APPENDIX—SUPPLEMENTARY MATERIAL

A Some results used in the proofs

Fact 1 (Chernoff-Hoeffding bound). *Let $X_1, \ldots, X_n$ be independent $0-1$ r.v.s with $E[X_i] = p_i$ (not necessarily equal). Let $X = \frac{1}{n}\sum_i X_i$, $\mu = E[X] = \frac{1}{n}\sum_{i=1}^n p_i$. Then, for any $0 < \lambda < 1-\mu$,*

$$\Pr(X \geq \mu + \lambda) \leq \exp\{-nd(\mu+\lambda, \mu)\},$$

and, for any $0 < \lambda < \mu$,

$$\Pr(X \leq \mu - \lambda) \leq \exp\{-nd(\mu-\lambda, \mu)\},$$

where $d(a,b) = a \ln \frac{a}{b} + (1-a)\ln\frac{(1-a)}{(1-b)}$.

Fact 2 (Chernoff–Hoeffding bound). *Let $X_1, ..., X_n$ be random variables with common range $[0,1]$ and such that $\mathbb{E}[X_t \mid X_1, ..., X_{t-1}] = \mu$. Let $S_n = X_1 + \ldots + X_n$. Then for all $a \geq 0$,*

$$\Pr(S_n \geq n\mu + a) \leq e^{-2a^2/n},$$

$$\Pr(S_n \leq n\mu - a) \leq e^{-2a^2/n}.$$

Fact 3.

$$F^{beta}_{\alpha,\beta}(y) = 1 - F^B_{\alpha+\beta-1,y}(\alpha-1),$$

for all positive integers α, β.

Formula 7.1.13 from Abramowitz and Stegun [1964] can be used to derive the following concentration for Gaussian distributed random variables.

Fact 4. *Abramowitz and Stegun [1964] For a Gaussian distributed random variable Z with mean m and variance σ^2, for any z,*

$$\frac{1}{4\sqrt{\pi}} \cdot e^{-7z^2/2} < \Pr(|Z-m| > z\sigma) \leq \frac{1}{2}e^{-z^2/2}.$$

B Thompson Sampling with Beta Distribution

B.1 Proof of Lemma 3

Let τ_k denote the time at which k^{th} trial of arm i happens. Let $\tau_0 = 0$. Note that $\tau_k > T$ for $k > k_i(T)$. Also, $T \leq \tau_T$. Then,

$$\begin{aligned}
&\sum_{t=1}^{T} \Pr(i(t)=i, \overline{E^\mu_i(t)}) \\
\leq\; & \mathbb{E}\left[\sum_{k=0}^{T-1}\sum_{t=\tau_k+1}^{\tau_{k+1}} I(i(t)=i) I(\overline{E^\mu_i(t)})\right] \\
=\; & \mathbb{E}\left[\sum_{k=0}^{T-1} I(\overline{E^\mu_i(\tau_k+1)}) \sum_{t=\tau_k+1}^{\tau_{k+1}} I(i(t)=i)\right] \\
=\; & \mathbb{E}\left[\sum_{k=0}^{T-1} I(\overline{E^\mu_i(\tau_k+1)})\right] \\
\leq\; & 1 + \mathbb{E}\left[\sum_{k=1}^{T-1} I(\overline{E^\mu_i(\tau_k+1)})\right] \\
\leq\; & 1 + \sum_{k=1}^{T-1} \exp(-kd(x_i,\mu_i)) \\
\leq\; & 1 + \frac{1}{d(x_i,\mu_i)}
\end{aligned}$$

The second last inequality follows from the observation that the event $\overline{E^\mu_i(t)}$ was defined as $\hat{\mu}_i(t) > x_i$, At time τ_k+1 for $k \geq 1$, $\hat{\mu}_i(\tau_k+1) = \frac{S_i(\tau_k+1)}{k+1} \leq \frac{S_i(\tau_k+1)}{k}$, where latter is simply the average of the outcomes observed from k i.i.d. plays of arm i, each of which is a Bernoulli trial with mean μ_i. Using Chernoff-Hoeffding bounds (Fact 1), we obtain that $\Pr(\hat{\mu}_i(\tau_k+1) > x_i) \leq \Pr(\frac{S_i(\tau_k+1)}{k} > x_i) \leq e^{-kd(x_i,\mu_i)}$. □

B.2 Proof of Lemma 4

Below, we slightly abuse the notation for readability – the notation $\Pr(Beta(\alpha,\beta) > y_i)$ will represent the probability that a random variable distributed as $Beta(\alpha,\beta)$ takes a value greater than y_i.

$$\begin{aligned}
&\Pr\left(i(t)=i, \overline{E^\theta_i(t)} \,\middle|\, E^\mu_i(t), \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq\; & \Pr\left(\theta_i(t) > y_i \mid \hat{\mu}_i(t) \leq x_i, \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
=\; & \Pr\left(\theta_i(t) > y_i \right. \\
& \qquad \left. \mid S_i(t) \leq x_i(k_i(t)+1), \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
=\; & \Pr\left(Beta(S_i(t)+1, k_i(t)-S_i(t)+1) > y_i \right. \\
& \qquad \left. \mid S_i(t) \leq x_i(k_i(t)+1), \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq\; & \Pr\left(Beta(x_i(k_i(t)+1)+1, \right. \\
& \qquad \left. (1-x_i)(k_i(t)+1)) > y_i \mid \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq\; & e^{-L_i(T)d(x_i,y_i)} \\
=\; & \frac{1}{T}.
\end{aligned}$$

where the last inequality used (Fact 3) along with Chernoff-Hoeffding bounds (refer to Fact 1) to obtain that for any fixed $k_i(t) > L_i(T)$,

$$\begin{aligned}
&\Pr\left(Beta(x_i(k_i(t)+1)+1, (1-x_i)(k_i(t)+1))\right.\\
&\qquad \left.> y_i\right)\\
= \;& F^B_{k_i(t)+1,y_i}(x_i(k_i(t)+1))\\
\leq \;& e^{-(k_i(t)+1)d(x_i,y_i)}\\
\leq \;& e^{-L_i(T)d(x_i,y_i)}
\end{aligned}$$

which is smaller than $\frac{1}{T}$ because $L_i(T) = \frac{\ln T}{d(x_i,y_i)}$.

Then,

$$\begin{aligned}
&\sum_{t=1}^{T} \Pr\left(i(t)=i, \overline{E^\theta_i(t)}, E^\mu_i(t)\right)\\
= \;& \sum_{t=1}^{T} \Pr\left(i(t)=i, k_i(t)\leq L_i(T), \overline{E^\theta_i(t)}, E^\mu_i(t)\right)\\
&+\sum_{t=1}^{T} \Pr\left(i(t)=i, k_i(t) > L_i(T), \overline{E^\theta_i(t)}, E^\mu_i(t)\right)\\
\leq \;& \mathbb{E}\left[\sum_{t=1}^{T} I(i(t)=i, k_i(t)\leq L_i(T))\right]\\
&+\mathbb{E}\left[\sum_{t=1}^{T} \Pr\left(i(t)=i, \overline{E^\theta_i(t)}\right.\right.\\
&\qquad\qquad \left.\left.\mid E^\mu_i(t), \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right)\right]\\
\leq \;& L_i(T) + \sum_{t=1}^{T}\frac{1}{T}\\
= \;& L_i(T)+1.
\end{aligned}$$

□

B.3 Proof of Lemma 2

Let $k_1(t) = j, S_1(t) = s$. Let $y = y_i$. Then, $p_{i,t} = \Pr(\theta_1(t) > y) = F^B_{j+1,y}(s)$ (using Fact 3). Let $\tau_j + 1$ denote the time step after the $(j)^{th}$ play of arm 1. Then, $k_1(\tau_j+1) = j$, and

$$\mathbb{E}[\frac{1}{p_{i,\tau_j+1}}] = \sum_{s=0}^{j} \frac{f_{j,\mu_1}(s)}{F^B_{j+1,y}(s)}.$$

Let $\Delta' = \mu_1 - y$.

In the derivation below, we abbreviate $F^B_{j+1,y}(s)$ as $F_{j+1,y}(s)$.

For $j < \frac{8}{\Delta'}$: Let $R = \frac{\mu_1(1-y)}{y(1-\mu_1)}$, $D = y\ln\frac{y}{\mu_1} + (1-y)\ln\frac{1-y}{1-\mu_1}$. Note that $\mu_1 \geq y$, so that $R \geq 1$.

$$\begin{aligned}
&\sum_{s=0}^{j}\frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)}\\
\leq \;& \frac{1}{1-y}\sum_{s=0}^{j}\frac{f_{j,\mu_1}(s)}{F_{j,y}(s)}\\
\leq \;& \frac{1}{1-y}\sum_{s=0}^{\lfloor yj\rfloor}\frac{f_{j,\mu_1}(s)}{f_{j,y}(s)} + \frac{1}{1-y}\sum_{s=\lceil yj\rceil}^{j} 2f_{j,\mu_1}(s)\\
= \;& \frac{1}{1-y}\sum_{s=0}^{\lfloor yj\rfloor} R^s\frac{(1-\mu_1)^j}{(1-y)^j} + \frac{1}{1-y}\sum_{s=\lceil yj\rceil}^{j} 2f_{j,\mu_1}(s)\\
= \;& \frac{1}{1-y}\left(\frac{R^{\lfloor yj\rfloor+1}-1}{R-1}\right)\frac{(1-\mu_1)^j}{(1-y)^j}\\
&+\frac{1}{1-y}\sum_{s=\lceil yj\rceil}^{j} 2f_{j,\mu_1}(s)\\
\leq \;& \frac{1}{1-y}\left(\frac{R}{R-1}\right)R^{yj}\frac{(1-\mu_1)^j}{(1-y)^j} + \frac{2}{\Delta'}\\
= \;& \frac{\mu_1}{\Delta'}e^{-Dj} + \frac{2}{\Delta'}\\
\leq \;& \frac{3}{\Delta'}. \qquad (4)
\end{aligned}$$

For $j \geq \frac{8}{\Delta'}$: We will divide the sum $Sum(0,j) = \sum_{s=0}^{j}\frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)}$ into four partial sums and prove that

$$\begin{aligned}
Sum(0,\lfloor yj\rfloor - 1) &\leq \Theta\left(e^{-Dj}\tfrac{1}{(j+1)}\tfrac{1}{\Delta'^2}\right)\\
&\quad +\Theta(e^{-2\Delta'^2 j}),\\
Sum(\lfloor yj\rfloor, \lfloor yj\rfloor) &\leq 3e^{-Dj},\\
Sum(\lceil yj\rceil, \lfloor \mu_1 j - \tfrac{\Delta'}{2}j\rfloor) &\leq \Theta(e^{-\Delta'^2 j/2}),\\
Sum(\lceil \mu_1 j - \tfrac{\Delta'}{2}j\rceil, j) &\leq 1 + \tfrac{1}{e^{\Delta'^2 j/4}-1}.
\end{aligned}$$

Together, the above estimates will prove the required bound.

We use the following bounds on the cdf of Binomial distribution [Jeřábek, 2004, Prop. A.4].
For $s \leq y(j+1) - \sqrt{(j+1)y(1-y)}$,

$$F_{j+1,y}(s) = \Theta\left(\frac{y(j+1-s)}{y(j+1)-s}\binom{j+1}{s}y^s(1-y)^{j+1-s}\right). \qquad (5)$$

For $s \geq y(j+1) - \sqrt{(j+1)y(1-y)}$,

$$F_{j+1,y}(s) = \Theta(1). \qquad (6)$$

Bounding $Sum(0, \lfloor yj \rfloor - 1)$**.** Using the bounds just given, for any s,

$$\begin{aligned}
&\frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)} \\
\leq\;& \Theta\left(\frac{f_{j,\mu_1}(s)}{\frac{y(j+1-s)}{y(j+1)-s}\binom{j+1}{s}y^s(1-y)^{j+1-s}}\right) \\
&+\Theta(1) f_{j,\mu_1}(s) \\
=\;& \Theta\left(\left(1-\frac{s}{y(j+1)}\right)\cdot R^s \cdot \frac{(1-\mu_1)^j}{(1-y)^{j+1}}\right) \\
&+\Theta(1) f_{j,\mu_1}(s).
\end{aligned}$$

This gives

$$\begin{aligned}
&Sum(0, \lfloor yj \rfloor - 1) \\
\leq\;& \Theta\left(\frac{(1-\mu_1)^j}{(1-y)^{j+1}}\sum_{s=0}^{\lfloor yj \rfloor -1}\left(1-\frac{s}{y(j+1)}\right)\cdot R^s\right) \\
&+\Theta(1)\sum_{s=0}^{\lfloor yj \rfloor -1} f_{j,\mu_1}(s).
\end{aligned}$$

We now bound the first expression on the RHS.

$$\begin{aligned}
&\frac{(1-\mu_1)^j}{(1-y)^{j+1}}\sum_{s=0}^{\lfloor yj \rfloor -1}\left(1-\frac{s}{y(j+1)}\right)\cdot R^s \\
=\;& \frac{(1-\mu_1)^j}{(1-y)^{j+1}}\left(\frac{R^{\lfloor yj \rfloor}-1}{R-1}\right. \\
&\left.-\frac{1}{y(j+1)}\left(\frac{(\lfloor yj \rfloor -1)R^{\lfloor yj \rfloor}}{R-1} - \frac{R^{\lfloor yj \rfloor}-R}{(R-1)^2}\right)\right) \\
\leq\;& \frac{(1-\mu_1)^j}{(1-y)^{j+1}}\left(\frac{1}{y(j+1)}\frac{R^{\lfloor yj \rfloor}}{(R-1)^2}\right. \\
&\left.+\frac{(y(j+1)-\lfloor yj \rfloor+1)}{y(j+1)}\frac{R^{\lfloor yj \rfloor}}{(R-1)}\right) \\
\leq\;& \frac{(1-\mu_1)^j}{(1-y)^{j+1}}\frac{3}{y(j+1)}\frac{R^{\lfloor yj \rfloor+1}}{(R-1)^2} \\
\leq\;& e^{-Dj}\frac{3}{y(1-y)(j+1)}\frac{R}{(R-1)^2}
\end{aligned}$$

The last inequality uses

$$\frac{(1-\mu_1)^j}{(1-y)^j}R^{\lfloor yj \rfloor} \leq \frac{(1-\mu_1)^j}{(1-y)^j}R^{yj} = e^{-Dj}.$$

Now, $R-1 = \frac{\mu_1(1-y)}{y(1-\mu_1)} - 1 = \frac{\mu_1 - y}{y(1-\mu_1)}$. And, $\frac{R}{R-1} = \frac{\mu_1(1-y)}{\mu_1 - y}$. Therefore,

$$\begin{aligned}
&\frac{1}{y(1-y)(j+1)}\frac{R}{(R-1)^2} \\
=\;& \frac{1}{y(1-y)(j+1)}\cdot\frac{\mu_1(1-y)}{\mu_1-y}\cdot\frac{y(1-\mu_1)}{\mu_1-y} \\
=\;& \frac{1}{(j+1)}\frac{\mu_1(1-\mu_1)}{(\mu_1-y)^2}.
\end{aligned}$$

Substituting, we get

$$\begin{aligned}
&\frac{(1-\mu_1)^j}{(1-y)^{j+1}}\sum_{s=0}^{\lfloor yj \rfloor}\left(1-\frac{s}{y(j+1)}\right)\cdot R^s \\
\leq\;& e^{-Dj}\frac{1}{(j+1)}\frac{\mu_1(1-\mu_1)}{(\mu_1-y)^2}.
\end{aligned}$$

Substituting in (7)

$$\begin{aligned}
&Sum(0, \lfloor yj \rfloor - 1) \\
\leq\;& \Theta\left(e^{-Dj}\frac{1}{(j+1)}\frac{1}{\Delta'^2}\right) + \Theta(1)\sum_{s=0}^{\lfloor yj \rfloor -1} f_{j,\mu_1}(s) \\
\leq\;& \Theta\left(e^{-Dj}\frac{1}{(j+1)}\frac{1}{\Delta'^2}\right) + \Theta(e^{-2(\mu_1-y)^2 j}).
\end{aligned}$$

Bounding $Sum(\lfloor yj \rfloor, \lfloor yj \rfloor)$**.** We use $\frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)} \leq \frac{f_{j,\mu_1}(s)}{f_{j+1,y}(s)} = \left(1-\frac{s}{j+1}\right)R^s\frac{(1-\mu_1)^j}{(1-y)^{j+1}}$, to get

$$\begin{aligned}
&Sum(\lfloor yj \rfloor, \lfloor yj \rfloor) \\
=\;& \frac{f_{j,\mu_1}(\lfloor yj \rfloor)}{F_{j+1,y}(\lfloor yj \rfloor)} \\
\leq\;& \left(1-\frac{yj-1}{j+1}\right)R^{yj}\frac{(1-\mu_1)^j}{(1-y)^{j+1}} \\
\leq\;& \frac{(1-y+\frac{2}{j+1})}{1-y}R^{yj}\frac{(1-\mu_1)^j}{(1-y)^j} \\
\leq\;& 3e^{-Dj}. \qquad (7)
\end{aligned}$$

The last inequality uses $j \geq \frac{1}{\Delta'} \geq \frac{1}{1-y}$.

Bounding $Sum(\lceil yj \rceil, \lfloor \mu_1 j - \frac{\Delta'}{2}j \rfloor)$**.** Now, if $j > \frac{1}{\Delta'}$,then $\sqrt{(j+1)y(1-y)} > \sqrt{y} > y$, so $y(j+1) - \sqrt{(j+1)y(1-y)} < yj \leq \lceil yj \rceil$. Therefore, (using the bounds by Jeřábek [2004] given in Equation (6)) for $s \geq \lceil yj \rceil$, $F_{j+1,y}(s) = \Theta(1)$. Using this observation, we derive the following.

$$\begin{aligned}
& Sum(\lceil yj \rceil, \lfloor \mu_1 j - \frac{\Delta'}{2} j \rfloor) \\
= & \sum_{s=\lceil yj \rceil}^{\lfloor \mu_1 j - \frac{\Delta'}{2} j \rfloor} \frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)} \\
= & \Theta\left(\sum_{s=\lceil yj \rceil}^{\lfloor \mu_1 j - \frac{\Delta'}{2} j \rfloor} f_{j,\mu_1}(s) \right) \\
\leq & \Theta(e^{-2\left(\mu_1 j - \lfloor \mu_1 j - \frac{\Delta'}{2} j \rfloor\right)^2 / j}) \\
= & \Theta(e^{-\Delta'^2 j/2}), \qquad (8)
\end{aligned}$$

where the inequality follows using Chernoff-Hoeffding bounds (refer to Fact 2).

Bounding $Sum(\lceil \mu_1 j - \frac{\Delta'}{2} j \rceil, j)$**.** For $s \geq \lceil \mu_1 j - \frac{\Delta'}{2} j \rceil = \lceil yj + \frac{\Delta'}{2} j \rceil$, again using Chernoff-Hoeffding bounds from Fact 2,

$$\begin{aligned}
F_{j+1,y}(s) & \geq 1 - e^{-2(yj + \frac{\Delta'}{2} j - y(j+1))^2/(j+1)} \\
& \geq 1 - e^{2\Delta'} e^{-\Delta'^2 j/2} \\
& \geq 1 - e^{\Delta'^2 j/4} e^{-\Delta'^2 j/2} \\
& = 1 - e^{-\Delta'^2 j/4}.
\end{aligned}$$

The last inequality uses $j \geq \frac{8}{\Delta'}$.

$$\begin{aligned}
Sum(\lceil \mu_1 j - \frac{\Delta'}{2} j \rceil, j) & = \sum_{s=\lceil \mu_1 j - \frac{\Delta'}{2} j \rceil}^{j} \frac{f_{j,\mu_1}(s)}{F_{j+1,y}(s)} \\
& \leq \frac{1}{1 - e^{-\Delta'^2 j/4}} \\
& = 1 + \frac{1}{e^{\Delta'^2 j/4} - 1}. \qquad (9)
\end{aligned}$$

Combining, we get for $j \geq \frac{8}{\Delta'}$,

$$\begin{aligned}
& \mathbb{E}[\frac{1}{p_{i,\tau_{j+1}}}] \\
\leq & 1 + \Theta(e^{-\Delta'^2 j/2} + \frac{1}{(j+1)\Delta'^2} e^{-Dj} + \frac{1}{e^{\Delta'^2 j/4} - 1})
\end{aligned}$$

□

C Thompson Sampling with Gaussian Distribution

C.1 Proof of Lemma 5

Proof. The proof of this lemma is similar to the proof of Lemma 4 in Appendix B.2.

Below, we slightly abuse the notation for readability – the notation $\Pr(\mathcal{N}(m, \sigma^2) > y_i)$ will represent the probability that a random variable distributed as $\mathcal{N}(m, \sigma^2)$ takes a value greater than y_i.

$$\begin{aligned}
& \Pr\left(i(t) = i, \overline{E_i^\theta(t)} \,\middle|\, E_i^\mu(t), \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq & \Pr\left(\theta_i(t) > y_i \mid \hat{\mu}_i(t) \leq x_i, \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
= & \Pr\left(\mathcal{N}(\hat{\mu}_i(t), \frac{1}{k_i(t)+1}) > y_i \right. \\
& \left. \mid \hat{\mu}_i(t) \leq x_i, \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq & \Pr\left(\mathcal{N}(x_i, \frac{1}{k_i(t)+1}) > y_i \,\middle|\, \mathcal{F}_{t-1}, k_i(t) > L_i(T)\right) \\
\leq & \frac{1}{2} e^{-\frac{(L_i(T))(y_i - x_i)^2}{2}} \\
= & \frac{1}{T\Delta_i^2},
\end{aligned}$$

where the last inequality used the concentration of Gaussian distribution (refer to Fact 4) to obtain that for any fixed $k_i(t) > L_i(T)$,

$$\begin{aligned}
\Pr\left(\mathcal{N}(x_i, \frac{1}{k_i(t)+1}) > y_i\right) & \leq \frac{1}{2} e^{-\frac{(k_i(t)+1)(y_i - x_i)^2}{2}} \\
& \leq \frac{1}{2} e^{-\frac{(L_i(t))(y_i - x_i)^2}{2}}
\end{aligned}$$

which is smaller than $\frac{1}{T\Delta_i^2}$ because $L_i(T) = \frac{2\ln(T\Delta_i^2)}{(y_i - x_i)^2}$. Now,

$$\begin{aligned}
& \sum_{t=1}^{T} \Pr\left(i(t) = i, \overline{E_i^\theta(t)}, E_i^\mu(t)\right) \\
= & \sum_{t=1}^{T} \Pr\left(i(t) = i, \overline{E_i^\theta(t)}, E_i^\mu(t), k_i(t) \leq L_i(T)\right) \\
& + \sum_{t=1}^{T} \Pr\left(i(t) = i, \overline{E_i^\theta(t)}, E_i^\mu(t), k_i(t) > L_i(T)\right) \\
\leq & \mathbb{E}\left[\sum_{t=1}^{T} I(i(t) = i, k_i(t) \leq L_i(T))\right] \\
& + \sum_{t=1}^{T} \mathbb{E}\left[\Pr\left(i(t) = i, \overline{E_i^\theta(t)} \right.\right. \\
& \left.\left. \mid E_i^\mu(t), k_i(t) > L_i(T), \mathcal{F}_{t-1}\right)\right] \\
\leq & L_i(T) + \sum_{t=1}^{T} \frac{1}{T\Delta_i^2} \\
\leq & L_i(T) + \frac{1}{\Delta_i^2}.
\end{aligned}$$

□

C.2 Proof of Lemma 6

Let Θ_j denote a $\mathcal{N}(\hat{\mu}_1(\tau_j+1), \frac{1}{k_1(\tau_j+1)})$ distributed Gaussian random variable. Let G_j be a geomertic random variable denoting the number of consecutive independent trials until $\Theta_j > y_i$. Then, observe that $p_{i,\tau_j+1} = \Pr(\Theta_j > y_i)$, and

$$\frac{1}{p_{i,\tau_j+1}} - 1 = \mathbb{E}[G_j] = \sum_{r=1}^{\infty} \Pr(G_j \geq r)$$

We will bound the expected value of G_j by a constant for all j. Consider any integer $r \geq 1$. Let $z = \sqrt{\ln r}$, let random variable MAX_r denote the maximum of r independent samples of Θ_j. We abbreviate $\hat{\mu}_1(\tau_j+1)$ as $\hat{\mu}_1$ and $k_1(\tau_j+1)$ as k_1 in the following. Then

$$\begin{aligned}
& \Pr(G_j < r) \\
\geq\ & \Pr(MAX_r > y_i) \\
\geq\ & \Pr\left(MAX_r > \hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \,\middle|\, \hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \geq y_i\right) \\
& \qquad \cdot \Pr\left(\hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \geq y_i\right)
\end{aligned}$$

The following anti-concentration bound can be derived for the Gaussian r.v. Z with mean μ and std deviation σ, using Formula 7.1.13 from Abramowitz and Stegun [1964]

$$\Pr(Z > \mu + z\sigma) \geq \frac{1}{\sqrt{2\pi}} \frac{z}{z^2+1} e^{-z^2/2}.$$

This gives

$$\begin{aligned}
& \Pr\left(MAX_r > \hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \,\middle|\, \hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \geq y_i\right) \\
\geq\ & 1 - \left(1 - \frac{1}{\sqrt{2\pi}} \frac{z}{(z^2+1)} e^{-z^2/2}\right)^r \\
=\ & 1 - \left(1 - \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\ln r}}{(\ln r + 1)} \frac{1}{\sqrt{r}}\right)^r \\
\geq\ & 1 - e^{-\frac{r}{\sqrt{4\pi r \ln r}}}.
\end{aligned}$$

Also, using Chernoff-Hoeffding bounds (refer to Fact 2),

$$\Pr(\hat{\mu}_1 + \frac{z}{\sqrt{k_1}} \geq \mu_1) \geq 1 - e^{-2z^2} = 1 - \frac{1}{r^2}.$$

Therefore, substituting,

$$\begin{aligned}
\Pr(G_j < r) &\geq (1 - e^{-\sqrt{\frac{r}{4\pi \ln r}}}) \cdot (1 - \frac{1}{r^2}) \\
&\geq 1 - \frac{1}{r^2} - e^{-\sqrt{\frac{r}{4\pi \ln r}}}.
\end{aligned}$$

$$\begin{aligned}
E[G_j] &= \sum_{r\geq 1} (1 - \Pr(G_j < r)) \\
&\leq \sum_{r \geq 1} \frac{1}{r^2} + e^{-\sqrt{\frac{r}{2\pi \ln r}}} \\
&\leq e^{11} + \sum_r 2\frac{1}{r^2} \\
&\leq e^{11} + 4,
\end{aligned}$$

The second last inequality in above uses the fact that for $r \geq e^{11}$, $e^{-\sqrt{\frac{r}{2\pi \ln r}}} \leq \frac{1}{r^2}$.