These are the supplementary materials for the ICML 2013 paper entitled: Learning Multiple Behaviors from Unlabeled Demonstrations in a Latent Controller Space.

1 Sparse EM for a known number of clusters

Given the model presented in Section 3 of the paper, it is possible to derive an EM-algorithm for a known number of clusters K in the controller space. The complete penalized log-likelihood of the model corresponding to the MAP estimate given the prior on $\mathcal{B}$ can be written as:

$$\begin{aligned}
&\mathcal{L}(Y_{1:m}, X_{1:m}, C \mid \mathcal{B}) = \\
&\quad \textstyle\sum_{i=1}^{m} \Big(\log p(c_i = k) + \sum_{t=1}^{T} \log p(Y \mid X, c_i = k, \beta_k) \Big) \\
&\qquad\qquad + \textstyle\sum_k \log p(\beta_k),
\end{aligned}$$

where $\mathcal{B}$ is just a vector containing all β_k, $k \in 1..K$. The previous expression decomposes into two different terms. The first term corresponds to the likelihood of the trajectory given its assignment to one of the K clusters. The model assumes that the different points of the trajectory are independent of each other. This is clearly a simplification since we are estimating gradients and, therefore, error accumulates over time. However, this simplification allows us to evaluate the demonstrated trajectories directly over the potential created by the parameters β_k of each cluster and eliminates the need to simulate trajectories for the different clusters. The second term is the sparse term that encodes the penalty and does not directly depend on the correspondences. As in the DPMM algorithm, we used a Laplacian prior that results in Lasso estimates.

According to the previous observations, the previous model is very similar to the standard Gaussian mixture. Thus, the EM algorithm proceeds iteratively over two steps. First, the E-Step computes the expectation of the hidden variables c_i given the measurements and the current parameters. Given the parameters β_k, the E-step is not affected by the prior $p(\beta_k)$ and is simply proportional to the likelihood of each trajectory given the component parameters . The M-step, on the other hand, minimizes the sum of the square errors with a L1 norm and weighted by the expectation of the correspondences $\mathbb{E}(c_i)$ computed in the E-Step. Given the equivalence between the probabilistic model and the convex formulation we perform the maximization step using a general convex optimization solver.

2 Expression for $\int_\beta f(\cdot|\beta)p(\beta \mid \sigma^2)$

This appendix provides the expression for $q_0 = \int_\beta G_0(\beta)f(\cdot|\beta)\, d\beta$ when the likelihood model is a Multivariate Normal and the prior is a Laplacian distribution. After resolving the integral over β we get the following expression

$$\begin{aligned}
q_0 &= \left(\frac{\lambda}{2\sigma}\right)^p \frac{1}{\sqrt[T]{2\pi\sigma^2}} exp\left(-\frac{Y^TY}{2\sigma^2}\right) I \\
I &= \prod_{i=1}^{\hat{p}} (T_{1i} + T_{2i}) \prod_{i=\hat{p}+1}^{p} -\frac{2}{b_i} \\
T_{1i} &= exp\left(\frac{(b_i - e_i)^2}{2d_i}\right) \sqrt{\frac{\pi}{2d_i}} erfc\left(\frac{-(b_i - e_i)}{\sqrt{2d_i}}\right) \\
T_{2i} &= exp\left(\frac{(b_i + e_i)^2}{2d_i}\right) \sqrt{\frac{\pi}{2d_i}} erfc\left(\frac{-(b_i + e_i)}{\sqrt{2d_i}}\right)
\end{aligned}$$

where $\hat{p}$ is the rank of matrix $A = \frac{1}{\sigma^2}X^TX$ and d_i are the eigenvalues of matrix A. The integral requires to decompose matrix $A = SDS^{-1}$ to compute the vector $E^T = \frac{Y^T}{\sigma^2}XS$ and the vector $B^T = (\frac{-\lambda}{\sigma}, \cdots, \frac{-\lambda}{\sigma})R$, where matrix R is defined as each element of R is equal to the absolute value of each element of S.

3 Derivation of q_0

In the DPMM q_0 is defined as

$$q_0 = \int_{\boldsymbol{\beta}} G_0(\boldsymbol{\beta}) f(\boldsymbol{Y}|\boldsymbol{\beta})\ d\boldsymbol{\beta} \tag{1}$$

in our case

$$G_0(\boldsymbol{\beta}) = p(\boldsymbol{\beta}|\lambda) = \prod_{j=1}^{p} \frac{\lambda}{2\sqrt{\sigma^2}} exp\left(\frac{-\lambda|\beta_j|}{\sqrt{\sigma^2}}\right) \tag{2}$$

$$f(Y|\boldsymbol{\beta}) = N(\boldsymbol{Y}|\mathbf{X}\boldsymbol{\beta}, \sigma^2 I_{T\times T}) = \frac{1}{\sqrt[T]{2\pi\sigma^2}} exp\left(-\frac{1}{2}\frac{(\boldsymbol{Y}-\mathbf{X}\boldsymbol{\beta})^T(\boldsymbol{Y}-\mathbf{X}\boldsymbol{\beta})}{\sigma^2}\right) \tag{3}$$

where $\boldsymbol{Y}$ is a zero mean row vector $\mathbf{Y} = (y_1, \cdots, y_T)^T$, $\mathbf{X}$ is a data matrix, $\mathbf{X} \in \mathbb{R}^{T\times p}$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)^T$, while λ and σ are positive hyperparameters. Manipulating the numerator of the exponent

$$(\boldsymbol{Y}-\mathbf{X}\boldsymbol{\beta})^T(\boldsymbol{Y}-\mathbf{X}\boldsymbol{\beta}) = \boldsymbol{Y}^T\boldsymbol{Y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} - 2\boldsymbol{Y}^T\mathbf{X}\boldsymbol{\beta} \tag{4}$$

replacing (2),(3) and (4) in (1) results

$$q_0 = \int_{\boldsymbol{\beta}} \left[\prod_{j=1}^{p} \frac{\lambda}{2\sqrt{\sigma^2}} exp\left(\frac{-\lambda|\beta_j|}{\sqrt{\sigma^2}}\right)\right] \frac{1}{\sqrt[T]{2\pi\sigma^2}} exp\left(-\frac{\boldsymbol{Y}^T\boldsymbol{Y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} - 2\boldsymbol{Y}^T\mathbf{X}\boldsymbol{\beta}}{2\sigma^2}\right) d\boldsymbol{\beta} \tag{5}$$

$$q_0 = \left(\frac{\lambda}{2\sigma}\right)^p \frac{1}{\sqrt[T]{2\pi\sigma^2}} exp\left(-\frac{\boldsymbol{Y}^T\boldsymbol{Y}}{2\sigma^2}\right) \int_{-\infty}^{\infty} exp\left(\frac{-\lambda}{\sigma}\sum_{j=1}^{p}|\beta_j|\right) exp\left(-\frac{\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}}{2\sigma^2} + \frac{Y^T\mathbf{X}\boldsymbol{\beta}}{\sigma^2}\right) d\boldsymbol{\beta} \tag{6}$$

At this point we introduce some definitions in order to simplify notation.

$$\mathbf{A} = \frac{1}{\sigma^2}\mathbf{X}^T\mathbf{X} \tag{7}$$

$$\mathbf{J}^T = \frac{\boldsymbol{Y}^T}{\sigma^2}\mathbf{X}. \tag{8}$$

Substituting (7) and (8) in (6) we get

$$I = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\boldsymbol{\beta}^T\mathbf{A}\boldsymbol{\beta} + \mathbf{J}^T\boldsymbol{\beta} + \left(\frac{-\lambda}{\sigma}\sum_{j=1}^{p}|\beta_j|\right)\right) d\boldsymbol{\beta}. \tag{9}$$

The term $\boldsymbol{\beta}^T\mathbf{A}\boldsymbol{\beta}$ can be expressed as $\sum_{i=1}^{p}\sum_{j=1}^{p} A_{ij}\ \beta_i\beta_j$ where we realize that coupled terms make the integral difficult. In order to decoupled these terms and express the integral as a product of unidimensional integrals need to use the following change of variable.

Because matrix $\mathbf{A}$ is by construction (7) a symmetric square matrix with real coefficients, the spectral theorem ensures that eingenvalues of $\mathbf{A}$ are positive reals and that it exists an orthogonal matrix $\boldsymbol{S}$ such that $\mathbf{A}$ can be decomposed in the form $\mathbf{A} = \mathbf{SDS^{-1}} = \mathbf{SDS^T}$ with $\mathbf{D} = \mathbf{S^{-1}AS}$ a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$ denoted as d_i. Using this property, we can apply the following change of variable

$$\begin{aligned} \boldsymbol{\beta} &= \mathbf{Sz} \\ \beta_j &= \sum_{k=1}^{p} S_{jk} z_k \\ |\beta_j| &= \sum_{k=1}^{p} |S_{jk}||z_k| \\ d\beta_1 d\beta_2 \cdots d\beta_p &= |\mathbf{S}| dz_1 dz_2 \cdots dz_p, \end{aligned} \tag{10}$$

where the Jacobian of the transformation is just the matrix $\mathbf{S}$. Since $\mathbf{S}$ is an orthogonal matrix, the Jacobian determinant is $|\mathbf{S}| = 1$ and $d\beta_1 \cdots d\beta_p = dz_1 \cdots dz_p$. Now we can rewrite de equation (9) as

$$I = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{S^{-1}ASz} + \mathbf{J}^T\mathbf{Sz} + \left(\frac{-\lambda}{\sigma}\sum_{j=1}^{p}\sum_{k=1}^{p}|S_{jk}||z_k|\right)\right) dz_1 \cdots dz_2 \tag{11}$$

$$I = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{Dz} + \mathbf{J}^T\mathbf{Sz} + \left(\frac{-\lambda}{\sigma}\sum_{j=1}^{p}\sum_{k=1}^{p}|S_{jk}||z_k|\right)\right) dz_1 \cdots dz_2. \tag{12}$$

At this point we can simplify furher more the notation by defining

$$\begin{aligned} \mathbf{E^T} &= \mathbf{J^TS} \qquad (13) \\ \mathbf{C}^T &= (-\lambda/\sigma, \cdots, -\lambda/\sigma) \\ R_{jk} &= |S_{jk}| \\ \tilde{\mathbf{z}} &= (|z_1|, |z_2|, \cdots, |z_p|)^T \\ -\frac{\lambda}{\sigma}\sum_{j=1}^{p}\sum_{k=1}^{p}|S_{jk}||z_k| &= \mathbf{C}^T\mathbf{R}\tilde{\mathbf{z}} \\ \mathbf{B}^T &= \mathbf{C}^T\mathbf{R}. \end{aligned}$$

With this simplificatin equation(12) looks

$$I = \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{Dz} + \mathbf{E}^T\mathbf{z} + \mathbf{B}^T\tilde{\mathbf{z}}\right) dz_1 \cdots dz_2. \tag{14}$$

Because $\mathbf{D}$ is diagonal the exponent without using matrix notation is

$$(-\frac{1}{2}d_1 z_1^2 + e_1 z_1 + b_1|z_1|) + \cdots + (-\frac{1}{2}d_p z_p^2 + e_p z_p + b_p|z_p|). \tag{15}$$

Now the integration variables are decoupled and the integral (15) can be expressed as a product of integrals

$$I = \prod_{i=1}^{p} \int_{-\infty}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + e_i z_i + b_i|z_i|)\ dz_i. \tag{16}$$

Because the matrix $\mathbf{A}$ is semi-definite positive and its eigenvalues d_i are greater or equal to zero we can say the in general there will be $\hat{p}$ eigenvalues greater than zero with $\hat{p} = rank(A)$ and $p - \hat{p}$ eigenvalues equal to zero, so we can rewrite

$$I = \prod_{i=1}^{\hat{p}} \int_{-\infty}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + e_i z_i + b_i|z_i|) dz_i \prod_{i=\hat{p}+1}^{p} \int_{-\infty}^{\infty} exp\ (e_i z_i + b_i|z_i|) dz_i \tag{17}$$

$$I = \underbrace{\prod_{i=1}^{\hat{p}} I_{1i}}_{d_i>0}\ \underbrace{\prod_{i=\hat{p}+1}^{p} I_{2i}}_{d_i=0}\ . \tag{18}$$

We have to solve two type of integrals, we will start with those corresponding to eigenvalues $d_i > 0$.

$$\begin{aligned} I_{1i} &= \int_{-\infty}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + e_i + b_i|z_i|) dz_i \qquad (19) \\ &= \int_{-\infty}^{0} exp\ (-\frac{1}{2}d_i z_i^2 + (e_i - b_i) z_i) dz_i + \int_{0}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + (e_i + b_i) z_i) dz_i \\ &= \int_{0}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + (b_i - e_i) z_i) dz_i + \int_{0}^{\infty} exp\ (-\frac{1}{2}d_i z_i^2 + (b_i + e_i) z_i) dz_i \\ &= T_{1i} + T_{2i}, \end{aligned}$$

where we have made use of $|z_i| = z_i \ \ \forall z_i \geq 0$ while $|z_i| = -z_i \ \ \forall z_i < 0$. Besides, we have change the integral $\int_{-\infty}^{0}$ in $\int_{0}^{\infty}$ with the simple variable change $z_i = -z_i$ so the problem is just to solve an integral of the next type with K_1 and K_2 arbitrary constants

$$\int_{0}^{\infty} exp\ (-\frac{1}{2}K_1 z^2 + K_2 z)\ dz. \tag{20}$$

To solve the integral we are going to complete the square of the exponent and make a new change of variable

$$
\begin{aligned}
-\frac{1}{2}K_1 z^2 + K_2 z &= -\frac{1}{2}K_1\left(z - \frac{K_2}{K_1}\right)^2 + \frac{K_2^2}{2K_1} \\
z - \frac{K_2}{K_1} &= t \\
dz &= dt \\
z = \infty &\rightarrow t = \infty \\
z = 0 &\rightarrow t = -\frac{K_2}{K_1}.
\end{aligned} \tag{21}
$$

Now (20) is

$$
exp\left(\frac{K_2^2}{2K_1}\right)\int_{\frac{-K_2}{K_1}}^{\infty} exp(-\frac{1}{2}K_1 t^2)\ dt \tag{22}
$$

With another change of variable it is possible to reduce the integral term in (22) to the complementary error function

$$
erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-z^2}\ dz. \tag{23}
$$

The proposed change of variable is

$$
\begin{aligned}
t &= \sqrt{\frac{2}{K_1}}z \\
dt &= \sqrt{\frac{2}{K_1}}dz \\
t = \infty &\rightarrow z = \infty \\
t = -\frac{K_2}{K_1} &\rightarrow z = -\frac{K_2}{K_1}\sqrt{\frac{K_1}{2}} = -\frac{K_2}{\sqrt{2K_1}}.
\end{aligned} \tag{24}
$$

Substituting (23) and multiplying for $\frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}$ we get

$$
exp\left(\frac{K_2^2}{2K_1}\right)\sqrt{\frac{2}{K_1}}\frac{\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{-K_2}{\sqrt{2K_1}}}^{\infty} e^{-z^2}\ dz\right] = \tag{25}
$$

$$
exp\left(\frac{K_2^2}{2K_1}\right)\sqrt{\frac{\pi}{2K_1}}erfc\left(\frac{-K_2}{\sqrt{2K_1}}\right). \tag{26}
$$

We can solve now $I_{1i} = T_{1i} + T_{21}$ by coefficient identification $K_1 = d_i$ for T_{1i} and T_{2i} while $K_2 = b_i - e_i$ for T_{1i} and$K_2 = b_i + e_i$ for T_{2i}.

To solve I_{2i}, the integral corresponding to eigenvalues $d_i = 0$ we simply write

$$
\begin{aligned}
I_{2i} &= \int_{-\infty}^{\infty} exp\ (e_i z_i + b_i|z_i|)dz_i \\
&= \int_{-\infty}^{0} exp((e_i - b_i)z_i)dz_i + \int_0^{\infty} exp((e_i + b_i)z_i)dz_i \\
&= \int_0^{\infty} exp((b_i - e_i)z_i)dz_i + \int_0^{\infty} exp((b_i + e_i)z_i)dz_i.
\end{aligned} \tag{27}
$$

Both are of the type

$$\int_0^{\infty} e^{Kz}\, dz = \frac{1}{K} e^{Kz} \Big|_0^{\infty} \tag{28}$$

The integral converges only if the constant K is negative and in this case the result is $-1/K$. In our case we have to ensure that $b_i - e_i$ and $b_i + e_i$ are always negative for $i = \hat{p}+1, \cdots, p$. In the equation set (13) we see that $\mathbf{B}^T = \mathbf{C}^T\mathbf{R}$ with $\mathbf{R}$ a matrix of positive values because each element is the absolute value of $\mathbf{S}$ and that $\mathbf{C}^T = (-\lambda/\sigma, \cdots, -\lambda/\sigma)$ being λ and σ positive hyperparameters. Therefore, we conclude that $b_i < 0 \;\; i = 1, 2, \cdots, p$, whereas the components e_i comes from $\mathbf{E^T} = \mathbf{J^T S}$. In this case it holds that $\mathbf{J^T S} = 0 \;\; i = \hat{p}+1, \cdots, p$ because the vector $\mathbf{J^T}$ is orthogonal to eigenvectors associated to null eigenvalues. As a result $b_i - e_i = b_i + e_i = b_i \;\; i = \hat{p}+1, \cdots, p$ and finally $I_{2i} = -2/b_i$.

As a summary

$$\begin{aligned}
q_0 &= \left(\frac{\lambda}{2\sigma}\right)^p \frac{1}{\sqrt[T]{2\pi\sigma^2}} exp\left(-\frac{Y^TY}{2\sigma^2}\right) I && (29)\\
I &= \prod_{i=1}^{\hat{p}} I_{1i} \prod_{i=\hat{p}+1}^{p} I_{2i} && (30)\\
I_{1i} &= T_{1i} + T_{2i} && (31)\\
T_{1i} &= exp\left(\frac{(b_i - e_i)^2}{2d_i}\right) \sqrt{\frac{\pi}{2d_i}} erfc\left(\frac{-(b_i - e_i)}{\sqrt{2d_i}}\right) && (32)\\
T_{2i} &= exp\left(\frac{(b_i + e_i)^2}{2d_i}\right) \sqrt{\frac{\pi}{2d_i}} erfc\left(\frac{-(b_i + e_i)}{\sqrt{2d_i}}\right) && (33)\\
I_{2i} &= -\frac{2}{b_i} && (34)
\end{aligned}$$