APPENDIX – SUPPLEMENTARY MATERIAL

A. Proofs

Proof of Properties 3.2c (Max-subadditivity).
Consider the linear combination of two stationary processes x_t and y_t,

$$z_t = \alpha x_t + \beta y_t$$

It holds $\mathbb{V} z_t = \alpha^2 \sigma_x^2 + \beta^2 \sigma_y^2 + 2\alpha\beta cov(x_t, y_t)$. If x_t and y_s are uncorrelated for all $s \neq t$ then $\mathbb{V} z_t = \alpha^2 + \beta^2$ (if x_t and y_t are both unit-variance processes). To have a unit-variance sum we need $\beta = \sqrt{1-\alpha^2}$:

The spectrum of z_t equals (if they are uncorrelated)

$$S_z(\lambda) = \alpha^2 S_x(\lambda) + \beta^2 S_y(\lambda) \leq \alpha S_x(\lambda) + \beta S_y(\lambda) \quad (29)$$

It therefore holds

$$\begin{aligned} \Omega(z_t) &= \Omega(\alpha x_t + \beta y_t) \\ &= 1 - H\left(\alpha^2 S_x(\lambda) + \beta^2 S_y(\lambda)\right) \\ &\leq 1 - \left(\alpha^2 H(S_x(\lambda)) + \beta^2 H(S_y(\lambda))\right), \end{aligned}$$

where the last inequality follows by (29). Plugging in $\beta = \sqrt{1-\alpha^2}$ gives

$$\begin{aligned} &\alpha^2 - \alpha^2 H(S_x(\lambda)) + (1-\alpha^2) - (1-\alpha^2) H(S_y(\lambda)) \\ &= \alpha^2 (1 - H(S_x(\lambda))) + (1-\alpha^2)(1 - H(S_y(\lambda))) \\ &= \alpha^2 \Omega(x_t) + (1-\alpha^2)\Omega(y_t) \\ &\leq \max\left(\Omega(x_t), \Omega(y_t)\right), \end{aligned}$$

which completes the proof.

□

Proof of Proposition 4.1. For every $\mathbf{w}$,

$$\begin{aligned} \mathbf{w}^\top \overline{\widehat{S}_U^{(i)}} \mathbf{w} &= -\frac{1}{T} \sum_{j=0}^{T-1} \mathbf{w}^\top \left[\widehat{S}_U(\omega_j) \cdot \ell(\mathbf{w}_i; \omega_j) \right] \mathbf{w} \\ &= \frac{1}{T} \sum_{j=0}^{T-1} \underbrace{-\ell(\mathbf{w}_i; \omega_j)}_{\geq 0} \cdot \underbrace{\mathbf{w}^\top \widehat{S}_U(\omega_j) \mathbf{w}}_{\geq 0} \geq 0. \end{aligned}$$

□