Supplementary Materials

A Proof of Theorem 2.1

Let $b^*$ be an optimal solution for a given instance of the problem. We first examine properties of GreedyProcedure that hold for arbitrary $b_0$. We then show that there exists some $b_0$ in the enumeration step such that GreedyProcedure returns $b$ with $f(b) \geq (1 - 1/e) f(b^*)$, which proves Theorem 2.1.

Let us fix an initial solution $b_0$, and analyze behavior of GreedyProcedure with input $b_0$. We denote by $b_i$ the tentative solution $b$ at the beginning of the $i$th iteration and denote by $s_i$ and $k_i$ $s$ and $k$ chosen in the $i$th iteration, respectively. Assume that GreedyProcedure first has not updated the tentative solution $b$ in the $Lth$ trial. Equivalently, let $L$ be the minimum number such that $b_L = b_{L+1}$ and $b_i < b_{i+1}$ for $i = 1, \ldots, L - 1$. Note that if such a situation never happens during the execution of GreedyProcedure, define $L$ to be the number of iterations.

Lemma A.1. Without loss of generality, we may assume that $b(s_L) + k_L \leq b^*(s_L)$.

Proof. Suppose that $b(s_L) + k_L > b^*(s_L)$. Let us consider a modified instance in which the capacity of $s_L$ is decreased to $b(s_L) + k_L - 1$. The optimal value is unchanged by this modification because $b^*$ is still feasible and optimal. Furthermore, GreedyProcedure returns the same solution (with respect to same $b_0$). Thus it suffices to analyze the algorithm in the modified instance. Repeating this argument completes the proof of this lemma.

Consider the $i$th iteration of the algorithm. For simplicity, we denote $\Delta_i(b_i, s_i, k_i)$ by $\Delta_i$, $\delta(b_i, s_i, k_i)$ by $\delta_i$, and $w(s_i)$ by $w_i$. Note that $f(b_i) = f(b_0) + \sum_{i=1}^{i-1} \Delta_i$ for $i = 1, \ldots, L$. Let $B' := B - w \cdot b_0$.

Lemma A.2. For $i = 1, \ldots, L$, we have $\Delta_i \geq \frac{w_i k_i}{B'} (f(b^*) - f(b_i))$.

Proof. Let us denote $b_i \vee b^* = b_i + \sum \alpha_s \chi_s$, where the sum is taken over $s$ in $\text{supp}^+ (b^* - b_i)$ and $\alpha_s := b^*(s) - b_i(s)$. Since $b_i \vee b^*(s) \chi_s = b_i + \alpha_s \chi_s$ for each $s \in \text{supp}^+ (b^* - b_i)$, (4) implies

$$f(b_i \vee b^*) \leq f(b_i) + \sum_{s \in \text{supp}^+ (b^* - b_i)} (f(b_i + \alpha_s \chi_s) - f(b_i))$$

$$\leq f(b_i) + \sum_{s \in \text{supp}^+ (b^* - b_i)} w(s) \alpha_s \delta_i,$$

where the last inequality follows from the fact that $(f(b_i + k_s \chi_s) - f(b_i)) / (w(s) k) \leq \delta_i$ for all $s \in S$ and $k$. Since $\sum w(s) \alpha_s$ is at most $B'$ and $f(b^*) \leq f(b_i \vee b^*)$ by the monotonicity of $f$, it holds that $f(b^*) \leq f(b_i) + \delta_i B'$. Therefore, we obtain $\delta_i \geq (f(b^*) - f(b_i)) / B'$.

Applying Lemma A.2 repeatedly, we have the following lemmas.
Lemma A.3. For \( i = 1, \ldots, L \), we have \( f(b^*) - f(b_i) \leq (f(b^*) - f(b_0)) \cdot \prod_{j=1}^{i-1} (1 - w_j k_j / B') \).

Proof. We prove this lemma by induction on \( i \). For \( i = 1 \), then the inequality holds because \( \Delta_1 \geq w_1 k_1 (f(b^*) - f(b_0)) / B' \) by Lemma A.2. For \( i > 1 \), we have

\[
\begin{align*}
f(b^*) - f(b_0) - \sum_{j=1}^{i} \Delta_j &= f(b^*) - f(b_0) - \sum_{j=1}^{i-1} \Delta_j - \Delta_i \\
&\leq f(b^*) - f(b_0) - \sum_{j=1}^{i-1} \Delta_j - \frac{w_{i} k_{i}}{B'} \left( f(b^*) - f(b_0) - \sum_{j=1}^{i-1} \Delta_j \right) \\
&= \left( 1 - \frac{w_{i} k_{i}}{B'} \right) \left( f(b^*) - f(b_0) - \sum_{j=1}^{i-1} \Delta_j \right) \\
&\leq \left( 1 - \frac{w_{i} k_{i}}{B'} \right) \cdot (f(b^*) - f(b_0)) \cdot \prod_{j=1}^{i-1} \left( 1 - \frac{w_{j} k_{j}}{B'} \right),
\end{align*}
\]

(by the induction hypothesis)

which completes the proof. \( \square \)

Lemma A.4. It holds that \( f(b^*) - f(b_i) \leq (f(b^*) - f(b_0)) / e \).

Proof. Let \( \varphi(x) = \ln(1 - x) \). Note that \( \varphi \) is concave in \([0, 1]\). By Jensen’s inequality, we have \( \sum_{i=1}^{L} \varphi(x_i) / L \leq \varphi(\sum_{i=1}^{L} x_i / L) \) for \( x_1, \ldots, x_L \in [0, 1] \). Putting \( x_i := w_i k_i / B' \), we obtain

\[
\frac{1}{L} \sum_{i=1}^{L} \ln \left( 1 - \frac{w_i k_i}{B'} \right) \leq \ln \left( 1 - \frac{1}{L} \sum_{i=1}^{L} \frac{w_i k_i}{B'} \right) \leq \ln \left( 1 - \frac{1}{L} \right),
\]

where the last inequality follows since \( \sum_{i=1}^{L} w_i k_i \geq B' \) and \( \varphi \) is a monotonically decreasing function. Thus we have

\[
\prod_{i=1}^{L} \left( 1 - \frac{w_i k_i}{B'} \right) \leq \left( 1 - \frac{1}{L} \right)^L \leq \frac{1}{e}.
\]

Combining this fact and Lemma A.3 completes the proof. \( \square \)

We now move to proving that the greedy procedure returns a \((1 - 1/e)\)-approximate solution for some \( b_0 \) in the enumeration step. If the optimal solution \( b^* \) satisfies \( \text{supp}^+(b^*) \leq 3 \), it can be found in the enumeration step. Therefore, we assume that all the optimal solutions have more than three positive components. Let \( s_1^*, s_2^*, s_3^* \) be elements in \( S \) satisfying:

\[
s_i^* \in \arg\max_{s \in S \setminus \{s_1^*, \ldots, s_{i-1}^* \}} \Delta \left( \bigvee_{j=1}^{i-1} b^*(s_j) \chi_{s_j}, s, b^*(s) \right)
\]

for \( i = 1, 2, 3 \). Since the size of the support of \( b^* \) is more than three, such \( s_1^*, s_2^* \) and \( s_3^* \) clearly exist.
Lemma A.5. For the feasible solution $b_0$ with the support $\{s^*_1, s^*_2, s^*_3\}$ satisfying $b_0(s^*_i) = b^*(s^*_i)$ for $1 \leq i \leq 3$, we have $\Delta_L \leq f(b_0)/3$.

Proof. It follows that $\Delta_L = f(b_L + k_L \chi_{s_L}) - f(b_L) \leq f(b_L \lor b^*(s_L)\chi_{s_L}) - f(b_L)$ since $b_L(s_L) + k_L \leq b^*(s_L)$. By Lemma A.4, $\Delta_L$ is at most $f(b^*(s_L)\chi_{s_L}) - f(0) = f(b^*(s^*_1)\chi_{s^*_1})$, and hence $\Delta_L \leq f(b^*(s^*_1)\chi_{s^*_1}) = \Delta(0, s^*_1, b^*(s^*_1))$ by the choice of $s^*_1$. Similarly, by the choices of $s^*_2$ and $s^*_3$, we have $\Delta_L \leq \Delta(b^*(s^*_1)\chi_{s^*_1}, s^*_2, b^*(s^*_2))$ and $\Delta_L \leq \Delta(b^*(s^*_1)\chi_{s^*_1} \lor b^*(s^*_2)\chi_{s^*_2}, s^*_3, b^*(s^*_3))$. Adding these inequalities, we obtain $\Delta_L \leq f(b_0)/3$. □

We are now ready to prove Theorem 2.1. Since $f(b) \geq f(b_0) + \sum_{i=1}^L \Delta_i - \Delta_L$, Lemmas [A.4] and [A.5] imply that

$$f(b) \geq (1 - 1/3)f(b_0) + (1 - 1/e)(f(b^*) - f(b_0)) \geq (1 - 1/e)f(b^*)$$

for the initial solution $b_0$ described in Lemma [A.5]. This completes the proof of Theorem 2.1.

B Proof of Lemmas

B.1 Proof of Lemma 2.2

Proof. If $k \leq x(s)$ then both sides are 0. If $x(s) < k \leq y(s)$ then the inequality $\Delta$ is equivalent to $f(x \lor k \chi_s) - f(x) \geq 0$, which is valid by monotonicity. Lastly, if $k > y(s)$ then we have $f(x \lor k \chi_s) + f(y) \geq f(y \lor k \chi_s) + f(x)$ by submodularity, which directly implies $\Delta$. □

B.2 Proof of Lemma 2.3

Proof. We prove this lemma by induction on the size of $\text{supp}^+(y - x)$. If $|\text{supp}^+(y - x)| = 0$, that is, $x \lor y = x$, then (3) is trivial. Suppose that there is an index $s \in \text{supp}^+(y - x)$. We define $y_1 = y - y(s)\chi_s$. Then $y = y_1 \lor y(s)\chi_s$ and $y_1 \lor y(s)\chi_s = 0$ hold. By submodularity of $x \lor y_1$ and $x \lor y(s)\chi_s$, we have

$$f(x \lor y_1) + f(x \lor y(s)\chi_s) \geq f(x \lor y) + f((x \lor y_1) \land (x \lor y(s)\chi_s)).$$

Since it holds that

$$(x \lor y_1) \land (x \lor y(s)\chi_s) = x \lor (y_1 \land y(s)\chi_s) = x \lor 0 = x,$$

the above inequality implies that

$$f(x \lor y_1) + f(x \lor y(s)\chi_s) - f(x) \geq f(x \lor y).$$

Therefore, applying the induction hypothesis to $x$ and $y_1$, we obtain (3). Thus the statement holds. □
B.3 Proof of Lemma 4.1

Proof. By simple algebra, we can easily check that

\[ f(b + \chi_s) - f(b) = p_s^{(b_s+1)} \sum_{t \in \Gamma(s)} \prod_{v \in \Gamma(t)} (1 - p_v^{(i)}), \]  

(3)

\[ f(b + 2\chi_s) - f(b + \chi_s) = (1 - p_s^{(b_s+1)})(1 - p_s^{(b_s+2)}) \sum_{t \in \Gamma(s)} \prod_{v \in \Gamma(t)} (1 - p_v^{(i)}). \]  

(4)

Let \( \alpha := \sum_{t \in \Gamma(s)} \prod_{v \in \Gamma(t)} (1 - p_v^{(i)}) \) for simplicity of notations. Then we have

\[ f(b + \chi_s) - f(b) - (f(b + 2\chi_s) - f(b + \chi_s)) = p_s^{(b_s+1)}(1 - p_s^{(b_s+1)}) \alpha \geq \alpha(p_s^{(b_s+2)} - p_s^{(b_s+2)}(1 - p_s^{(b_s+2)}))^2 \geq 0. \]

C Pseudocode of Algorithm

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Algorithm 3 SIMPLEGREEDYPROCEDUREFORCOMPETITORMODEL
1: for each \( t \in T \) do
2:   Let \( \lambda := 1 \)
3:   for \( k = 1 \) to \( 1 + \max_{s \in \Gamma(t)} \tilde{b}(s) \) do
4:     Let \( \phi_k(t) := 1 \) and \( \lambda_{t,k} := \lambda \left( 1 - \prod_{s \in \Gamma(t):\tilde{b}(s) \geq k} (1 - \tilde{p}_s^{(k)}) \right) \).
5:     Let \( \lambda := \lambda \prod_{s \in \Gamma(t):\tilde{b}(s) \geq k} (1 - \tilde{p}_s^{(k)}) \).
6:   end for
7: end for
8: Let \( \bar{b} := 0 \).
9: for \( i = 1 \) to \( B \) do
10:   Choose \( s \) maximizing \( \sum_{t \in \Gamma(s)} \sum_k \lambda_{t,k} r_s^{(b_s+1)} \phi_k(t) \).
11:   Let \( b(s) := b(s) + 1 \).
12: for \( t \in \Gamma(s) \) do
13:   for \( k = 1 \) to \( 1 + \max_{s \in \Gamma(t)} \tilde{b}(s) \) do
14:     Let \( \phi_k(t) := (1 - r_s^{(b(s))}) \phi_k(t) \).
15:   end for
16: end for
17: end for
18: return \( \bar{b} \)
```