\boldsymbol{R}^{+}||_2^{j-1}||\boldsymbol{r}_i^{(1)}||_{\boldsymbol{R}} + \ldots + ||\boldsymbol{R}^{+}||_2||\boldsymbol{r}_i^{(j-1)}||_{\boldsymbol{R}} + ||\boldsymbol{r}_i^{(j)}||_{\boldsymbol{R}}
\end{aligned} \tag{16}$$

Now note, by repeatedly applying (15),

$$\begin{aligned}
||\boldsymbol{r}_i^{(k)}||_{\boldsymbol{R}} &\leq \epsilon||\boldsymbol{R}^+\boldsymbol{z}_i^{(k-1)}||_{\boldsymbol{R}} \\
&\leq \epsilon||\boldsymbol{R}^+||_2||\boldsymbol{z}_i^{(k-1)}||_{\boldsymbol{R}} \\
&\leq \epsilon||\boldsymbol{R}^+||_2\left(||\boldsymbol{R}^+\boldsymbol{z}_i^{(k-2)}||_{\boldsymbol{R}} + ||\boldsymbol{r}_i^{(k-1)}||_{\boldsymbol{R}}\right) \\
&\leq \epsilon||\boldsymbol{R}^+||_2\left(||\boldsymbol{R}^+||_2||\boldsymbol{z}_i^{(k-2)}||_{\boldsymbol{R}} + \epsilon||\boldsymbol{R}^+\boldsymbol{z}_i^{(k-2)}||_{\boldsymbol{R}}\right), \\
&\leq \epsilon||\boldsymbol{R}^+||_2^2||\boldsymbol{z}_i^{(k-2)}||_{\boldsymbol{R}} + \text{h.o.t.} \\
&\vdots \\
&\leq \epsilon||\boldsymbol{R}^+||_2^{k-1}||\boldsymbol{z}_i^{(1)}||_{\boldsymbol{R}} + \text{h.o.t.} \\
&\leq \epsilon||\boldsymbol{R}^+||_2^{k-1}||\boldsymbol{R}^+\boldsymbol{e}_{s_i}||_{\boldsymbol{R}} + \text{h.o.t.} \\
&\leq \epsilon||\boldsymbol{R}^+||_2^{k-\frac{1}{2}} + h.o.t.,
\end{aligned} \tag{17}$$

where h.o.t. denotes terms in ϵ^2 or greater. And so plugging this into (16) gives,

$$\begin{aligned}
||\boldsymbol{z}_i^{(j)} - (\boldsymbol{R}^+)^j\boldsymbol{e}_{s_i}||_{\boldsymbol{R}} &\leq j\epsilon||\boldsymbol{R}^+||_2^{(j-\frac{1}{2})} + \text{h.o.t.} \\
||\boldsymbol{z}_i^{(j)} - (\boldsymbol{R}^+)^j\boldsymbol{e}_{s_i}||_{\boldsymbol{R}^j} &\leq j\epsilon||\boldsymbol{R}||_2^{(j-1)/2}||\boldsymbol{R}^+||_2^{(j-\frac{1}{2})} + \text{h.o.t.} \\
||\boldsymbol{z}_i^{(p)} - \boldsymbol{Q}^+\boldsymbol{e}_{s_i}||_{\boldsymbol{Q}} &\leq p\epsilon||\boldsymbol{R}||_2^{(p-1)/2}||\boldsymbol{R}^+||_2^{(p-\frac{1}{2})} + \text{h.o.t.}
\end{aligned}$$

Now let $\boldsymbol{Z} := \begin{pmatrix} \boldsymbol{z}_1^{(p)} & \ldots & \boldsymbol{z}_{\widehat{n}}^{(p)} \end{pmatrix}$ and

$$\boldsymbol{A} := \boldsymbol{Z}^\top\boldsymbol{Q}\boldsymbol{Z} = \boldsymbol{Z}^\top\boldsymbol{R}^p\boldsymbol{Z},$$

and note that $\boldsymbol{A}$ can be computed with $\mathcal{O}(ps\widehat{n} + \widehat{n}^2 n)$ operations since $\boldsymbol{R}$ has s non-zero entries. Now note that,

$$\begin{aligned}
|\widehat{Q}_{ij}^+ - A_{ij}| &= |Q_{s_is_j}^+ - A_{ij}| \\
&= |\boldsymbol{e}_{s_i}^\top\boldsymbol{Q}^+\boldsymbol{e}_{s_j} - \boldsymbol{z}_i^{(p)^\top}\boldsymbol{Q}\boldsymbol{z}_j^{(p)}| \\
&= |(\boldsymbol{Q}^+\boldsymbol{e}_{s_i} - \boldsymbol{z}_i^{(p)})^\top\boldsymbol{Q}\boldsymbol{Q}^+\boldsymbol{e}_{s_j} + (\boldsymbol{Q}^+\boldsymbol{e}_{s_j} - \boldsymbol{z}_j^{(p)})^\top\boldsymbol{Q}\boldsymbol{Q}^+\boldsymbol{e}_{s_i} - (\boldsymbol{Q}^+\boldsymbol{e}_{s_i} - \boldsymbol{z}_i^{(p)})^\top\boldsymbol{Q}(\boldsymbol{Q}^+\boldsymbol{e}_{s_j} - \boldsymbol{z}_j^{(p)})| \\
&\leq ||\boldsymbol{Q}^+\boldsymbol{e}_{s_i} - \boldsymbol{z}_i^{(p)}||_{\boldsymbol{Q}}\sqrt{\boldsymbol{e}_{s_j}{}^\top\boldsymbol{Q}^+\boldsymbol{e}_{s_j}} + ||\boldsymbol{Q}^+\boldsymbol{e}_{s_j} - \boldsymbol{z}_j^{(p)}||_{\boldsymbol{Q}}\sqrt{\boldsymbol{e}_{s_i}{}^\top\boldsymbol{Q}^+\boldsymbol{e}_{s_i}} \\
&\quad + ||\boldsymbol{Q}^+\boldsymbol{e}_{s_i} - \boldsymbol{z}_i^{(p)}||_{\boldsymbol{Q}}||\boldsymbol{Q}^+\boldsymbol{e}_{s_j} - \boldsymbol{z}_j^{(p)}||_{\boldsymbol{Q}} \\
&\leq 2p\epsilon||\boldsymbol{R}||_2^{(p-1)/2}||\boldsymbol{R}^+||_2^{\frac{3p-1}{2}} + \text{h.o.t.},
\end{aligned}$$

which, after setting ϵ' such that $\epsilon = 2p\epsilon'||\boldsymbol{R}||_2^{(p-1)/2}||\boldsymbol{R}^+||_2^{\frac{3p-1}{2}}$, we have that in time complexity,

$$\begin{aligned}
&\mathcal{O}(p\widehat{n}s\log n(\log\log n)^2\log\frac{1}{\epsilon'} + p\widehat{n}^2 s) \\
&\qquad = \mathcal{O}(p\widehat{n}s\log n(\log\log n)^2(\log p + p\log||\boldsymbol{R}||_2||\boldsymbol{R}^+||_2 + \log\frac{1}{\epsilon}) + \widehat{n}^2 n)
\end{aligned}$$

the guarantee,

$$|\widehat{Q}_{ij}^+ - A_{ij}| \leq \epsilon + \text{h.o.t.},$$

and the higher order terms can be removed as in the proof of Theorem 4.3. □