Appendix: Proofs

In this section we give detailed proofs of the assertions made in Section 4 and 5.

7.1. Proof of Proposition 3

Recall, for any $\mathbf{t} \in \mathbb{R}^d$, for $\Phi^{-1}(\mathbf{t})$, we mean $\left(\Phi_1^{-1}(t_1), \ldots, \Phi_d^{-1}(t_d)\right) \in \mathbb{R}^d$, where $\Phi_j(\cdot)$ is the CDF of $p_j(\cdot)$.

From $f_{\mathbf{u}}(\mathbf{t}) = e^{-i\mathbf{u}^T\Phi^{-1}(\mathbf{t})}$, for any $j = 1, \ldots, d$, we have

$$\frac{\partial f(\mathbf{t})}{\partial \mathbf{t}_j} = (-i)\frac{u_j}{p_j(\Phi_j^{-1}(t_j))} e^{-i\mathbf{u}^T\Phi_j^{-1}(\mathbf{t})} \,.$$

Then it is straightforward to get the following,

$$\frac{\partial^d f(\mathbf{t})}{\partial t_1 \cdots \partial t_d} = \prod_{j=1}^{d} \left((-i)\frac{u_j}{p_j(\Phi_j^{-1}(t_j))}\right) e^{-i\mathbf{u}^T\Phi_j^{-1}(\mathbf{t})} \,.$$

In (11), when $I = [d]$,

$$\begin{aligned}
\int_{[0,1]^{|I|}} \left|\frac{\partial f}{\partial \mathbf{u}_I}\right| d\mathbf{t}_I &= \int_{[0,1]^d} \left|\frac{\partial^d f(\mathbf{t})}{\partial t_1 \cdots \partial t_d}\right| dt_1 \cdots dt_d \\
&= \int_{[0,1]^d} \left|\prod_{j=1}^{d} \left((-i)\frac{u_j}{p_j(\Phi_j^{-1}(t_j))}\right) e^{-i\mathbf{u}^T\Phi_j^{-1}(\mathbf{t})}\right| dt_1 \cdots dt_d \\
&= \int_{[0,1]^d} \prod_{j=1}^{d} \left|\frac{u_j}{p_j(\Phi_j^{-1}(t_j))}\right| dt_1 \cdots dt_d \\
&= \prod_{j=1}^{d} \left(\int_{[0,1]} \left|\frac{u_j}{p_j(\Phi_j^{-1}(t_j))}\right| dt_j\right) \,. \qquad (25)
\end{aligned}$$

With a change of variable, $\Phi_j(t_j) = v_j$, for $j = 1, \ldots, d$, (25) becomes

$$\prod_{j=1}^{d} \left(\int_{[0,1]} \left|\left(\frac{u_j}{p_j(\Phi_j^{-1}(t_j))}\right)\right| dt_j\right) = \prod_{j=1}^{d} \left(\int_{\mathbb{R}} |u_j| dv_j\right) = \infty \,.$$

As this is a term in (11), we know that $V_{HK}[f_{\mathbf{u}}(\mathbf{t})]$ is unbounded.

7.2. Proof of Proposition 4

We make the assumption that,

$$\kappa = \sup_{\mathbf{x} \in \mathbb{R}^d} h(\mathbf{x}, \mathbf{x}) < \infty \,. \qquad (26)$$

We need the following lemmas, across which we will share some notation.

Lemma 12. *Under assumption 26, if $f \in \mathcal{H}$, where $\mathcal{H}$ is an RKHS with kernel $h(\cdot, \cdot)$, the integral $\int_{\mathbb{R}^d} f(\mathbf{x})p(\mathbf{x})d\mathbf{x}$ is finite.*

Proof. For notational convenience, we note that

$$\int_{\mathbb{R}^d} f(\mathbf{x})p(\mathbf{x})d\mathbf{x} = \mathbb{E}\left[f(X)\right] ,$$

where $\mathbb{E}\left[\cdot\right]$ denotes expectation and X is a random variable distributed according to the probability density p on $\mathbb{R}^d$.

Now consider a linear functional T that maps f to $\mathbb{E}[f(X)]$ i.e.

$$T[f] = \mathbb{E}[f(X)] \,. \tag{27}$$

The linear functional T is a bounded linear functional on the RKHS $\mathcal{H}$. To see this:

$$\begin{aligned} |\mathbb{E}[f(X)]| &\leq \mathbb{E}[|f(X)|] \quad \text{(Jensen's Inequality)} \\ &\leq \mathbb{E}[|\langle f, h(X,\cdot)\rangle_{\mathcal{H}}|] \quad \text{(Reproducing property)} \\ &\leq \|f\|_{\mathcal{H}} \mathbb{E}[\|h(X,\cdot)\|_{\mathcal{H}}] \quad \text{(Cauchy-Schwartz)} \\ &\leq \|f\|_{\mathcal{H}} \mathbb{E}\left[\sqrt{h(X,X)}\right] = \sqrt{\kappa} < \infty \,. \quad \text{(Assumption (26))} \end{aligned}$$

This shows that the integral $\int_{\mathbb{R}^d} f(\mathbf{x})p(\mathbf{x})d\mathbf{x}$ exists. □

Lemma 13. *The function $m(\cdot) = \int_{\mathbb{R}^d} h(\cdot, \mathbf{x})p(\mathbf{x})d\mathbf{x} \in \mathcal{H}$. In addition, for any $f \in \mathcal{H}$,*

$$\mathbb{E}[f(X)] = \int f(\mathbf{x})p(\mathbf{x})d\mathbf{x} = \langle f, m\rangle_{\mathcal{H}} \,. \tag{28}$$

Proof. From the Riesz Representation Theorem, every bounded linear functional on $\mathcal{H}$ admits an inner product representation. Therefore, for T defined in Eqn. 27, there exists $m \in \mathcal{H}$ such that,

$$T[f] = \mathbb{E}[f(X)] = \langle f, m\rangle_{\mathcal{H}} \,.$$

Therefore we have, $\langle f, m\rangle_{\mathcal{H}} = \int f(\mathbf{x})p(\mathbf{x})d\mathbf{x}$ for all $f \in \mathcal{H}$. For any $\mathbf{z}$, choosing $f(\cdot) = h(\mathbf{z}, \cdot)$, where $h(\cdot,\cdot)$ is the kernel associated with $\mathcal{H}$, from the reproducing property we see that,

$$\langle h(\mathbf{z},\cdot), m\rangle_{\mathcal{H}} = m(\mathbf{z}) = \int h(\mathbf{z}, \mathbf{x})p(\mathbf{x})d\mathbf{x} \,.$$

Hence, $\int_{\mathbb{R}^d} h(\cdot, \mathbf{x})p(\mathbf{x})d\mathbf{x} = m(\cdot) \in \mathcal{H}$. Eqn. (28) follows from $\mathbb{E}[f(X)] = \langle f, m\rangle_{\mathcal{H}}$. □

The proof of Proposition 4 follows from the existence Lemmas above, and the following steps.

$$\begin{aligned} \epsilon_{S,p}[f] &= \left|\int_{\mathbb{R}^d} f(\mathbf{x})p(\mathbf{x})d\mathbf{x} - \frac{1}{s}\sum_{l=1}^{s} f(\mathbf{w}_l)\right| \\ &= \left|\langle f, m\rangle_{\mathcal{H}} - \frac{1}{s}\sum_{l=1}^{s}\langle f, h(\mathbf{w}_l,\cdot)\rangle_{\mathcal{H}}\right| \\ &= \left|\langle f, m - \frac{1}{s}\sum_{l=1}^{s}(\mathbf{w}_l,\cdot)\rangle_{\mathcal{H}}\right| \\ &\leq \|f\|_{\mathcal{H}} \left\|m - \frac{1}{s}\sum_{l=1}^{s} h(\mathbf{w}_l,\cdot)\right\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}\, D_{h,p}(S) \,, \end{aligned}$$

where $D_{h,p}(S)$ is given as follows,

$$\begin{aligned} D_{h,p}(S)^2 &= \left\|m - \frac{1}{s}\sum_{l=1}^{s} h(\mathbf{w}_l,\cdot)\right\|_{\mathcal{H}}^2 \\ &= \langle m, m\rangle_{\mathcal{H}} - \frac{2}{s}\sum_{l=1}^{s}\langle m, h(\mathbf{w}_l,\cdot)\rangle_{\mathcal{H}} + \frac{1}{s^2}\sum_{l=1}^{s}\sum_{j=1}^{s}\langle h(\mathbf{w}_l,\cdot), h(\mathbf{w}_j,\cdot)\rangle_{\mathcal{H}} \\ &= \mathbb{E}[m(X)] - \frac{2}{s}\sum_{l=1}^{s}\mathbb{E}[h(\mathbf{w}_l,\cdot)] + \frac{1}{s^2}\sum_{l=1}^{s}\sum_{j=1}^{s} h(\mathbf{w}_l, \mathbf{w}_j) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(\omega,\phi)p(\omega)p(\phi)d\omega d\phi - \frac{2}{s}\sum_{l=1}^{s}\int_{\mathbb{R}^d} h(\mathbf{w}_l,\omega)p(\omega)d\omega + \frac{1}{s^2}\sum_{l=1}^{s}\sum_{j=1}^{s} h(\mathbf{w}_l,\mathbf{w}_j) \,. \end{aligned} \tag{29}$$

7.3. Proof of Theorem 6

We apply (17) to the particular case of $h = \mathrm{sinc}_{\mathbf{b}}$. We have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(\omega,\phi)p(\omega)p(\phi)d\omega d\phi &= \pi^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{j=1}^{d}\frac{\sin(b_j(\omega_j-\phi_j))}{\omega_j-\phi_j}p_j(\omega_j)p_j(\phi_j)d\omega d\phi \\ &= \pi^{-d}\prod_{j=1}^{d}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\sin(b_j(\omega_j-\phi_j))}{\omega_j-\phi_j}p_j(\omega_j)p_j(\phi_j)d\omega_j d\phi_j\ ,\end{aligned}$$

and

$$\begin{aligned}\sum_{l=1}^{s}\int_{\mathbb{R}^d} h(w_l,\omega)p(\omega)d\omega &= \pi^{-d}\sum_{l=1}^{s}\int_{\mathbb{R}^d}\prod_{j=1}^{d}\frac{\sin(b_j(w_{lj}-\omega_j))}{w_{lj}-\omega_j}p_j(\omega_j)d\omega \\ &= \pi^{-d}\sum_{l=1}^{s}\prod_{j=1}^{d}\int_{\mathbb{R}^d}\frac{\sin(b_j(w_{lj}-\omega_j))}{w_{lj}-\omega_j}p_j(\omega_j)d\omega_j\ .\end{aligned}$$

So we can consider each coordinate on its own.

Fix j. We have

$$\begin{aligned}\int_{\mathbb{R}}\frac{\sin(b_j x)}{x}p_j(x)d &= \int_{\mathbb{R}}\int_0^{b_j}\cos(\beta x)p_j(x)d\beta dx \\ &= \frac{1}{2}\int_{-b_j}^{b_j}\int_{\mathbb{R}} e^{i\beta x}p(x)dxd\beta \\ &= \frac{1}{2}\int_{-b_j}^{b_j}\varphi_j(\beta)d\beta\ .\end{aligned}$$

The interchange in the second line is allowed since the $p_j(x)$ makes the function integrable (with respect to x).

Now fix $w \in \mathbb{R}$ as well. Let $h_j(x,y) = \sin(b_j(x-y))/\pi(x-y)$. We have

$$\begin{aligned}\int_{\mathbb{R}} h_j(\omega,w)p_j(\omega)d\omega &= \pi^{-1}\int_{\mathbb{R}}\frac{\sin(b_j(\omega-w))}{\omega-w}p_j(\omega)d\omega \\ &= \pi^{-1}\int_{\mathbb{R}}\frac{\sin(b_j x)}{x}p_j(x+w)dx \\ &= (2\pi)^{-1}\int_{-b_j}^{b_j}\varphi_j(\beta)e^{iw\beta}d\beta\ ,\end{aligned}$$

where the last equality follows from first noticing that the characteristic function associated with the density function $x \mapsto p_j(x+w)$ is $\beta \mapsto \varphi(\beta)e^{iw\beta}$, and then applying the previous inequality.

We also have,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\sin(b_j(x-y))}{x-y}p_j(x)p_j(y)dxdy &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{b_j}\cos(\beta(x-y))p_j(x)p_j(y)d\beta dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-b_j}^{b_j}e^{i\beta(x-y)}p_j(x)p_j(y)d\beta dxdy \\ &= \frac{1}{2}\int_{-b_j}^{b_j}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i\beta(x-y)}p_j(x)p_j(y)dxdyd\beta \\ &= \frac{1}{2}\int_{-b_j}^{b_j}\left(\int_{\mathbb{R}}e^{i\beta x}p_j(x)dx\right)\left(\int_{\mathbb{R}}e^{-i\beta y}p_j(y)dy\right)d\beta \\ &= \frac{1}{2}\int_{-b_j}^{b_j}\varphi_j(\beta)\varphi_j(\beta)^*d\beta \\ &= \frac{1}{2}\int_{-b_j}^{b_j}|\varphi_j(\beta)|^2d\beta\ .\end{aligned}$$

The interchange at the third line is allowed because of $p_j(x)p_j(y)$. In the last line we use the fact that the φ_j is Hermitian.

7.4. Proof of Theorem 9

Let $b > 0$ be a scalar, and let $u \in [-b, b]$ and $z \in \mathbb{R}$. We have,

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-iux}\frac{\sin(b(x-z))}{\pi(x-z)}dx &= e^{-iuz}\int_{-\infty}^{\infty}e^{-i2\pi\frac{u}{2b}y}\frac{\sin(\pi y)}{\pi y}dy \\ &= e^{-iuz}\,\mathrm{rect}(u/2b) \\ &= e^{-iuz}\ .\end{aligned}$$

In the above, rect is the function that is 1 on $[-1/2, 1/2]$ and zero elsewhere.

The last equality implies that for every $\mathbf{u} \in \square\mathbf{b}$ and every $\mathbf{x} \in \mathbb{R}^d$ we have

$$f_{\mathbf{u}}(\mathbf{x}) = \int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{y})\,\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{x})d\mathbf{y}\ .$$

We now have for every $\mathbf{u} \in \square\mathbf{b}$,

$$\begin{aligned}\epsilon_{S,p}[f_{\mathbf{u}}] &= \left|\int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{x})p(\mathbf{x})d\mathbf{x} - \frac{1}{s}\sum_{i=1}^{s} f(\mathbf{w}_i)\right| \\ &= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{y})\,\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{x})d\mathbf{y}p(\mathbf{x})d\mathbf{x} - \frac{1}{s}\sum_{i=1}^{s}\int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{y})\,\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{w}_i)d\mathbf{y}\right| \\ &= \left|\int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{y})\left[\int_{\mathbb{R}^d}\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{x})p(\mathbf{x})d\mathbf{x} - \frac{1}{s}\sum_{i=1}^{s}\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{w}_i)\right]d\mathbf{y}\right|\ .\end{aligned}$$

Let us denote

$$r_S(\mathbf{y}) = \int_{\mathbb{R}^d}\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{x})p(\mathbf{x})d\mathbf{x} - \frac{1}{s}\sum_{i=1}^{s}\mathrm{sinc}_{\mathbf{b}}(\mathbf{y},\mathbf{w}_i)\ .$$

So,

$$\epsilon_{S,p}[f_{\mathbf{u}}] = \left|\int_{\mathbb{R}^d} f_{\mathbf{u}}(\mathbf{y})r_S(\mathbf{y})d\mathbf{y}\right|\ .$$

The function r_S is in $L2$ so it has a Fourier transform $\hat{r}_S$. The above formula is exactly the value of $\hat{r}_S$ at $\mathbf{u}$. That is,

$$\epsilon_{S,p}[f_{\mathbf{u}}] = |\hat{r}_S(\mathbf{u})|\ .$$

Now,

$$\begin{aligned}
\mathbb{E}_{f\sim\mathcal{U}(\mathcal{F}_{\square\mathbf{b}})}\left[\epsilon_{S,p}[f]^2\right] &= \mathbb{E}_{\mathbf{u}\sim\mathcal{U}(\square\mathbf{b})}\left[\epsilon_{S,p}[f_{\mathbf{u}}]^2\right] \\
&= \int_{\mathbf{u}\in\square\mathbf{b}} |\hat{r}_S(\mathbf{u})|^2 \left(\prod_{j=1}^d b_j\right)^{-1} d\mathbf{u} \\
&= \left(\prod_{j=1}^d b_j\right)^{-1} \|\hat{r}_S\|_{L2}^2 \\
&= \frac{(2\pi)^d}{\prod_{j=1}^d b_j}\|r_S\|_{PW_{\mathbf{b}}}^2 \\
&= \frac{(2\pi)^d}{\prod_{j=1}^d b_j} D_p^{\square}(S)^2 \,.
\end{aligned}$$

The equality before the last follows from Plancherel formula and the equality of the norm in $PW_{\mathbf{b}}$ to the $L2$-norm. The last equality follows from the fact that r_S is exactly the expression used in the proof of Proposition 4 to derive $D_p^{\square}$.

7.5. Proof of Corollary 10

In this case, $p(\mathbf{x}) = \prod_{j=1}^d p_j(x_j)$ where $p_j(x_j)$ is the density function of $\mathcal{N}(0, 1/\sigma_j)$. The characteristic function associated with p_j is $\varphi_j(\beta) = e^{-\frac{\beta^2}{2\sigma_j^2}}$. We apply (21) directly.

For the first term, since

$$\begin{aligned}
\int_0^{b_j} |\varphi_j(\beta)|^2 d\beta &= \int_0^{b_j} e^{-\frac{\beta^2}{\sigma_j^2}} d\beta \\
&= \sigma_j \int_0^{b_j/\sigma_j} e^{-y^2} dy \\
&= \frac{\sigma_j\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{b_j}{\sigma_j}\right) \,,
\end{aligned}$$

we have

$$(\pi)^{-d}\prod_{j=1}^d \int_0^{b_j} |\varphi_j(\beta)|^2 d\beta = \prod_{j=1}^d \frac{\sigma_j}{2\sqrt{\pi}} \operatorname{erf}\left(\frac{b_j}{\sigma_j}\right) \,. \tag{30}$$

For the second term, since

$$\begin{aligned}\int_{-b_j}^{b_j}\varphi_j(\beta)e^{iw_{lj}\beta}d\beta &= \int_{-b_j}^{b_j}e^{-\frac{\beta_j^2}{2\sigma^2}+iw_{lj}\beta}d\beta\\ &= e^{-\frac{\sigma_j w_{lj}}{2}}\int_{-b_j}^{b_j}e^{-\left(\frac{\beta}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)^2}d\beta\\ &= \sqrt{2}\sigma_j e^{-\frac{\sigma_j^2 w_{lj}^2}{2}}\int_{-\frac{b_j}{\sqrt{2}\sigma_j}}^{\frac{b_j}{\sqrt{2}\sigma_j}}e^{-\left(y-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)^2}dy\\ &= \sqrt{2}\sigma_j e^{-\frac{\sigma_j^2 w_{lj}^2}{2}}\int_{-\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}}^{\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}}e^{-z^2}dz\\ &= \frac{\sqrt{\pi}\sigma_j}{\sqrt{2}}e^{-\frac{\sigma_j^2 w_{lj}^2}{2}}\left(\operatorname{erf}\left(-\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)-\operatorname{erf}\left(\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)\right)\\ &= \sqrt{2\pi}\sigma_j e^{-\frac{\sigma_j^2 w_{lj}^2}{2}}\operatorname{Re}\left(\operatorname{erf}\left(-\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)\right)\ ,\end{aligned}$$

we have

$$\frac{2}{s}(2\pi)^{-d}\sum_{l=1}^{s}\prod_{j=1}^{d}\int_{-b_j}^{b_j}\varphi_j(\beta)e^{i\mathbf{w}_{lj}\beta}d\beta = \frac{2}{s}\sum_{l=1}^{s}\prod_{j=1}^{d}\frac{\sigma_j}{\sqrt{2\pi}}e^{-\frac{\sigma_j^2 w_{lj}^2}{2}}\operatorname{Re}\left(\operatorname{erf}\left(-\frac{b_j}{\sqrt{2}\sigma_j}-i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)\right)\ . \tag{31}$$

Combining (30), (31) and (21), (23) follows.

7.6. Proof of Proposition 11

Before we compute the derivative, we prove two auxiliary lemmas.

Lemma 14. *Let $\mathbf{x}\in\mathbb{R}^d$ be a variable and $\mathbf{z}\in\mathbb{R}^d$ be fixed vector. Then,*

$$\frac{\partial\,\mathrm{sinc}_{\mathbf{b}}(\mathbf{x},\mathbf{z})}{\partial x_j} = b_j\,\mathrm{sinc}'_{b_j}(x_j,z_j)\prod_{q\neq j}\mathrm{sinc}_{b_q}(x_q,z_q)\ . \tag{32}$$

We omit the proof as it is a simple computation that follows from the definition of $\mathrm{sinc}_{\mathbf{b}}$.

Lemma 15. *The derivative of the scalar function $f(x)=\operatorname{Re}\left[e^{-ax^2}\operatorname{erf}(c+idx)\right]$, for real scalars a,c,d is given by,*

$$\frac{\partial f}{\partial x} = -2axe^{-ax^2}\mathit{Real}\left[\operatorname{erf}(c+idx)\right]+\frac{2d}{\sqrt{\pi}}e^{-ax^2}e^{d^2x^2-c^2}\sin(2cdx)\ .$$

Proof. Since

$$\begin{aligned}f(x) &= \frac{1}{2}\left(e^{-ax^2}\operatorname{erf}(c+idx)+\left(e^{-ax^2}\operatorname{erf}(c+idx)\right)^*\right)\\ &= \frac{1}{2}\left(e^{-ax^2}\operatorname{erf}(c+idx)+e^{-ax^2}\operatorname{erf}(c-idx)\right)\ ,\end{aligned} \tag{33}$$

it suffices to compute the the derivative $g(x)=e^{-ax^2}\operatorname{erf}(c+idx)$.

Let $k(x)=\operatorname{erf}(c+idx)$. We have

$$g'(x) = -2axe^{-ax^2}k(x)+e^{-ax^2}k'(x)\ . \tag{34}$$

Since

$$
\begin{aligned}
k(x) &= \operatorname{erf}(c + idx) \\
&= \frac{2}{\sqrt{\pi}} \int_0^{c+idx} e^{-z^2} dz \\
&= \frac{2}{\sqrt{\pi}} \left(\int_0^{c} e^{-z^2} dz + \int_c^{c+idx} e^{-z^2} dz \right) \\
&= \frac{2}{\sqrt{\pi}} \left(\int_0^{c} e^{-y^2} dy + (id) \int_0^{x} e^{-(c+idt)^2} dt \right) ,
\end{aligned}
\tag{35}
$$

we have

$$
k'(x) = \frac{2}{\sqrt{\pi}} e^{-(c+idx)^2} = \frac{2d}{\sqrt{\pi}} e^{d^2x^2-c^2} (\sin(2cdx) + i\cos(2cdx)) \, . \tag{36}
$$

We now have

$$
\begin{aligned}
f'(x) &= \frac{1}{2} \left(g'(x) + (g^*(x))' \right) \\
&= \frac{1}{2} \left(g'(x) + (g'(x))^* \right) \\
&= \frac{1}{2} \left(-2axe^{-ax^2} (k(x) + k^*(x)) + e^{-ax^2} (k'(x) + (k'(x))^*) \right) \\
&= \frac{1}{2} \left(-4axe^{-ax^2} \operatorname{Re}\left[\operatorname{erf}\left(c + idx\right)\right] + e^{-ax^2} \frac{4d}{\sqrt{\pi}} e^{d^2x^2-c^2} \sin(2cdx) \right) \\
&= -2axe^{-ax^2} \operatorname{Re}\left[\operatorname{erf}\left(c + idx\right)\right] + \frac{2d}{\sqrt{\pi}} e^{-ax^2} e^{d^2x^2-c^2} \sin(2cdx) \, .
\end{aligned}
\tag{37}
$$

□

Proof of Proposition 11. For the first term in (23), that is $\frac{1}{s^2}\sum_{m=1}^{s}\sum_{r=1}^{s}\operatorname{sinc}_{\mathbf{b}}(\mathbf{w}_m, \mathbf{w}_r)$, to compute the partial derivative of w_{lj}, we only have to consider when at least m or r is equal to l. If $m = j = l$, by definition, the corresponding term in the summation is one. Hence, we only have to consider the case when $m \neq r$. By symmetry, it is equivalent to compute the partial derivative of the following function $\frac{2}{s^2}\sum_{m=1, m\neq l}^{s}\operatorname{sinc}_{\mathbf{b}}(\mathbf{w}_l, \mathbf{w}_m)$. Applying Lemma 14, we get the first term in (24).

Next, for the last term in (23), we only have consider the term associated with one in the summation and the term associated with j in the product. Since $\left(\frac{\sigma_j}{\sqrt{2\pi}}\right) e^{-\frac{\sigma_j^2 w_{lj}^2}{2}} \operatorname{Re}\left(\operatorname{erf}\left(\frac{b_j}{\sigma_j\sqrt{2}} - i\frac{\sigma_j w_{lj}}{\sqrt{2}}\right)\right)$ satisfies the formulation in Lemma 15, we can simply apply Lemma 15 and get its derivative with respect to w_{lj}.

(24) follows by combining these terms. □