# Multi-period Trading Prediction Markets: Supplementary Material

## A. Algorithm for the Multi-period Trading Markets (the Split Version)

We split the market algorithm into two routines, one for the market maker and one for the agent, respectively. We do this to emphasise the fact that each agent in the market has its *own objective* (achieving the optimal portfolio based on its unique preferences), plus a communication with the market maker.

**Algorithm 1** The market maker in a multi-period market

**Input:** time period $T$
- **for** $t = 1$ **to** $T$ **do**
  - publish a pricing rule $c_t(\cdot)$
  - collect trading request $\{\Delta X_{n,t}\}$ from agents
  - choose agent $a_t$ and make trade, $\Delta X_t \equiv \Delta X_{a_t,t}$
  - update $c_t(\cdot) \to c_{t-1}(\cdot)$
- **end for**
- close the market

**Output:** $\{a_t\}, \{\Delta X_t\}$

**Algorithm 2** An agent $n \in A$ in a multi-period market

**Input:** initial portfolio $\{w_{n,0}, X_{n,0}\}$, risk measure $\rho_n(\cdot)$, starting point $t = 1$
- **repeat**
  - receive the pricing rule for time $t$
  - calculate $\{\Delta X_{n,t}, -c_t(\Delta X_{n,t})\}$ for this round using **Select**$(\{w_{n,t-1}, X_{n,t-1}\}, \rho_n(\cdot), c_t(\cdot))$, and send its trading request to the market maker
  - **if** trade happens **then**
    - $X_{n,t} = X_{n,t-1} + \Delta X_{n,t}$
    - $w_{n,t} = w_{n,t-1} - c_t(\Delta X_{n,t})$
  - **else**
    - $X_{n,t} = X_{n,t-1}, w_{n,t} = w_{n,t-1}, \Delta X_{n,t} = 0$
  - **end if**
  - $t = t + 1$
- **until** market is closed

**Output:** $\{w_{n,t}, X_{n,t}\}_{t=1,2,\ldots}, \{\Delta X_{n,t}\}_{t=1,2,\ldots}$

## B. Complete Proof of Proposition 1

**Proposition 1** (The global objective of a market)**.** *A multi-period market with a path-independent pricing rule market maker aims to minimise the global objective*

$$L = c(Y) + \sum_{n\in A} \rho_n(X_n), \quad Y = \sum_{n\in A} X_n, \tag{1}$$

*by performing a sequential optimisation algorithm, which is implemented by the trading process: define $\varphi_{a_t}(\Delta X'_t) \equiv \rho_{a_t}(X_{a_t,t-1} + \Delta X'_t + w_{a_t,t-1} - c_t(\Delta X'_t))$ and for each time $t$*

$$\Delta X_t = \arg\min_{\Delta X'_t} \varphi_{a_t}(\Delta X'_t), \tag{2}$$

$$X_{n,t} = X_{n,t-1} + \mathbb{1}(n = a_t)\Delta X_t, \tag{3}$$

$$w_{n,t} = w_{n,t-1} - \mathbb{1}(n = a_t)c_t(\Delta X_t), \tag{4}$$

$$Y_t = Y_{t-1} + \Delta X_t, \tag{5}$$

*If the algorithm converges at time $t'$, i.e. $\Delta X_t = 0$ for all $t > t'$, then $\{X_{n,t'}\}, Y_{t'}$ achieves a local minimum of the objective $L$ in (1).*

*Proof.* At time $t$ only agent $a_t$ will trade with the market maker, so $\Delta X_t = \Delta X_{a_t,t}$. At time $t$, for any agent $n$ all quantities calculated before $t$ can be treated as constants as they could no longer be modified. Therefore, the functional that is minimised in (2) has the same optimal point with the following functional

$$l_t(\Delta X'_t) = \rho_{a_t}(X_{a_t,t-1} + \Delta X'_t + w_{a_t,t-1} - c_t(\Delta X'_t)) - \rho_{a_t}(X_{a_t,t-1} + w_{a_t,t-1}). \tag{6}$$

Apply the property of translation invariance to $l_t$, we have

$$l_t(\Delta X'_t) = \rho_{a_t}(X_{a_t,t-1} + \Delta X'_t) - \rho_{a_t}(X_{a_t,t-1}) + c_t(\Delta X'_t). \tag{7}$$

Sum over all $l_t$'s and denote this summation by $L_T$, which is a functional. Then

$$\min_{\{\Delta X'_t\}} L_T = \min_{\{\Delta X'_t\}} \sum_{t=1}^{T} l_t(\Delta X'_t) = \sum_{t=1}^{T} \min_{\Delta X'_t} l_t(\Delta X'_t) = \sum_{t=1}^{T} l_t(\Delta X_t). \tag{8}$$

Here $\Delta X_t$'s are the optimal point obtained from (2). Substitute (7) to (8)

$$\sum_{t=1}^{T} l_t(\Delta X_t) = \sum_{t=1}^{T} \rho_{a_t}(X_{a_t,t-1} + \Delta X_t) - \rho_{a_t}(X_{a_t,t-1}) + \sum_{t=1}^{T} c_t(\Delta X_t). \tag{9}$$

Note that at time $t$ for any agent $n \neq a_t$ it makes no trade $\Delta X_{n,t} = 0$, and so

$$\rho_n(X_{n,t-1}+\Delta X_{n,t})-\rho_n(X_{n,t-1}) = 0, \quad \forall n \neq a_t. \quad (10)$$

The first summation on RHS thus becomes

$$\begin{aligned} &\sum_{t=1}^{T} \rho_{a_t}(X_{a_t,t-1} + \Delta X_t) - \rho_{a_t}(X_{a_t,t-1}) \\ =& \sum_{t=1}^{T} \sum_{n \in A} \rho_n(X_{n,t-1} + \Delta X_{n,t}) - \rho_n(X_{n,t-1}) \\ =& \sum_{n \in A} \rho_n(X_{n,T}) - \rho_n(X_{n,0}). \end{aligned} \quad (11)$$

Since the pricing rule is path-independent, the second summation on RHS is

$$\sum_{t=1}^{T} c_t(\Delta X_t) = \sum_{t=1}^{T} c_t(Y_t) - c_t(Y_{t-1}) = c(Y_t) - c(0), \quad (12)$$

where $Y_t = \sum_{\tau=1}^{t} \Delta X_\tau$ and $Y_0 = 0$. Since $X_{n,0} = 0$ and for any $t$ and $n \neq a_t$ $\Delta X_{n,t} = 0$, we have

$$\begin{aligned} Y_t &= \sum_{\tau=1}^{t} \Delta X_\tau = \sum_{\tau=1}^{t} \Delta X_{a_\tau,\tau} = \sum_{\tau=1}^{t} \sum_{n \in A} \Delta X_{n,\tau} \\ &= \sum_{n \in A} \sum_{\tau=1}^{t} \Delta X_{n,\tau} = \sum_{n \in A} X_{n,t}, \quad \forall t > 0. \end{aligned} \quad (13)$$

Finally, substitute (11) (12) and (13) to (8) and merge the rest terms we can end up with

$$\min_{\{\Delta X_t\}} L_T = \min_{\{\Delta X_t\}} c(Y_T) + \sum_{n \in A} \rho_n(X_{n,T}) - C, \quad (14)$$

where $Y_T = \sum_{n \in A} \Delta X_{n,T}$ and $C = c(0) + \sum_{n \in A} \rho_n(0)$ is a constant. This is a sequential minimisation scheme for $\min L$. Finally, if the market converges at time $T$, we have $X_n = X_{n,T}$ and $Y = Y_T$, leading to a local minimal point of $L$. $\square$

## C. Another Example of Constructing Risk Measures from Expected Utilities

As another example, consider the HARA utility

$$u_H(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \quad a > 0, \frac{ax}{1-\gamma} + b > 0. \quad (15)$$

The resultant convex risk measure is the one who has the following penalty functional

$$\alpha(Q) = \frac{\gamma}{a}\eta^{-1/\eta}(-u_0)^{1/\gamma}\,\mathbb{E}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^{\eta}\right]^{1/\eta} + (1-\gamma)\frac{b}{a}, \quad (16)$$

where $1/\eta + 1/\gamma = 1$.

## D. Bayesian Updates for Gaussians

Here we explain the connection between the markets with Bayesian updates in detail (the second example in Section 7). To estimate a univariate Gaussian $\mathcal{N}(\mu, \sigma_1)$ all we need is the sufficient statistics calculated from a set of $N$ data points $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$. For clarity of exposition let's assume that we only care about the Bayesian updates of the mean parameter $\mu$, and think $\sigma_1$ is a prefixed constant. Introduce a Conjugate prior on the mean

$$p(\mu \mid \mu_0, \sigma_0) \propto \exp\left(-\frac{1}{\theta_0}\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right), \quad (17)$$

where $\theta_0^{-1}$ is so-called the pseudo count. The posterior is

$$\begin{aligned} p(\mu \mid \mathcal{D}, \mu_0, \sigma_0) &\propto p(\mu \mid \mu_0, \sigma_0) p(\mathcal{D} \mid \mu, \sigma_1) \\ &\propto \exp\left(-\frac{1}{\theta_0}\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right)\exp\left(-N\frac{(\mu-\bar{x})^2}{2\sigma_1^2}\right) \\ &\propto \exp\left(-\frac{1}{\theta_0}\frac{(\mu-\mu_0)^2}{2\sigma_0^2} - \frac{1}{\theta_1}\frac{(\mu-\mu_1)}{2\sigma_1^2}\right), \end{aligned} \quad (18)$$

where $\mu_1 = \bar{x}$ denotes the sample mean of the data set, and $\theta_1 = N^{-1}$. If our goal is to calculate the MAP distribution then we have an optimisation problem

$$L = \min_{\mu \in \mathbb{R}} \frac{1}{\theta_0}\frac{(\mu-\mu_0)^2}{2\sigma_0^2} + \frac{1}{\theta_1}\frac{(\mu-\mu_1)^2}{2\sigma_1^2}. \quad (19)$$

Let

$$F_0(\mu) \equiv \frac{1}{\theta_0}\frac{(\mu-\mu_0)^2}{2\sigma_0^2}, \quad F_1(\mu) \equiv \frac{1}{\theta_1}\frac{(\mu-\mu_1)^2}{2\sigma_1^2}, \quad (20)$$

and thus we have $L = \min_{\mu \in \mathbb{R}} F_0(\mu) + F_1(\mu)$. Since $F_0$ and $F_1$ are convex, we could apply the Fenchel's duality to the problem $L$, which gives us the following dual problem

$$-L' = \min_{s \in \mathbb{R}} F_0^*(s) + F_1^*(-s), \quad (21)$$

where $F_0^*$ is the Legendre-Fenchel transform of $F_0$

$$F_0^*(s) = \sup_{\mu \in \mathbb{R}} s\mu - F_0(s) = s\mu_0 + \frac{1}{2}\sigma_0^2\theta_0 s^2, \quad (22)$$

and similarly $F_1^*(s) = s\mu_1 + \frac{1}{2}\sigma_1^2\theta_1 s^2$. Choose the hyper-parameter $\mu_0 = 0, \sigma_0 = 1$, and we finally have

$$\begin{aligned} -L' &= \min_{s \in \mathbb{R}} \frac{\theta_0}{2}s^2 + \left(-s\mu_1 + \frac{1}{2}\sigma_1^2\theta_1 s^2\right) \\ &= \min_{s \in \mathbb{R}} c(x) + \rho_1(s). \end{aligned} \quad (23)$$

This is exactly the agent's objective. Since $s$ and $\mu$ are dual to each other, the market performs the Bayesian update (MAP estimate) in the dual space of the mean parameters.