Large-margin Weakly Supervised Dimensionality Reduction (Supplementary Material)

1. Proof of Theorem 1

Theorem 1. *Fix $\theta \geq 0$. For any preference pair (z_1, z_2) in low-dimensional space $\mathcal{Z} \subset \mathbb{R}^d$, which can be partitioned into K disjoint sets, denoted by $\{C_{i=1}\}_{i=1}^K$, assume that $\|z_1 - z_2\| \in [a, b]$. Given a linear preference learning algorithm $\mathcal{A}$ $\{w : z \to \mathbb{R}\}$ and $\|w\| \leq W$, we have for any **s** $\subset \mathcal{Z}$,*

$$\begin{aligned}
&|\ell(\mathcal{A}_s, z_1, z_2) - \ell(\mathcal{A}_s, s_1, s_2)| \leq W\sqrt{2b^2 - 2a^2\cos(\theta)} \\
&\forall\, i, j = 1, \cdots, K \ : s_1, z_1 \in C_i \text{ and } s_2, z_2 \in C_j, \\
&\cos(z_1 - z_2, s_1 - s_2) \geq \cos(\theta).
\end{aligned}$$

Hence $\mathcal{A}$ is $(K, W\sqrt{2b^2 - 2a^2\cos(\theta)})$-robust.

Proof. We can partition $\mathcal{Z}$ into K disjoint sets so that if preference pairs (s_1, s_2) and (z_1, z_2) are close, then

$$\cos(z_1 - z_2, s_1 - s_2) \geq \cos(\theta).$$

Therefore,

$$\begin{aligned}
&|\ell(w, z_1, z_2) - \ell(w, s_1, s_2)| \\
&= |[1 - \langle w, z_1 - z_2\rangle]^+ - [1 - \langle w, s_1 - s_2\rangle]^+| \\
&\leq |\langle w, (z_1 - z_2) - (s_1 - s_2)\rangle| \\
&\leq W\|(z_1 - z_2) - (s_1 - s_2)\|.
\end{aligned}$$

For the norm term, we have

$$\begin{aligned}
&\|(z_1 - z_2) - (s_1 - s_2)\|^2 \\
&= \|z_1 - z_2\|^2 + \|s_1 - s_2\|^2 - 2\langle z_1 - z_2, s_2 - s_2\rangle \\
&\leq 2b^2 - 2a^2\cos(\theta).
\end{aligned}$$

By combining the above results, the proof is completed. □

2. Proof of Theorem 2

Theorem 2. *If a preference learning algorithm $\mathcal{A}$ is $(K, \epsilon(\cdot))$-robust and the training sample **s** is composed of n preference pairs $\{p_i = (s_1, s_2)\}_{i=1}^n$ whose examples are generated from μ, then for any $\delta > 0$, with probability at least $1 - \delta$, we have,*

$$|\mathcal{L}(\mathcal{A}_s) - \ell_{emp}(\mathcal{A}_s)| \leq \epsilon(s) + 2B\sqrt{\frac{2K\ln 2 + 2\ln(1/\delta)}{n}}.$$

Proof. Let N_i be the set of index of points of **s** that fall into the C_i. $(|N_1|, \cdots, |N_K|)$ is a IID random variable with parameters n and $(\mu(C_1), \cdots, \mu(C_K))$. We have

$$\begin{aligned}
&|\mathcal{L}(\mathcal{A}_s) - \ell_{emp}(\mathcal{A}_s)| \\
&= |\sum_{i=1}^K \sum_{j=1}^K E_{z_1, z_2 \sim \mu}(\ell(\mathcal{A}_s, z_1, z_2)|z_1 \in C_i, z_2 \in C_2)\mu(C_i)\mu(C_j) - \frac{1}{n^2}\sum_{i=1}^n \ell(\mathcal{A}_s, p_{(i,1)}, p_{(i,2)})|
\end{aligned}$$

$$
\begin{aligned}
&\leq |\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\mu(C_i)\mu(C_j) \\
&-\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\mu(C_i)\frac{N_j}{n}| \\
&+|\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\mu(C_i)\frac{N_j}{n} - \frac{1}{n^2}\sum_{i=1}^{n}\ell(\mathcal{A}_s,p_{(i,1)},p_{(i,2)})| \\
&\leq |\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\mu(C_i)(\mu(C_j)-\frac{N_j}{n})| \\
&+|\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\mu(C_i)\frac{N_j}{n} \\
&-\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\frac{N_i}{n}\frac{N_j}{n}| \\
&+|\sum_{i=1}^{K}\sum_{j=1}^{K} E_{z_1,z_2\sim\mu}(\ell(\mathcal{A}_s,z_1,z_2)|z_1\in C_i, z_2\in C_2)\frac{N_i}{n}\frac{N_j}{n}\frac{1}{n^2}\sum_{i=1}^{n}\ell(\mathcal{A}_s,p_{(i,1)},p_{(i,2)})| \\
&\leq B|\sum_{j=1}^{K}(\mu(C_j)-\frac{N_j}{n})| + B|\sum_{i=1}^{K}(\mu(C_i)-\frac{N_i}{n})| + |\frac{1}{n^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{a\in N_i}\sum_{b\in N_j}\max_{z\in C_i}\max_{z'\in C_j}|\ell(\mathcal{A}_s,z,z^{'})-\ell(\mathcal{A}_s,s_a,s_b)|| \\
&\leq \epsilon(s) + 2B\sum_{i=1}^{K}|\frac{N_i}{n}-\mu(C_i)| \\
&\leq \epsilon(s) + 2B\sqrt{\frac{2K\ln 2 + 2\ln(1/\delta)}{n}}.
\end{aligned}
$$

The first and second inequalities are due to the triangle inequality, and the third inequality is because of $\sum_{i=1}^{K}\mu(C_i)=1$ and $\sum_{i=1}^{K}\frac{N_j}{n}=1$. Finally, the last inequality is the application of Proposition 1. □