Supplementary material for "Top-k Selection based on Adaptive Sampling of Noisy Preferences"

A. Proof of Theorem 1

Proof. First, we apply the Hoeffding theorem (Hoeffding, 1963) concerning the sum of random variables to $\bar{y}_{i,j}$. Note that $Y_{i,j}$ is a random variable with support $[0,1]$ thus its range is 1. An equivalent formulation of the Hoeffding theorem is as follows: For any $0 < \delta < 1$, the interval

$$\Bigg[\underbrace{\bar{y}_{i,j} - \sqrt{\frac{1}{2n_{i,j}}\log\frac{2}{\delta}}}_{\ell_{i,j}}, \underbrace{\bar{y}_{i,j} + \sqrt{\frac{1}{2n_{i,j}}\log\frac{2}{\delta}}}_{u_{i,j}}\Bigg] \tag{12}$$

contains $y_{i,j}$ with probability at least $1-\delta$.

According to this, by setting the confidence level to $\delta/K^2 n_{\max}$, for any i, j and round $n_{i,j}$, the probability that $y_{i,j}$ is not included in $[\ell_{i,j}, u_{i,j}]$ is at most $\delta/(K^2 n_{\max})$. Thus, with probability at least $1-\delta$, $y_{i,j} \in [\ell_{i,j}, u_{i,j}]$ for every i and j throughout the whole run of the PBR-CO algorithm. Therefore, if the PBR-CO returns an index set $\widehat{I}$ of options and $n_{i,j} \le n_{\max}$ for all $i, j \in [K]$ then $\widehat{I}$ is the solution set of (5) with probability with at least $1-\delta$.

For the expected sample complexity bound, let $\widetilde{n}_{i,j}$ be

$$\left\lceil \frac{1}{2\Delta_{i,j}^2} \log \frac{2K^2 n_{\max}}{\delta} \right\rceil \ .$$

Based on (12), when for some i and j, $Y_{i,j}$ is sampled for at least $\widetilde{n}_{i,j}$ times, then $[\ell_{i,j}, u_{i,j}]$ does not contain $1/2$ with probability at most $\delta/(K^2 n_{\max})$ if we assume that $y_{i,j} \neq 1/2$, and thus $\Delta_{i,j} = y_{i,j} - 1/2 \neq 0$. Furthermore, if for some o_i all the preferences against other options are decided (i.e., $\ell_{i,j} > 1/2$ or $u_{i,j} < 1/2$ for all j), then $Y_{i,1}, \dots, Y_{i,K}$ will not be sampled any more (see Procedure 2, line 7).

Putting these observations together, the claim follows from the union bound. □

B. Proof of Theorem 2

Proof. One can get a confidence interval for $\bar{y}_i = \frac{1}{K-1}\sum_{j\neq i} \bar{y}_{i,j}$ based on the confidence intervals of $\bar{y}_{j,i}$. More precisely, put $c_i = \frac{1}{K-1}\sum_{j \neq i} c_{i,j}$, and observe that, as $y_{i,j} \in [y_{i,j} - c_{i,j}, y_{i,j} + c_{i,j}]$ with probability at least $1-\delta/(2K^2 n_{\max})$ for any $1 \le j \le K$ in any of the $n_{\max}$ rounds, the interval

$$\Big[\underbrace{\bar{y}_i - c_i}_{\ell_i}, \underbrace{\bar{y}_i + c_i}_{u_i}\Big]$$

contains y_i with probability at least $1 - \delta/(2Kn_{\max})$ in any round. Therefore, denoting by $\mathcal{E}$ the event that each y_i is within the confidence interval of $\bar{y}_i$ throughout the whole run of algoritm PBR-SE, it holds that $\mathbf{P}[\mathcal{E}] \ge 1 - \delta/2$. Also note that whenever $\mathcal{E}$ holds then $\bar{y}_i - c_i \le y_i \le y_j \le \bar{y}_j + c_j$ for $1 \le i < j \le K$, and thus none of $o_1, \dots, o_{K-\kappa}$ gets selected and none of $o_{K-\kappa+1}, \dots, o_K$ gets discarded.

If for some $i \le K - \kappa$ and $j \ge K - \kappa + 1$ the number of samples for each of $y_{i,1}, \dots, y_{i,K}$ and $y_{j,1}, \dots, y_{j,K}$ is at least $\left\lceil \left(\frac{4}{y_i - y_j}\right)^2 \log \frac{2K^2 n_{\max}}{\delta} \right\rceil$ then event $\mathcal{E}$ implies $\bar{y}_i + c_i < y_i + 2c_i \le y_j - 2c_j < \bar{y}_j - c_j$ by Hoeffding's bound. Therefore, with probability at least $1-\delta$, after sampling $y_{i,k}$ at least b_i times for $i = 1, \dots, K-\kappa$ and $k = 1, \dots, K$ and sampling $y_{j,k}$ at least b_j times for $j = K-\kappa+1, \dots, K$ and $k = 1, \dots, k$, algorithm PBR-SE selects $o_{K-\kappa+1}, \dots, o_K$, discards $o_1, \dots, o_{K-\kappa}$, and thus terminates and returns the optimal solution. □

B.1. The pseudo-code of PBR framework regarding the setup described in Section 6

In this setup there are given a set of random variables $X_i, \ldots, X_K$ as input. Each random variable X_i takes values in a set Ω that is a partially ordered by a preference relation $\preceq$.

Procedure 5 PBR$(X_1, \ldots, X_K, \kappa, n_{\max}, \delta)$

$B = D = \emptyset$ ▷ Set of chosen and discarded options
$A = \{(i,j) |\ 1 \le i,j \le K, i \ne j\}$
for $i = 1 \to K$ **do**
  $n_i = 0$
while $\forall (n_i \le n_{\max}) \wedge (|A| > 0)$ **do**
  for all i appearing in A **do**
    $x_i^{(n_i)} \sim X_i$
    $n_i = n_i + 1$ ▷ Draw a random sample
  for all $(i,j) \in A$ **do**
    Update $\bar{y}_{i,j}$ with the new samples according to (11)
    $c_{i,j} = \sqrt{\frac{1}{2\min(n_i,n_j)} \log \frac{2K^2 n_{\max}}{\delta}}$
    $u_{i,j} = \bar{y}_{i,j} + c_{i,j}\ ,\ \ell_{i,j} = \bar{y}_{i,j} - c_{i,j}$
  ▷ For the implementation of sampling strategies see Section 4
  $(A, B) =$ **SSCO**$(A, \bar{\mathbf{Y}}, K, \kappa, \mathbf{U}, \mathbf{L})$ ▷ Sampling strategy for $\prec^{\mathrm{CO}}$
  $(A, B, D) =$ **SSSE**$(A, \bar{\mathbf{Y}}, K, \kappa, \mathbf{U}, \mathbf{L}, D)$ ▷ Sampling strategy for $\prec^{\mathrm{SE}}$
  $(A, B) =$ **SSRW**$(\bar{\mathbf{Y}}, K, \kappa, \mathbf{C})$ ▷ Sampling strategy for $\prec^{\mathrm{RW}}$
return B