6. Proofs

6.1. Proof of Theorem 1

Since $\iint I_{[-z',\infty)}(z)d\mathscr{E}[F(z)F(z')] < \infty$, then

$$\begin{aligned}\mathscr{E}[P(Z \geq -Z')] &= \mathscr{E}\left[\iint I_{[-z',\infty)}(z)dF(z)F(z')\right] \quad (11)\\ &= \iint I_{[-z',\infty)}(z)d\mathscr{E}[F(z)F(z')].\end{aligned}$$

Let us define the following sets:

$$\begin{aligned}A = (-\infty, \min(z,z')],\ \ B = (-\infty, \max(z,z')],\\ C = (\min(z,z'), \max(z,z')]\end{aligned}$$

then

$$\begin{aligned}\mathscr{E}[F(z)F(z')] &= \mathscr{E}[P(A)P(B)] = \mathscr{E}[P(A)(P(A)+P(C))]\\ &= \mathscr{E}[P(A)^2] + \mathscr{E}[P(A)P(C)].\end{aligned}$$

From the property of the Dirichlet distribution, it results that

$$\begin{aligned}&\mathscr{E}[P(A)^2] + \mathscr{E}[P(A)P(C)]\\ &= \frac{\mathscr{E}[P(A)](1+\alpha(\mathscr{Z})\mathscr{E}[P(A)])}{\alpha(\mathscr{Z})+1} + \frac{\alpha(\mathscr{Z})\mathscr{E}[P(A)]\mathscr{E}[P(C)]}{\alpha(\mathscr{Z})+1}.\end{aligned}$$

By some algebraic manipulations, it follows that

$$\begin{aligned}\mathscr{E}[F(z)F(z')] &= \frac{\mathscr{E}[P(A)]}{\alpha(\mathscr{Z})+1} + \frac{\alpha(\mathscr{Z})\mathscr{E}[P(A)]\mathscr{E}[P(B)]}{\alpha(\mathscr{Z})+1}\\ &= \frac{1}{\alpha(\mathscr{Z})+1}G(\min(z,z')) + \frac{\alpha(\mathscr{Z})}{\alpha(\mathscr{Z})+1}G(z)G(z').\end{aligned}$$

6.2. Proof of Theorem 2

From (2), setting $\alpha(\mathscr{Z}) = s \to 0$ it can be easily seen that

$$\mathscr{E}\left[P(Z \geq -Z')\right] = \iint I_{[-z',\infty)}(z)dG(\min(z,z')). \quad (12)$$

Note that, $G_0(\min(z,z'))$ is a singular distribution on the cartesian product $Z \times Z'$. Hence, we can write $dG_0(\min(z,z')) = \delta_z(z')dG_0(z)dz'$. As example consider the multivariate Normal distribution

$$\begin{bmatrix} Z \\ Z' \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right),$$

which tends to $\delta_z(z')N(z,0,1)$ for $\rho \to 1$. Since $dG_0(\min(z,z')) = \delta_z(z')dG_0(z)$, then we have $\mathscr{E}\left[P(Z \geq -Z')\right] = \int I_{[0,\infty)}(z)dG_0(z)$.

The posterior is found for $\alpha(\mathscr{Z}) = s+n \to n$ and

$$G = G_n = \frac{1}{n}\sum_{i=1}^{n} I_{[Z_i,\infty)}.$$

Then we can write the integral in (2) as follows :

$$\begin{aligned}&\mathscr{E}[P(Z \geq -Z')|Z^n]\\ &= \frac{1}{n(n+1)}\left[\int I_{[-z',\infty)}(z)\sum_{i=1}^{n}\delta_{Z_i}(z)\sum_{i=1}^{n}\delta_{Z_i}(z')dzdz'\right.\\ &\left. + \iint I_{[-z',\infty)}(z)d\left(\sum_{i=1}^{n} I_{[Z_i,\infty)}(\min(z,z'))\right)\right]\end{aligned}$$

Given that

$$I_{[a,\infty)}(\min(z,z')) = \begin{cases} 1, & z,z' \geq a,\\ 0, & otherwise\end{cases} \quad (13)$$

which implies that $dI_{[a,\infty)}(\min(z,z')) = \delta_{(a,a)}(z,z')dzdz'$, then,

$$\iint I_{[-z',\infty)}(z)d\sum_{i=1}^{n} I_{[Z_i,\infty)}(\min(z,z')) = \sum_{i=1}^{n} I_{[0,\infty)}(Z_i). \quad (14)$$

It also holds that

$$\iint I_{[-z',\infty)}(z)\sum_{i=1}^{n}\delta_{Z_i}(z)\sum_{i=1}^{n}\delta_{Z_j}(z')dzdz' = \sum_{i=1}^{n}\sum_{j=1}^{n} I_{[-Z_j,\infty)}(Z_i). \quad (15)$$

Therefore,

$$\begin{aligned}&\mathscr{E}[P(Z \geq -Z')|Z^n]\\ &= \frac{\left[\sum_{i=1}^{n}\sum_{j=1}^{n} I_{[-Z_j,\infty)}(Z_i) + \sum_{i=1}^{n} I_{[0,\infty)}(Z_i)\right]}{n(n+1)}\end{aligned}$$

6.3. Proof of Theorem 3

From (2), setting $\alpha(\mathscr{Z}) = s$ and $dG = \delta_{Z_0}$ it can be easily seen that

$$\begin{aligned}\mathscr{E}[P(Z \geq -Z')] = 0 \quad \text{if } Z_0 < 0\\ \mathscr{E}[P(Z \geq -Z')] = 1 \quad \text{if } Z_0 > 0\end{aligned}$$

and thus those are the lower and upper bounds of $\mathscr{E}[P(Z \geq -Z')]$.

When $\alpha(\mathscr{Z}) = s+n$ and $G = G_n$ we can write the integral in (2) as follows :

$$\begin{aligned}\mathscr{E}[P(Z \geq -Z')|Z^n] = &\frac{1}{(s+n)(s+n+1)}\\
&\Big[s^2 \iint I_{[-z',\infty)}(z) dG_0(z) dG_0(z')\\
&+ s \iint I_{[-z',\infty)}(z) \sum_{i=1}^n \delta_{Z_i}(z') dG_0(z) dz'\\
&+ s \iint I_{[-z',\infty)}(z) \sum_{i=1}^n \delta_{Z_i}(z) dG_0(z') dz\\
&+ \iint I_{[-z',\infty)}(z) \sum_{i=1}^n \delta_{Z_i}(z) \sum_{j=1}^n \delta_{Z_j}(z') dz dz'\\
&+ \iint I_{[-z',\infty)}(z) d\left(\frac{s}{s+n} G_0(\min(z,z'))+ \right.\\
&\left.\left. + \sum_{i=1}^n I_{[z_i,\infty)}(\min(z,z'))\right)\right].\end{aligned} \tag{16}$$

Assuming that $dG_0 = \delta_b$, with $b \neq Z_1, \ldots, Z_n$, one has

$$\iint I_{[-z',\infty)}(z) dG_0(z) dG_0(z') = I_{[0,\infty)}(b), \tag{17}$$

$$\iint I_{[-z',\infty)}(z) \sum_{i=1}^n \delta_{Z_i}(z') dG_0(z) dz' = \sum_{i=1}^n I_{[-z_i,\infty)}(b), \tag{18}$$

$$\iint I_{[-z',\infty)}(z) \sum_{i=1}^n \delta_{Z_i}(z) dG_0(z') dz = \sum_{i=1}^n I_{[-z_i,\infty)}(b), \tag{19}$$

$$\iint I_{[-z',\infty)}(z) d\left(G_0(\min(z,z'))\right) = I_{[0,\infty)}(b), \tag{20}$$

where the last result follows from (13). Therefore from (16–20) and (14–15), the posterior expectation is

$$\begin{aligned}\mathscr{E}[P(Z \geq -Z')|Z^n] = &\frac{1}{(s+n)(s+n+1)}\\
&\Big[s^2 I_{[0,\infty)}(b) + 2s \sum_{i=1}^n I_{[-Z_i,\infty)}(b)\\
&+ \sum_{i=1}^n \sum_{j=1}^n I_{[-Z_j,\infty)}(Z_i) + s I_{[0,\infty)}(b) + \sum_{i=1}^n I_{[0,\infty)}(Z_i)\Big].\end{aligned}$$

Its extrema are found when both $I_{[0,\infty)}(b)$ and $I_{[-z_i,\infty)}(b)$ are equal to, respectively, 0 (minimum) and 1 (maximum). These extrema have been obtained assuming that $G_0 = \delta_{Z_0} = \delta_b$. However, this result holds for any other choice of G_0, since $G_0 = \delta_b$ for a proper choice of b minimizes/maximizes the terms involving G_0 in (16).

6.4. Proof of Theorem 4

The posterior lower and upper probabilities of $P(Z \geq -Z') > a$ are obtained in correspondence of the DP priors with atomic base distribution $dG_0 = \delta_{Z_0}$ given in Theorem 3. This can be proven by using a stick-breaking construction of DP from a generic G_0 and showing that the lower and upper probabilities $\underline{\mathscr{P}}, \overline{\mathscr{P}}$ are obtained for $dG_0 = \delta_{Z_0}$ for a suitable choice of Z_0. Those priors give posterior DPs with base distribution

$$dG_n = \frac{s}{s+n}\delta_{Z_0} + \frac{1}{s+n}\sum_{j=1}^n \delta_{Z_j}$$

The fact that a sample from dG_n is given by $dF_n = w_0\delta_{Z_0} + \sum_{j=1}^n w_j \delta_{Z_j}$ follows from the definition of Dirichlet process and the discreteness of the support of G_n, by taking the partition $(\{Z_0\}, \{Z_1\}, \ldots, \{Z_n\}, \mathbb{R} \setminus \{Z_0, \ldots, Z_n\})$; the vector of probabilities $(P(\{Z_0\}), P(\{Z_1\}), \ldots, P(\{Z_n\}), P(\mathbb{R} \setminus \{Z_0, \ldots, Z_n\}))$ has a Dirichlet distribution with parameters $(s, 1, \ldots, 1, 0)$, and thus $(P(\{Z_0\}), P(\{Z_1\}), \ldots, P(\{Z_n\})) \sim Dir(s, 1, \ldots, 1)$. Let F_n be a sample from $DP(s+n, dG_n)$, the probability of $P(Z \geq -Z') > a$ is:

$$\mathscr{P}\left[P(Z \geq -Z') > a|Z^n\right] = \mathscr{P}\left[P_n > a\right],$$

with

$$P_n = \iint I_{[-z',\infty)}(z) d(F_n(z) F_n(z')).$$

By $dF_n = w_0\delta_{Z_0} + \sum_{j=1}^n w_j \delta_{Z_j}$, one has

$$\begin{aligned}P_n = &\iint I_{[-z',\infty)}(z)(w_0\delta_{Z_0}(z) + \sum_{j=1}^n w_j \delta_{Z_j}(z))\cdot\\
&\cdot (w_0\delta_{Z_0}(z') + \sum_{j=1}^n w_j \delta_{Z_j}(z')) dz dz'\\
= &w_0^2 I_{[0,\infty)}(Z_0) + 2w_0 \sum_{i=1}^n w_i I_{[-Z_i,\infty)}(Z_0) + \sum_{i=1}^n \sum_{j=1}^n w_i w_j I_{[-Z_i,\infty)}(Z_j).\end{aligned}$$

which gives

$$P_n = \sum_{i=1}^n \sum_{j=1}^n w_i w_j I_{[-Z_i,\infty)}(Z_j) \qquad \text{if } Z_0 < -\max Z_i$$

$$P_n = w_0(2-w_0) + \sum_{i=1}^n \sum_{j=1}^n w_i w_j I_{[-Z_i,\infty)}(Z_j) \text{ if } Z_0 > \max Z_i.$$