Proof of Lemma 2.

Proof. Let $\boldsymbol{\theta}$ be the coordinate of $p \in \mathcal{U} \subset \mathcal{M}^n$ under some smooth coordinate map $\varphi_{\mathcal{U}}$ and $\boldsymbol{\gamma}$ be the coordinate of $f(p) \in \mathcal{U}' \subset \mathcal{N}^m$ under some smooth coordinate map $\varphi_{\mathcal{U}'}$. Since p' approaches p, we have $\boldsymbol{\theta}' = \boldsymbol{\theta} + d\boldsymbol{\theta}$, where $\boldsymbol{\theta}'$ is the coordinate of p' under the coordinate map $\varphi_{\mathcal{U}}$ and $d\boldsymbol{\theta}$ is an infinitesimal small change approaching $\mathbf{0}$. Furthermore, since $f : \mathcal{M}^n \longrightarrow \mathcal{N}^m$ is a differential map, the function $\varphi_{\mathcal{U}'} \circ f \circ \varphi_{\mathcal{U}}^{-1} : \Theta \longrightarrow \Gamma$, which we will denote by g, is also differentiable. According to the Taylor expansion, we have $\boldsymbol{\gamma}' = g(\boldsymbol{\theta}') = g(\boldsymbol{\theta} + d\boldsymbol{\theta}) = g(\boldsymbol{\theta}) + \bigtriangledown g(\boldsymbol{\theta}) d\boldsymbol{\theta} + R_g(d\boldsymbol{\theta}, \boldsymbol{\theta}) = \boldsymbol{\gamma} + \mathbf{J} d\boldsymbol{\theta} + R_g(d\boldsymbol{\theta}, \boldsymbol{\theta})$, where $\boldsymbol{\gamma}'$ is the coordinate of $f(p')$ under the coordinate map $\varphi_{\mathcal{U}'}$, $\mathbf{J}$ is the Jacobian matrix of the function g at point $\boldsymbol{\theta}$ and $R_g(d\boldsymbol{\theta}, \boldsymbol{\theta})$ is the remainder term of linear approximation. Finally, according to the definition of pullback metric, we have $\lim_{d\boldsymbol{\theta} \to \mathbf{0}} \frac{d_{\mathbf{G}(\boldsymbol{\gamma})}(\boldsymbol{\gamma}', \boldsymbol{\gamma})}{d_{\mathbf{G}^*(\boldsymbol{\theta})}(\boldsymbol{\theta}', \boldsymbol{\theta})} = \lim_{d\boldsymbol{\theta} \to \mathbf{0}} \frac{(\mathbf{J} d\boldsymbol{\theta} + R_g(d\boldsymbol{\theta}, \boldsymbol{\theta}))^T \mathbf{G}(\boldsymbol{\gamma}) (\mathbf{J} d\boldsymbol{\theta} + R_g(d\boldsymbol{\theta}, \boldsymbol{\theta}))}{d\boldsymbol{\theta}^T \mathbf{J}^T \mathbf{G}(\boldsymbol{\gamma}) \mathbf{J} d\boldsymbol{\theta}} = 1$. This ends the proof. $\square$