Modeling Temporal Evolution and Multiscale Structure in Networks Supplementary Material

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Proof of claims c1-c5

Proof Recall the following claims regarding regarding the THRM

c1 invariance under relabeling of vertices

c2 invariance under temporal translation of all coordinates, $(it) \mapsto (i, t + \Delta t)$

c3 marginal consistency when integrating a single temporal observation (it), see eq.(1)

c4 correlation of the hierarchies at epochs t and s decrease as $|t - s|$ increase

c5 marginally distributed as the HRM at each epoch.

where marginal consistency indicate the condition

$$\sum_{x_{n+1}} p_{n+1}(x_1, \ldots, x_n, x_{n+1}) = p_n(x_1, \ldots, x_n) \qquad (1)$$

The proof of these claims mainly relate to the properties of the prior, The first two claims follow easily in that the generative procedure makes no specific reference to vertex labeling and only depend on the ordering of temporal states, and the last claim follow from the change-point prior being less correlated across longer time spans. The fifth claim follows easily from the fourth by projecting out all temporal observations not at the given epoch one at a time.

To show C4, consider two models m, $m+1$ defined on temporal states $\mathcal{S}^m, \mathcal{S}^{m+1}$. By assumption, the two models agree on all parameters aside those relating to the new state (it) such that $\mathcal{S}^m_{it} = 0, \mathcal{S}^{m+1}_{it} = 1$. There are three cases for how the new temporal state can enter: (i) *appearance*, if $S^m_{i,t-1} = \mathcal{S}^m_{i,t+1} = 0$, (ii-iii) *future/past expansion*, if $\mathcal{S}^m_{i,t-1} = 1$ or $\mathcal{S}^m_{i,t+1} = 1$. Before showing the marginalization condition we list the following properties relating to the HRM and Gibbs fragmentation trees

P1 Let $T_{B\cup i}$ be the GFT obtained from T_B by the addition of the extra vertex i. Then trivially by projectivity $\sum_{t_i} p(T_{B\cup\{i\}}) = p(T_B)$ where by $\sum_{t_i}$ implies summation over all single insert operations of vertex i to the tree.

P2 Given a HRM of m vertices parameterized by $\boldsymbol{\theta} = (\boldsymbol{A}, \boldsymbol{\eta})$ and T_m. Assume a new vertex is added to T_m giving the tree T_{m+1} and write $\boldsymbol{\theta} \cup \boldsymbol{\theta}' = (\boldsymbol{A} \cup \boldsymbol{a}, \boldsymbol{\eta} \cup \boldsymbol{\eta}')$ for the updated set of parameters. The projective condition of the HRM implies (notice we are conditioning on T_m on the RHS):

$$\sum_{\boldsymbol{\theta}'} p(\boldsymbol{\theta}, \boldsymbol{\theta}' | T_{m+1}) = p(\boldsymbol{\theta} | T_m).$$

Returning to the main result, let $\boldsymbol{\theta}^{\setminus t}, \boldsymbol{\eta}^{\setminus t}$ be the data and parameters excluding temporal epoch t. Let T_{m+1} be the giant tree obtained by adding the new vertex (it) to T_m and recall C_s is the set of vertex-observations at time slice s. Assume first case 1, *appearance*. The projective condition becomes:

$$\begin{aligned}
&\sum_{\boldsymbol{\theta}', t_{m+1}} p(\boldsymbol{\theta}^{\setminus t}, \boldsymbol{\theta}^t, \boldsymbol{\theta}', T_{m+1}) \\
&= \sum_{t_{m+1}, \boldsymbol{\theta}'} p(\boldsymbol{\theta}^{\setminus t} | T_{m+1}) p(\boldsymbol{\theta}^t, \boldsymbol{\theta}' | T_{m+1}) p(T_{m+1}) \\
&= \sum_{t_{m+1}} p(\boldsymbol{\theta}^{\setminus t} | T_m) p(\boldsymbol{\theta}^t | T_m) p(T_{m+1}) = p(\boldsymbol{\theta}, T_m)
\end{aligned}$$

The first equality sign follows since at time slices $\neq t$, the model is projected onto sets not containing the new state (it), the projected GFT do not change and the first term reduce to being conditioned on T_m. Also, since T_{m+1} is obtained from T_m by addition of a single vertex, this also holds for the projection onto $C_t \cup c_{(it)}$, thus the sum disappear by (P2). The final equality follow by (P1).

Consider case 2, *future expansion*. Here there is also introduced a flip-variable F_{it}. If $F_{it} = 0$ the giant tree is not changed, but if $F_{it} = 1$ a new vertex is inserted in the giant tree. In either case the epoch t will contain an extra vertex-observation. We first compute

$$\sum_{F_{it}=0,1} \sum_{\text{pars.}|F_{it}} p(\cdots) =$$

$$(1-\gamma)p(\boldsymbol{\theta}^{\setminus t}|T_m) \sum_{\boldsymbol{\theta}'} p(\boldsymbol{\theta}^t, \boldsymbol{\theta}'|\text{proj}_{C_t \cup c_{(it)}} T_m) p(T_m) +$$

$$\gamma \sum_{\boldsymbol{\theta}', t_{(it)}} p(\boldsymbol{\theta}^{\setminus t}|T_{m+1}) p(\boldsymbol{\theta}^t, \boldsymbol{\theta}'|\text{proj}_{C_t \cup c_{(it)}} T_{m+1}) p(T_{m+1})$$

By argument similar to before $p(\boldsymbol{\theta}^{\setminus t}|T_{m+1}) = p(\boldsymbol{\theta}^{\setminus t}|T_m)$. In the first case the tree $\text{proj}_{C_t \cup c_{(it)}} T_m$ is obtained from T_m by the addition of one vertex. In the second case the tree $\text{proj}_{C_t \cup c_{(it)}} T_{m+1}$ can be obtained from T_{C_t} by the addition of one extra vertex; thus in both cases the condition (P2) is fulfilled giving

$$p(\boldsymbol{\theta}^{\setminus t}|T_m)(1-\gamma) + \gamma p(\boldsymbol{\theta}^t|T_m) \sum_{t_{(it)}} p(T_{m+1})$$

from which the result follows by (P1). The final case 3, *past expansion* follows by an argument similar to the second case.