Dynamic Scaled Sampling for Deterministic Constraints
Supplemental Material

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Technical details

**Problem 3** (Posterior with bounded non-negative continuous variables) Let \( s \) be a target sum, \( \epsilon \) a positive number, and suppose there are \( k \) independent real-valued random variables \( X_1, \ldots, X_k \), such that \( X_i \sim p_i(x) \) and the prior probability densities \( p_i(x) > 0 \) if and only if \( x \in [0, \infty) \). \( \epsilon \) is coded in finite bits. \( p_i(\cdot) \) is represented in parametric form with finitely many parameters in constant bits. The goal is to find a sample \( (x_i) \) from the posterior probability density \( p(X_1, \ldots, X_k \mid \sum_{i=1}^k X_i = s) \) such that \( \sum_{i=1}^k X_i = s \) and the un-normalized probability density

\[
p(X_1 = x_1) \cdot p(X_2 = x_2) \cdots p(X_k = x_k) > \epsilon \quad (1)
\]

If such an \((x_i)\) does not exist, failure must be reported.

**Theorem 2:** Problem 3 is NP-hard.

**Proof.** Suppose that such a subset \( A \subseteq \{a_1, \ldots, a_k\} \) does exist. Let \( X_i = x_i = a_i \) if \( a_i \in A \) and \( X_i = x_i = 0 \) otherwise. Then by definition of \( p_i(x_i) \):

\[
\prod_{i=1}^k p(X_i = x_i) = \frac{1}{\prod_i c_i} \cdot \frac{1}{(2\pi)^k} \prod_i \left(1 + \exp\left(-\frac{a_i^2}{\delta^2}\right)\right) > \epsilon
\]

By the definition of \( \eta \) and \( \epsilon \), the above LHS is greater than \( \epsilon \). As a result, the sample \((X_1 = x_1 \ldots X_k = x_k)\) is a valid solution for Problem 3.

Conversely, if Problem 3 has a valid solution, we can show there is a subset \( A \) of \( \{a_1, \ldots, a_k\} \) summing up to \( s \). To show this, suppose that a sample \((X_1 = x_1, \ldots, X_k = x_k)\) such that \( \sum_{i=1}^k x_i = s \) and the un-normalized probability density \( \prod p(X_i = x_i) > \epsilon \). We claim \( |x_i| < \frac{1}{2\pi} \) or \( |x_i - a_i| < \frac{1}{2\pi} \) for each \( i = 1 \ldots k \).

To see this, note that if \( |x_j| \geq \frac{1}{2\pi} \) and \( |x_i - a_i| \geq \frac{1}{2\pi} \),

\[
\prod_{i=1}^k p(X_i = x_i) = \frac{1}{\prod_i c_i} \cdot \frac{1}{(2\pi)^k} \prod_i \left(1 + \exp\left(-\frac{a_i^2}{\delta^2}\right)\right) \leq \frac{1}{\prod_i c_i} \cdot \frac{1}{(2\pi)^k} \prod_i \left(1 + \exp\left(-\frac{a_i^2}{\delta^2}\right)\right) \cdot \exp\left(-\frac{1}{4\pi^2}\right) + \exp\left(-\frac{(a_i - 1/2k)^2}{\delta^2}\right) \\
< \epsilon
\]

Since

\[
\exp\left(-\frac{1}{4\pi^2}\right) + \exp\left(-\frac{(a_i - 1/2k)^2}{\delta^2}\right) < 2 \exp(-16) = \eta
\]

We can select a set \( A = \{a_i \mid x_i - a_i < \frac{1}{2\pi}\} \). It then follows \( \sum_{a_i \in A} a_i - \sum x_i < \frac{k}{2\pi} = \frac{1}{2} \). Therefore \( \sum_{a \in A} a = s \).

**Lemma 1.** \( \Delta^k \) is compact.

**Proof of Lem. 1**

**Lemma 2.** Let \( f \) be a continuous real valued function defined on compact set \( X \). Then \( f \) is bounded on \( X \).

**Proof of Lem. 2**

**Lemma 3.** The target joint distribution \( X_1, \ldots, X_k, \sum_{i=1}^k X_i = S \) has continuous probability density function \( p \) on \( R^k \) with support containing the standard \( k - 1 \) simplex of value \( S \). That is

\[
\text{supp}(X_1, \ldots, X_k, \sum_{i=1}^k X_i = S) \supseteq \Delta^{k-1}_{S}
\]
Proof of Lem. 3

Proof. By Assumption (1) we can write the joint density as

$$N(x_1, \ldots, x_k, s) = \left( \prod_{j=1}^{n} p(x_i) \right) I \left( \sum_{i=1}^{k} x_i = s \right)$$

This expression is continuous on $\Delta_S^{k-1}$ when $s = S$ so we have a continuous density on $\Delta_S^{k-1}$.

Lemma 4. If $S - \sum_{j=1}^{i-1} x_j > 0$ then

$$P_{q_i(\cdot | \eta_i)}(0 \leq X_i \leq S - \sum_{j=1}^{i-1} x_j) \geq \frac{1}{2}$$

Proof of Lem. 4

Proof. Note that

$$\mathbb{E}_{q_i(\cdot | \eta_i)}[X_i] = \int x_i q_i(x_i | \eta_i) dx_i = \eta_i = \frac{S - \sum_{j=1}^{i-1} x_j}{k - i + 1}$$

Using this equation, we calculate that

$$P_{q_i(\cdot | \eta_i)}(0 \leq X_i \leq S - \sum_{j=1}^{i-1} x_j)$$

$$= \int q_i(x_i | \eta_i) I(0 \leq x_i \leq \eta_i(k - i + 1)) dx_i$$

$$\geq 1 - \int q_i(x_i | \eta_i) I(x_i > 2 \eta_i)$$

$$= 1 - P_{q_i(\cdot | \eta_i)}(x_i > 2 \eta_i)$$

$$\geq 1 - \frac{\mathbb{E}_{q_i(\cdot | \eta_i)}[X_i]}{2 \eta_i} = 1 - \frac{\eta_i}{2 \eta_i} = \frac{1}{2}$$

Lemma 5. DYSC$(S, k, X_i, q_i)$ has nonzero probability density function $q$ on $\mathbb{R}^k$ with support equal to the standard $k - 1$ simplex of value $S$

$$\text{supp}(\text{DYSC}(S, k, X_i, q_i)) = \Delta_S^{k-1}$$

$q$ is continuous in the interior of $\Delta_S^{k-1}$, int$(\Delta_S^{k-1})$, and is lower bounded on $\Delta_S^{k-1}$ by some $c > 0$.

Proof of Lem. 5

Proof. We prove this result by constructing an explicit formula for $q$ on $\mathbb{R}^k$. Given $x \in \mathbb{R}^k$, as before let

$$\ell(x, S) = \min(k - 1, \max_j \sum_{i=1}^{j-1} |x_i| < 1)$$

We need the function $\ell(x, S)$ since Alg. 1 stops random sampling at $j$ if $j = k - 1$ or if $\sum_{j=1}^{j-1} x_j = S$. Using this function, we can write our formula,

$$q(x_1, \ldots, x_k) \propto$$

$$\left( \prod_{i=1}^{\ell(x, S)} q_i(x_i | \eta_i) I \left( 0 \leq x_i \leq S - \sum_{j=1}^{i-1} x_j \right) \right)$$

If $x \notin \Delta_S^{k-1}$, the indicator functions in the above formula guarantee that $q(x) = 0$. If $x \in \Delta_S^{k-1}$, then the above formula simplifies.

$$q(x_1, \ldots, x_i) = \frac{N(x_1, \ldots, x_k)}{D(x_1, \ldots, x_k)}$$

$$N(x) = \prod_{i=1}^{\ell(x, S)} q_i(x_i | \eta_i)$$

$$D(x) = \prod_{i=1}^{\ell(x, S)} \int q_i(y | \eta_i) I(0 \leq y \leq S - \sum_{j=1}^{i-1} x_j) dy$$

where $N$ and $D$ are respectively the numerator and denominator. Assumption (3) guarantees that $N(x)$ and $D(x)$ are positive for $x \in \Delta_S^{k-1}$ and continuous in any neighborhood where $\ell(x, S)$ is constant. Specifically note that $\ell(x, S)$ is continuous in int$(\Delta_S^{k-1})$ so $N(x)$ and $D(x)$ are continuous there as well. Before we prove continuity of $q$ on int$(\Delta_S^{k-1})$, we will need the following consequence of lemma 4

$$D(x_1, \ldots, x_k)$$

$$= \prod_{i=1}^{\ell(x, S)} \int q_i(y | \eta_i) I(0 \leq y \leq S - \sum_{j=1}^{i-1} x_j) dy$$

$$= \prod_{i=1}^{\ell(x, S)} P_{q_i(\cdot | \eta_i)}(0 \leq X_i \leq S - \sum_{j=1}^{i-1} x_j)$$

$$\geq \prod_{i=1}^{\ell(x, S)} \frac{1}{2} \geq \left( \frac{1}{2} \right)^k$$

Thus the denominator $D$ is lower bounded by a positive number. Now let $x, y \in \text{int}(\Delta_S^{k-1})$. We can calcu-
late

\[ |q(x) - q(y)| = \left| \frac{N(x)}{D(x)} - \frac{N(y)}{D(y)} \right| \]

\[ = \left| \frac{N(x)}{D(x)} - \frac{N(x)}{D(y)} + \frac{N(x)}{D(y)} - \frac{N(y)}{D(y)} \right| \]

\[ = \left| N(x) \left( \frac{1}{D(x)} - \frac{1}{D(y)} \right) + \frac{1}{D(y)} (N(x) - N(y)) \right| \]

\[ \leq N(x) \left| \frac{1}{D(x)} - \frac{1}{D(y)} \right| + \frac{1}{D(y)} |N(x) - N(y)| \]

\[ \leq \left| \frac{1}{D(x)} - \frac{1}{D(y)} \right| + 2^k |N(x) - N(y)| \]

Since \( \frac{1}{D} \) and \( N \) are both positive continuous functions on \( \text{int}(\Delta_{k-1}^S) \), we can make the right hand side arbitrarily small as \( |x - y| \to 0 \). It follows that \( q \) is continuous on \( \text{int}(\Delta_{k-1}^S) \). We now show that \( q \) is lower bounded on \( \Delta_{k-1}^S \). By Assumption (3), there exist continuous positive functions \( f_i \) on \([0, \infty)\) that lower bound \( q_i \) regardless of choice of \( \eta_i \). Then \( f = \prod_{i=1}^{k} f_i \) defines a positive continuous function on \( \Delta_{k-1}^S \). Since \( \Delta_{k-1}^S \) is compact, \( f \) has a lower bound \( c > 0 \) on \( \Delta_{k-1}^S \).

It follows that \( q \) is lower bounded by \( c \). \( \square \)