Supplementary Material

Table 2. Results for the 33 benchmark problems that were not solved (near) optimally by all solvers: comparison of the gap between $V^L(b_0)$ and $V^U(b_0)$, lower bound $V^L(b_0)$, upper bound $V^U(b_0)$, the number of α-vectors ($|\Gamma|$) to represent V^L and time (seconds) for runs terminated after 1,000 seconds or when the gap is less than one unit at the 3rd significant digit. The smallest gaps or the highest lower bounds among algorithms are labeled with red color. Note that HSVI2, SARSOP and GapMin's results were reported in Tables 1 and 2 in (Poupart et al., 2011).

Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$	Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$
aloha.10						**aloha.30**					
($\|S\| = 30, \|A\| = 9, \|Z\| = 3, \gamma = 0.999$)						($\|S\| = 90, \|A\| = 29, \|Z\| = 3, \gamma = 0.999$)					
HSVI2	9.0	535.4	544.4	4,729	997	HSVI2	38	1,212	1,249	2,062	1,000
SARSOP	9.5	535.2	544.7	48	1,000	SARSOP	74	1,177	1,252	86	999
GapMin ST	10.3	534.1	544.4	136	673	GapMin ST	113	1,136	1,249	44	800
GapMin LP	7.6	536.5	544.2	152	968	GapMin LP	111	1,136	1,247	46	799
PGVI	7.3	537.4	544.7	58	999	PGVI	19	1,231	1,250	342	994
cheng.D3-1						**cheng.D3-2**					
($\|S\| = 3, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)						($\|S\| = 3, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)					
HSVI2	11	6,417	6,428	16	997	HSVI2	10	8,240	8,250	8	404
SARSOP	15	6,417	6,432	10	1,000	SARSOP	12	8,240	8,252	6	1,000
GapMin ST	10	6,412	6,422	8	26	GapMin ST	10	8,235	8,245	3	15
GapMin LP	10	6,412	6,422	8	25	GapMin LP	10	8,235	8,245	3	22
PGVI	10	6,417	6,427	8	990	PGVI	10	8,240	8,250	4	59
cheng.D3-3						**cheng.D3-4**					
($\|S\| = 3, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)						($\|S\| = 3, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)					
HSVI2	105	7,457	7,562	13	991	HSVI2	41	5,827	5,868	15	993
SARSOP	129	7,457	7,585	8	999	SARSOP	48	5,827	5,875	5	1,000
GapMin ST	10	7,452	7,462	7	56	GapMin ST	10	5,822	5,832	8	78
GapMin LP	10	7,452	7,462	7	37	GapMin LP	10	5,822	5,832	5	37
PGVI	102	7,457	7,559	14	999	PGVI	39	5,827	5,866	19	999
cheng.D3-5						**cheng.D4-1**					
($\|S\| = 3, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)						($\|S\| = 4, \|A\| = 4, \|Z\| = 4, \gamma = 0.999$)					
HSVI2	26	8,673	8,698	63	990	HSVI2	167	6,715	6,882	19	999
SARSOP	34	8,673	8,706	10	1,000	SARSOP	180	6,715	6,894	10	1,000
GapMin ST	10	8,668	8,678	9	34	GapMin ST	10	6,710	6,720	11	553
GapMin LP	10	8,668	8,678	10	15	GapMin LP	10	6,711	6,721	11	288
PGVI	31	8,673	8,703	19	1,000	PGVI	171	6,715	6,886	11	999
cheng.D4-2						**cheng.D4-3**					
($\|S\| = 4, \|A\| = 4, \|Z\| = 4, \gamma = 0.999$)						($\|S\| = 4, \|A\| = 4, \|Z\| = 4, \gamma = 0.999$)					
HSVI2	63	8,381	8,443	22	995	HSVI2	55	7,661	7,715	20	997
SARSOP	71	8,378	8,450	8	999	SARSOP	60	7,660	7,721	11	1,000
GapMin ST	10	8,376	8,386	12	135	GapMin ST	10	7,656	7,666	10	91
GapMin LP	10	8,376	8,386	13	115	GapMin LP	10	7,656	7,666	10	68
PGVI	64	8,381	8,445	9	1,000	PGVI	55	7,661	7,715	11	999
cheng.D4-4						**cheng.D4-5**					
($\|S\| = 4, \|A\| = 4, \|Z\| = 4, \gamma = 0.999$)						($\|S\| = 4, \|A\| = 4, \|Z\| = 4, \gamma = 0.999$)					
HSVI2	65	7,670	7,735	18	997	HSVI2	91	7,884	7,975	35	994
SARSOP	69	7,669	7,738	6	1,000	SARSOP	96	7,884	7,980	14	1,000
GapMin ST	10	7,665	7,675	16	313	GapMin ST	10	7,879	7,889	19	415
GapMin LP	10	7,665	7,675	11	109	GapMin LP	10	7,879	7,889	17	197
PGVI	64	7,670	7,734	16	1,000	PGVI	90	7,884	7,974	18	998
cheng-D5-1						**cit**					
($\|S\| = 5, \|A\| = 3, \|Z\| = 3, \gamma = 0.999$)						($\|S\| = 284, \|A\| = 4, \|Z\| = 28, \gamma = 0.990$)					
HSVI2	59	6,549	6,608	19	996	HSVI2	0.095	0.743	0.838	3,739	975
SARSOP	64	6,549	6,613	9	999	SARSOP	0.049	0.791	0.840	3,108	967
GapMin ST	10	6,544	6,554	1	26	GapMin ST	0.838	0.000	0.838	1	802
GapMin LP	10	6,544	6,554	1	25	GapMin LP	0.838	0.000	0.838	1	855
PGVI	54	6,549	6,603	12	999	PGVI	0.016	0.822	0.838	6,748	990
tiger-grid ($\|S\| = 81, \|A\| = 4, \|Z\| = 3, \gamma = 0.990$)						GapMin ST	0.106	2.296	2.402	386	912
HSVI2	0.388	2.138	2.525	3,394	990	GapMin LP	0.132	2.271	2.402	255	923
SARSOP	0.262	2.267	2.529	945	997	PGVI	0.215	2.293	2.508	1,947	991

Table 3. Results continued (1,000 seconds limit).

Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$	Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$
ejs1						**ejs2**					
$(\|S\| = 3, \|A\| = 4, \|Z\| = 2, \gamma = 0.999)$						$(\|S\| = 2, \|A\| = 2, \|Z\| = 2, \gamma = 0.999)$					
HSVI2	7.8	421.3	429.1	13	991	HSVI2	91	1,781	1,872	8	997
SARSOP	48.8	421.3	470.1	9	1,000	SARSOP	115	1,781	1,896	7	1,000
GapMin ST	0.4	421.1	421.5	9	52	GapMin ST	10	1,777	1,787	6	22
GapMin LP	0.3	421.2	421.6	9	65	GapMin LP	10	1,776	1,786	6	13
PGVI	0.1	421.3	421.4	9	184	PGVI	83	1,781	1,864	7	1,000
ejs4						**fourth**					
$(\|S\| = 3, \|A\| = 2, \|Z\| = 2, \gamma = 0.999)$						$(\|S\| = 1,052, \|A\| = 4, \|Z\| = 28, \gamma = 0.990)$					
HSVI2	20.2	-133.6	-113.4	7	999	HSVI2	0.376	0.242	0.617	3,345	994
SARSOP	22.8	-133.6	-110.8	2	1,000	SARSOP	0.330	0.288	0.618	3,595	975
GapMin ST	1.0	-134.1	-133.1	2	26	GapMin ST	0.618	0.000	0.618	1	532
GapMin LP	1.0	-134.1	-133.1	2	13	GapMin LP	0.618	0.000	0.618	1	669
PGVI	15.5	-133.6	-118.0	2	1,000	PGVI	0.142	0.475	0.617	6,166	984
hallway2						**hallway**					
$(\|S\| = 92, \|A\| = 5, \|Z\| = 17, \gamma = 0.950)$						$(\|S\| = 60, \|A\| = 5, \|Z\| = 21, \gamma = 0.950)$					
HSVI2	0.525	0.361	0.886	2,393	997	HSVI2	0.250	0.945	1.195	1,367	996
SARSOP	0.525	0.374	0.898	262	992	SARSOP	0.210	0.995	1.206	456	998
GapMin ST	0.372	0.417	0.789	294	940	GapMin ST	0.078	1.008	1.086	290	765
GapMin LP	0.428	0.362	0.790	153	759	GapMin LP	0.085	1.003	1.089	159	845
PGVI	0.422	0.440	0.862	736	996	PGVI	0.176	1.004	1.180	838	1,000
iff						**learning.c2**					
$(\|S\| = 104, \|A\| = 4, \|Z\| = 22, \gamma = 0.999)$						$(\|S\| = 12, \|A\| = 8, \|Z\| = 3, \gamma = 0.999)$					
HSVI2	0.924	8.931	9.855	7,134	999	HSVI2	0.090	1.549	1.639	4,082	996
SARSOP	0.775	9.095	9.871	6,811	997	SARSOP	0.093	1.556	1.648	4,903	996
GapMin ST	0.722	9.214	9.936	544	785	GapMin ST	0.078	1.553	1.631	810	893
GapMin LP	0.660	9.261	9.920	532	940	GapMin LP	0.024	1.558	1.582	470	885
PGVI	0.657	9.233	9.890	12,795	999	PGVI	0.055	1.558	1.613	13,606	997
learning.c3						**learning.c4**					
$(\|S\| = 24, \|A\| = 12, \|Z\| = 3, \gamma = 0.999)$						$(\|S\| = 48, \|A\| = 16, \|Z\| = 3, \gamma = 0.999)$					
HSVI2	0.250	2.364	2.614	4,229	988	HSVI2	0.567	3.055	3.622	4,569	999
SARSOP	0.222	2.446	2.668	981	997	SARSOP	0.321	3.358	3.679	923	982
GapMin ST	0.214	2.442	2.655	446	944	GapMin ST	0.363	3.308	3.671	349	858
GapMin LP	0.180	2.441	2.622	515	947	GapMin LP	0.353	3.306	3.658	500	989
PGVI	0.192	2.450	2.642	9,423	982	PGVI	0.296	3.367	3.663	7,436	991
machine						**milos-aaai97**					
$(\|S\| = 256, \|A\| = 4, \|Z\| = 16, \gamma = 0.990)$						$(\|S\| = 20, \|A\| = 6, \|Z\| = 8, \gamma = 0.900)$					
HSVI2	3.49	63.18	66.66	662	982	HSVI2	18.31	49.15	67.46	3,965	998
SARSOP	3.57	63.18	66.75	150	998	SARSOP	19.61	49.74	69.35	3,699	997
GapMin ST	2.98	62.93	65.90	77	817	GapMin ST	17.67	49.89	67.55	1,212	774
GapMin LP	3.20	62.39	65.59	67	856	GapMin LP	15.42	49.97	65.39	581	730
PGVI	3.09	63.18	66.27	209	990	PGVI	17.40	50.06	67.46	10,096	980
query.s2						**mit**					
$(\|S\| = 9, \|A\| = 2, \|Z\| = 3, \gamma = 0.990)$						$(\|S\| = 204, \|A\| = 4, \|Z\| = 28, \gamma = 0.990)$					
HSVI2	4.2	490.7	495.0	1,366	992	HSVI2	0.094	0.791	0.885	5,539	1,000
SARSOP	5.5	490.7	496.3	113	999	SARSOP	0.067	0.819	0.885	2,820	999
GapMin ST	1.0	490.4	491.4	37	224	GapMin ST	0.039	0.845	0.884	152	806
GapMin LP	1.0	490.5	491.5	31	57	GapMin LP	0.055	0.828	0.883	120	859
PGVI	3.8	490.7	494.6	495	1,000	PGVI	0.028	0.857	0.885	6,218	999
query.s3						**pentagon**					
$(\|S\| = 27, \|A\| = 3, \|Z\| = 3, \gamma = 0.990)$						$(\|S\| = 212, \|A\| = 4, \|Z\| = 28, \gamma = 0.990)$					
HSVI2	26.2	546.8	573.1	1,203	997	HSVI2	0.192	0.634	0.826	4,361	997
SARSOP	28.1	546.8	574.8	112	999	SARSOP	0.131	0.696	0.827	3,196	971
GapMin ST	10.8	546.7	557.5	154	686	GapMin ST	0.826	0.000	0.826	1	990
GapMin LP	7.0	546.7	553.7	119	706	GapMin LP	0.826	0.000	0.826	1	893
PGVI	26.0	546.9	572.9	549	1,000	PGVI	0.072	0.754	0.826	5,842	996
query.s4						**sunysb**					
$(\|S\| = 81, \|A\| = 4, \|Z\| = 3, \gamma = 0.990)$						$(\|S\| = 300, \|A\| = 4, \|Z\| = 28, \gamma = 0.990)$					
HSVI2	51.9	569.5	621.4	2,846	999	HSVI2	0.240	0.557	0.796	4,370	997
SARSOP	54.3	569.1	623.4	166	1,000	SARSOP	0.323	0.475	0.798	3,537	986
GapMin ST	46.1	569.6	615.6	377	958	GapMin ST	0.796	0.000	0.796	1	930
GapMin LP	43.2	569.5	612.7	169	939	GapMin LP	0.796	0.000	0.796	1	974
PGVI	52.2	569.6	621.8	446	979	PGVI	0.137	0.659	0.796	5,974	987

Table 4. Results for 8 problems of the 18 problems that were not solved (near) optimally by any of the solvers, with 50,000 seconds limit. The smallest gaps or the highest lower bounds among algorithms are labeled with red color. Note that HSVI2, SARSOP and GapMin's results were reported in Table 3 in (Poupart et al., 2011).

Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$	Algorithm	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$Time$
cit						**hallway**					
($\|S\| = 284, \|A\| = 4, \|Z\| = 28, \gamma = 0.990$)						($\|S\| = 60, \|A\| = 5, \|Z\| = 21, \gamma = 0.950$)					
HSVI2	0.018	0.819	0.837	29,803	49,760	HSVI2	0.179	0.994	1.173	15,374	49,951
SARSOP	0.017	0.823	0.840	21,168	49,916	SARSOP	0.178	1.013	1.191	3,053	49,992
GapMin ST	0.023	0.814	0.837	739	48,931	GapMin ST	0.043	1.015	1.058	947	34,828
GapMin LP	0.015	0.822	0.836	648	45,473	GapMin LP	0.036	1.016	1.051	851	43,184
PGVI	0.007	0.831	0.838	32,041	49,861	PGVI	0.134	1.017	1.150	18,902	49,831
hallway2						**iff**					
($\|S\| = 92, \|A\| = 5, \|Z\| = 17, \gamma = 0.950$)						($\|S\| = 104, \|A\| = 4, \|Z\| = 22, \gamma = 0.999$)					
HSVI2	0.421	0.432	0.853	18,505	49,983	HSVI2	0.199	9.302	9.501	40,984	50,000
SARSOP	0.448	0.434	0.882	1,901	49,973	SARSOP	0.290	9.259	9.549	54,016	49,966
GapMin ST	0.262	0.461	0.723	1,647	46,687	GapMin ST	0.634	9.273	9.908	1,614	34,472
GapMin LP	0.226	0.468	0.694	1,135	36,766	GapMin LP	0.156	9.275	9.431	1,626	40,046
PGVI	0.340	0.485	0.825	3,846	49,999	PGVI	0.237	9.267	9.504	63,784	49,978
machine						**mit**					
($\|S\| = 256, \|A\| = 4, \|Z\| = 16, \gamma = 0.990$)						($\|S\| = 204, \|A\| = 4, \|Z\| = 28, \gamma = 0.990$)					
HSVI2	2.89	63.18	66.07	7,857	49,998	HSVI2	0.058	0.827	0.885	34,461	49,942
SARSOP	3.02	63.18	66.20	996	49,963	SARSOP	0.020	0.866	0.885	20,662	49,616
GapMin ST	1.67	63.17	64.84	139	49,261	GapMin ST	0.011	0.871	0.882	861	41,564
GapMin LP	1.14	63.17	64.30	173	49,036	GapMin LP	0.009	0.872	0.881	832	43,680
PGVI	2.45	63.18	65.63	592	49,987	PGVI	0.011	0.874	0.885	30,598	49,911
pentagon						**sunysb**					
($\|S\| = 212, \|A\| = 4, \|Z\| = 28, \gamma = 0.990$)						($\|S\| = 36, \|A\| = 5, \|Z\| = 17, \gamma = 0.950$)					
HSVI2	0.135	0.691	0.826	29,033	49,924	HSVI2	0.217	2.286	2.502	28,182	49,978
SARSOP	0.070	0.757	0.827	21,950	49,994	SARSOP	0.231	2.290	2.522	5,333	49,987
GapMin ST	0.825	0.000	0.825	1	44,437	GapMin ST	0.055	2.322	2.377	2,404	38,675
GapMin LP	0.150	0.675	0.824	425	40,436	GapMin LP	0.052	2.321	2.373	2,404	43,254
PGVI	0.026	0.800	0.826	29,292	49,845	PGVI	0.160	2.317	2.477	20,836	49,935

Table 5. Parameters for the 35 small benchmark problems that PGVI could find near optimal solutions within 1,000 seconds.

Problem	$\|S\|$	$\|A\|$	$\|Z\|$	γ	Problem	$\|S\|$	$\|A\|$	$\|Z\|$	γ
1d	4	2	2	0.750	4×3.95	11	4	6	0.950
4×5×2.95	39	4	4	0.950	bridge-repair	5	12	5	0.950
cheese.95	11	4	7	0.950	cheese-taxi	34	7	10	0.950
cheng.D3-1	3	3	3	0.999	cheng.D3-2	3	3	3	0.999
concert	2	3	2	0.999	ejs1	3	4	2	0.999
ejs3	2	2	2	0.999	ejs5	2	2	2	0.999
ejs6	2	2	2	0.999	ejs7	2	2	2	0.999
ejs-ft-counter	2	2	2	0.900	line4-2goals	4	2	1	0.999
marking	9	4	3	0.870	marking2	9	4	3	0.870
mcc-example1	4	3	3	0.750	mcc-example2	4	3	3	0.750
mini-hall2	13	3	9	0.950	network.95	7	4	2	0.950
network	7	4	2	0.950	paint.95	4	4	2	0.950
parr95.95	7	3	6	0.950	saci-s12-a6-z5.95	12	6	5	0.950
saci-s100-a10-z31	100	10	31	0.950	shuttle.95	8	3	5	0.950
stand-tiger.95	4	4	4	0.950	tiger.95	2	3	2	0.950
tiger.aaai	2	3	2	0.750	web-ad	4	3	5	0.950
web-mall	2	3	2	0.950	hanks.95	4	4	2	0.950
baseball	7,681	6	9	0.999					

Table 6. PGVI's results for the 35 small benchmark problems that it could find a near optimal solution (gap smaller than one unit at the third significant digit) within 1,000 seconds. The linear correlation coefficient between $|\Gamma|$ and $Time$ is -0.0451, the linear correlation coefficient between $|S|$ and $Time$ is -0.0406, the linear correlation coefficient between $|B^s|$ and $Time$ is 0.9842, and the linear correlation coefficient between $\hat{\mathcal{P}}^{g,U}_{f,L}(\delta)$ with $\delta = 10^{-6}$ and $Time$ is 0.9869.

Problem	Gap	$V^L(b_0)$	$V^U(b_0)$	$\|\Gamma\|$	$\|B^s\|$	$\hat{\mathcal{P}}^{g,U}_{f,L}(\delta)$	$Time$ (s)
1d	0.00	1.26	1.26	6	9	9	0.00
4×3.95	0.01	1.89	1.90	227	152	152	0.25
4×5×2.95	0.01	2.08	2.09	2,428	1,680	1,679	30.62
bridge-repair	63	40,421	40,484	7	22	21	0.02
cheese.95	0.01	3.48	3.49	56	12	12	0.02
cheese-taxi	0.00	2.48	2.48	174	33	33	0.02
cheng.D3-1	10	6,417	6,427	8	31,565	26,202	989.98
cheng.D3-2	10	8,240	8,250	6	6,920	4,802	85.06
concert	0.00	0	0	1	1	1	0.01
ejs1	1	421	422	9	12,089	11,359	448.04
ejs3	0	1,712	1,712	5	12	11	0.07
ejs5	0	0	0	1	1	1	0.00
ejs6	0	0	0	1	1	1	0.00
ejs7	0	0	0	1	1	1	0.00
ejs-ft-counter	0.00	-3.18	-3.18	3	2	2	0.00
line4-2goals	0.00	0.47	0.47	2	3	3	0.00
marking	0.00	2.75	2.75	50	16	16	0.01
marking2	0.00	2.75	2.75	48	16	16	0.01
mcc-example1	0.00	0.38	0.38	10	21	21	0.00
mcc-example2	0.00	0.38	0.38	10	21	21	0.00
mini-hall2	0.01	2.71	2.72	166	18	18	0.04
network.95	1	293	294	24	3,023	3,023	10.40
network	1	293	294	24	3,065	3,065	10.12
paint.95	0.00	3.29	3.29	14	157	153	0.09
parr95.95	0.01	7.19	7.20	8	7	7	0.01
saci-s12-a6-z5.95	0.03	14.80	14.83	9	58	58	0.02
saci-s100-a10-z31	0.06	16.56	16.62	2	29	29	0.41
shuttle.95	0.10	32.79	32.89	109	6	6	0.02
stand-tiger.95	0.08	50.38	50.46	136	40	40	0.09
tiger.95	0.05	19.36	19.41	12	21	19	0.02
tiger.aaai	0.00	1.93	1.93	5	7	6	0.00
web-ad	0.001	0.804	0.805	543	738	725	1.32
web-mall	0.01	6.90	6.91	13	63	60	0.03
hanks.95	0.01	3.29	3.30	14	157	153	0.14
baseball	0.001	0.641	0.642	424	258	257	4.02

Table 7. Performance comparison, SARSOP and PGVI. Here, $Time$ represents the total computation time, and T_0 represents the initialization time on each test problem.

Problem	Gap	$Time$ (seconds)			$Time - T_0$ (seconds)		
		SARSOP	PGVI	Speedup	SARSOP	PGVI	Speedup
FieldVisionRockSample[5,5]	0.47	9,764	2,570	3.80	9,764	2,570	3.80
Tracking	0.69	9,998	2,294	4.36	9,105	1,401	6.50
Homecare	3.43	9,987	1,819	5.49	9,577	1,409	6.80
3D Navigation	754,977	9,934	$< 1{,}719$	> 5.78	9,922	$< 1{,}707$	> 5.81