9. Further Details for Section 3

9.1. Proof Sketch of Theorem 1

The proof parallels the proof of Proposition 1 by Strehl et al. (2006) for MDPs, except the horizon (denoted by T in their paper) needs to be redefined:

$$H = \frac{1}{1-\gamma}\ln\frac{4}{\epsilon(1-\gamma)} + L + \frac{1}{\sqrt{C}}\ln\frac{2}{\delta'}.$$

This choice of H ensures that, for any epoch t, the non-stationary policy $\mathbf{A}_t$ in state s_t is either $\Theta(\epsilon)$-optimal, or will reach an unknown state in H steps with probability at least $\epsilon(1-\gamma)$. In either case, the algorithm will reach a next state between step $\frac{1}{1-\gamma}\ln\frac{4}{\epsilon(1-\gamma)}$ and H, since with probability at least $1-\delta'$, the waiting time of taking action a_t in state s_t is $L + \frac{1}{\sqrt{C}}\ln\frac{1}{\delta'}$ (Lemma 1). Taking a union bound over all possible non-ϵ-optimal steps (which is polynomial in ζ, $1/\epsilon$, $1/\delta$, and $1/(1-\gamma)$), that is, setting δ' to $\delta/\mathrm{poly}(\zeta, 1/\epsilon, 1/\delta, 1/(1-\gamma))$, we can prove the theorem as done in Strehl et al. (2006). Note that we need not take a union over all epochs, but only those where the decision is potentially non-ϵ-optimal; if $\mathbf{A}_t$ is ϵ-optimal in epoch t, it does not count towards the sample complexity anyway.

9.2. Definition of Known-state SMDP

Definition 3 *Let $M = \langle \mathcal{S}, \mathcal{A}, P, R, \gamma\rangle$ be an SMDP, Q is a state–action value function, and $\mathcal{K} \subseteq \mathcal{S}\times\mathcal{A}$ a set of "known" state–actions. Define the known state–action SMDP (with respect to $\mathcal{K}$) as $M_{\mathcal{K}} = \langle \mathcal{S}, \mathcal{A}, P_{\mathcal{K}}, R_{\mathcal{K}}, \gamma\rangle$, where*

$$\begin{aligned} P_{\mathcal{K}}(s',\tau|s,a) &= \begin{cases} P(s',\tau|s,a), & \textit{if } (s,a)\in\mathcal{K} \\ \mathbb{I}\left(s=s',\tau=1\right), & \textit{otherwise} \end{cases} \\ R_{\mathcal{K}}(s,a) &= \begin{cases} R(s,a), & \textit{if } (s,a)\in\mathcal{K} \\ (1-\gamma)Q(s,a), & \textit{otherwise.} \end{cases} \end{aligned}$$

In other words, the known state–action SMDP $M_{\mathcal{K}}$ has identical dynamics to M except in unknown state–actions where (i) the transitions are all self-loops with unit waiting time, and (ii) the Q-values are exact.

9.3. Proof of Theorem 2

Clearly, the construction leads to optimistic value functions, so the first condition of Theorem 1 holds.

We now consider when a state–action pair (s,a) becomes known. Define the *effective* transition probabilities by

$$P^S(s'|s,a) = \sum_\tau P(s',\tau|s,a)\gamma^\tau,$$

and the marginal distribution of waiting time by

$$P^T(\tau|s,a) = \sum_{s'} P(s',\tau|s,a).$$

We first generalize the simulation lemma (see, *e.g.*, Kearns & Singh (2002); Strehl et al. (2009)) for MDPs to SMDPs, giving a bound on the value function differences in terms of model estimation errors:

Lemma 5 *Let $M_i = \langle \mathcal{S}, \mathcal{A}, P_i, R_i, \gamma\rangle$ ($i = 1, 2$) be two SMDPs that differ only in reward and transition functions, and V_i^* and Q_i^* the respective optimal value functions. Let $\bar\gamma_{s,a}$ be the effective discount factor for (s,a) under M_2:*

$$\bar\gamma_{s,a} = \sum_\tau \gamma^\tau P_2^T(\tau|s,a).$$

and define the discount-adjusted model estimation error by

$$\begin{aligned} \varepsilon_{s,a} &= \frac{1}{1-\bar\gamma_{s,a}}\big(\left|R_1(s,a)-R_2(s,a)\right| \\ &\quad + V_{\max}\left\|P_1^S(\cdot|s,a) - P_2^S(\cdot|s,a)\right\|_1\big). \end{aligned}$$

Then, for any s and a,

$$\begin{aligned} |Q_1^*(s,a) - Q_2^*(s,a)| &\le \max_{s,a}\varepsilon_{s,a} \\ |V_1^*(s,a) - V_2^*(s,a)| &\le \max_{s,a}\varepsilon_{s,a} \end{aligned}$$

Proof Let (s,a) be the state–action pair that achieves maximum difference of $|Q_1^*(\cdot,\cdot) - Q_2^*(\cdot,\cdot)|$. To simplify notation, define

$$\begin{aligned} \varepsilon_R &= |R_1(s,a) - R_2(s,a)| \\ \varepsilon_P &= \left\|P_1^S(\cdot|s,a) - P_2^S(\cdot|s,a)\right\|_1 \\ \Delta &= |Q_1^*(s,a) - Q_2^*(s,a)| \end{aligned}$$

Then,

$$\begin{aligned}
\Delta &= |Q_1^*(s,a) - Q_2^*(s,a)| \\
&= \left| \left(R_1(s,a) + \sum_{s',\tau} \gamma^\tau P_1(s',\tau|s,a) V_1^*(s') \right) \right. \\
&\qquad \left. - \left(R_2(s,a) + \sum_{s',\tau} \gamma^\tau P_2(s',\tau|s,a) V_2^*(s') \right) \right| \\
&\leq |R_1(s,a) - R_2(s,a)| \\
&\quad + \left| \sum_{s',\tau} \gamma^\tau \left(P_1(s',\tau|s,a) - P_2(s',\tau|s,a) \right) V_1^*(s') \right| \\
&\quad + \left| \sum_{s',\tau} \gamma^\tau P_2(s',\tau|s,a) (V_1^*(s') - V_2^*(s') \right| \\
&\leq \varepsilon_R + V_{\max} \varepsilon_P + \Delta \left| \sum_{s',\tau} \gamma^\tau P_2(s',\tau|s,a) \right| \\
&= (\varepsilon_R + V_{\max}\varepsilon_P) + \bar{\gamma}_{s,a} \Delta \\
&= (1 - \bar{\gamma}_{s,a}) \varepsilon_{s,a} + \bar{\gamma}_{s,a} \Delta .
\end{aligned}$$

Rearranging terms, we have

$$\Delta \leq \varepsilon_{s,a} \leq \max_{s',a'} \varepsilon_{s',a'} .$$

The case for V^* follows immediately from the following observation: for any state s,

$$\begin{aligned}
|V_1^*(s) - V_2^*(s)| &= \left| \max_a Q_1^*(s,a) - \max_a Q_2^*(s,a) \right| \\
&\leq \max_a |Q_1(s,a) - Q_2(s,a)| \ \leq \ \Delta .
\end{aligned}$$

□

Clearly, $R(s,a) \in [0, \frac{1}{1-\gamma}]$. Using a concentration argument based on Hoeffding's inequality, one can establish that $\mathbb{O}\left(1/(\varepsilon^2(1-\gamma)^2)\right)$ samples suffice to ensure ε accuracy in the reward estimate. Similarly, the effective transition probabilities $P(s'|s,a)$ can also be estimated within ε total variation with $\mathbb{O}\left(N_{sa}/\varepsilon^2\right)$ samples. Therefore, by setting ε appropriately, the accuracy condition in Theorem 1 can be satisfied.

Finally, there are at most SA many state–actions, each becoming known when it is visited sufficiently often. The bounded-surprises condition in Theorem 1 thus holds.

Therefore all three conditions of Theorem 1 hold, and the result follows.

10. Further Details for Section 4

10.1. Proof Sketch of Lemma 3

Fix a non-ϵ-optimal option set $\mathcal{O}' \subset \mathcal{O}^*$ with $|\mathcal{O}'| \leq \bar{O}$. By assumption, it fails to represent a near-optimal policy for MDPs drawn i.i.d. from ν over $\mathcal{M}$. Following the same argument for Lemma 1 of Brunskill & Li (2013), $p_{\min}^{-1} \ln \frac{C}{\delta}$ many tasks suffices to reveal the non-ϵ-optimality of $\mathcal{O}'$, with probability at least $1 - \delta/C$. Taking a union bound over all C subsets of $\mathcal{O}^*$ up to size $\bar{O}$, one finishes the proof of the lemma.

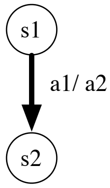
10.2. Proof Sketch of Lemma 4

For convenience, define $\epsilon_1 = (\epsilon - \varepsilon)/4$. The proof relies on three major steps, each holding with probability at least $1 - \delta$.

- *The MDP models are all estimated to sufficient accuracy:* The condition together with Lemma 2 implies every state–action will be visited at least $\Omega(N V_{\max}^2 \epsilon_1^{-2} (1-\gamma)^{-2} \ln 1/\delta)$ times. Applying Hoeffding's inequality together with Lemma 8.5.5 of Kakade (2003), the reward and transition probabilities of every state–action pair are estimated with $\epsilon_1(1-\gamma)/V_{\max}$ accuracy. By the simulation lemma (*c.f.*, Kearns & Singh (2002); Strehl et al. (2009)), $\left|V_M^*(s) - V_{\hat{M}}^*(s)\right| < \epsilon_1$, and similarly, $\left|V_{M'}^*(s) - V_{\hat{M}'}^*(s)\right| < \epsilon_1$, where M and $\hat{M}$ are the underlying/estimated MDPs, and M' and $\hat{M}'$ the corresponding SMDPs induced by the discovered option set $\hat{\mathcal{O}}$.
- *The discovered option set $\hat{\mathcal{O}}$ is ϵ-optimal for all MDPs in $\mathcal{M}$:* Using the triangle inequality together with the two inequalities established in the previous step, we have

  $$\begin{aligned}
  & V_M^*(s) - V_{M'}^*(s) \\
  &\quad \leq \left|V_M^*(s) - V_{\hat{M}}^*(s)\right| + \left|V_{M'}^*(s) - V_{\hat{M}'}^*(s)\right| \\
  &\qquad + \left|V_{\hat{M}}^*(s) - V_{\hat{M}'}^*(s)\right| \\
  &\quad \leq 2\epsilon_1 + \left|V_{\hat{M}}^*(s) - V_{\hat{M}'}^*(s)\right| .
  \end{aligned}$$

  In the option-discovery step, $\hat{\mathcal{O}}$ must satisfy $V_{\hat{M}}^*(s) - V_{\hat{M}'}^*(s) \leq (\epsilon + \varepsilon)/2$. Therefore, $V_M^*(s) - V_{M'}^*(s) \leq 2\epsilon_1 + (\epsilon+\varepsilon)/2 = \epsilon$; that is, the option set $\hat{\mathcal{O}}$ is ϵ-optimal for all MDPs *encountered in phase 1*. According to Lemma 3, $\hat{\mathcal{O}}$ must also be ϵ-optimal for all MDPs in $\mathcal{M}$; otherwise, it will fail to represent ϵ-optimal policies in at least one MDP in phase 1.
- *There exists at least one option set that satisfies the criterion of Equation 2:* According to the assumption, there exists some option set $\bar{\mathcal{O}}$ that is ε-optimal for $\mathcal{M}$: for any M and any s, $V_M^*(s) - V_{M'}^*(s) < \varepsilon$, where



	m1	m2	m3	m4
s1	a1	a1	a2	a2
s2	a4	a4	a5	a5

Figure 1. Example for sample complexity calculation illustration. The table shows the ϵ-optimal actions for each MDP. There are 5 actions but a3 is never optimal for any MDP.

M' is the SMDP induced by M and $\bar{\mathcal{O}}$. Using the triangle inequality as well as the accuracy guarantee established in step 1, one gets

$$\begin{aligned} V^*_{\hat{M}}(s) - V^*_{\hat{M}'}(s) &< V^*_M(s) + \epsilon_1 - V^*_{M'}(s) + \epsilon_1 \\ &< \varepsilon + 2\epsilon_1 \\ &= (\epsilon + \varepsilon)/2. \end{aligned}$$

In other words, $\bar{\mathcal{O}}$ will satisfy Equation 2.

The overall failure probability is at most δ: All three steps above hold with high probability. The first two steps require a union bound over all possible subsets of $\mathcal{O}^*$ with size up to $\bar{O}$. There are $C = \mathbb{O}\left((O^*)^{\bar{O}}\right)$ many such subsets. It suffices to set $\delta \leftarrow \delta/C$ for the union bound to complete the whole proof.

10.3. Proof of Theorem 3

The sample complexity can be divided into two terms, corresponding to tasks in phase 1 and in phase 2, respectively. The sample complexity of the MDP tasks in phase 1 is simply the number of tasks in phase 1, T_1, multiplied by the sample complexity of the E^3 algorithm.

11. Further Details for Section 5

We now illustrate the process of evaluting the bound on the sample complexity benefit with the small example shown in Figure 1. In this example there are 2 states and 4 MDPs, and each MDP has a single ϵ-optimal action in each state, shown in the Figure's table. Assume that state s_1 deterministically transitions to s_2. Before introducing an option, there were 4 state–action combinations $(s_1, a_1), (s_1, a_2), (s_2, a_4), (s_2, a_5)$ needed to cover the ϵ-optimal policies of each MDP, resulting in a sample complexity bound of $\mathbb{O}\left(\frac{4}{(1-\gamma)^6}\right)$. Now consider adding the option whose initiation state is s_1 and that takes action a_2 in state s_1 and action a_5 in state s_2. The length of this option is always 2, so from the prior section the option's contribution to the sample complexity is $\mathbb{O}\left(\frac{1}{(1-\gamma^2)^2(1-\gamma)^3}\left(2 + \frac{1}{1-\gamma}\right)\right)$. This option covers MDPs m_3 and m_4. To cover s_1 and s_2 for the remaining uncovered MDPs requires just 2 primitive state–action pairs, with a resulting $\mathbb{O}\left(\frac{2}{(1-\gamma)^6}\right)$ contribution to the sample complexity bound. Therefore, introducing the option will reduce this upper bound on the sample complexity if

$$\begin{aligned} \frac{1}{(1-\gamma^2)^2(1-\gamma)^3}(2 + \frac{1}{1-\gamma}) + \frac{2}{(1-\gamma)^6} &< \frac{4}{(1-\gamma)^6} \\ \Leftrightarrow \qquad 5 &< 6\gamma + \gamma^2 \end{aligned}$$

which holds for large γ, such as $\gamma = 0.9$. The algorithm evaluates this expression for the input γ, and keeps the option if the expression holds.