A Proof of Expression in Equation (6)

In this section, we restate the results provided in [3, 16] in order to obtain eq.(6). We follow the well-known Lagrangian duality approach as in Appendix A in [14].

The following result is provided in [3, 16]. Let $\Omega(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be the family of all distributions with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. For a fixed weight vector $\mathbf{a}$ and constant b, we have:

$$\sup_{\mathcal{Z} \in \Omega(\boldsymbol{\mu}, \boldsymbol{\Sigma})} \mathbb{P}_{\mathbf{z} \sim \mathcal{Z}}[\mathbf{a}^{\mathrm{T}}\mathbf{z} \geq b] = \frac{1}{1+d^2}$$
$$\text{where} \quad d^2 = \inf_{\mathbf{a}^{\mathrm{T}}\mathbf{z} \geq b} (\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})$$

Let $\mathbf{a} = -\mathbf{w}$ and $b = 0$. We have:

$$\sup_{\mathcal{Z} \in \Omega(\boldsymbol{\mu}, \boldsymbol{\Sigma})} \mathbb{P}_{\mathbf{z} \sim \mathcal{Z}}[\mathbf{w}^{\mathrm{T}}\mathbf{z} \leq 0] = \frac{1}{1+d^2}$$
$$\text{where} \quad d^2 = \inf_{\mathbf{w}^{\mathrm{T}}\mathbf{z} \leq 0} (\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})$$

Note that if $\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} \leq 0$, then we can just take $\mathbf{z} = \boldsymbol{\mu}$ and obtain $d^2 = 0$, which is certainly the optimum because $d^2 \geq 0$ due to positive definiteness of $\boldsymbol{\Sigma}$. In what follows, we assume $\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} > 0$, as required in eq.(6).

We are interested in the value of d^2. That is, we seek for a closed-form solution of the *primal* problem:

$$\min_{\mathbf{w}^{\mathrm{T}}\mathbf{z} \leq 0} (\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu}) \tag{36}$$

which has the following Lagrangian:

$$\mathcal{L}(\mathbf{z}, \lambda) = (\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu}) + \lambda \mathbf{w}^{\mathrm{T}}\mathbf{z}$$

By optimality arguments (i.e. $\partial\mathcal{L}/\partial\mathbf{z} = \mathbf{0}$), we have that $\mathcal{L}$ is minimized at $\mathbf{z}^* = -\frac{\lambda}{2}\boldsymbol{\Sigma}\mathbf{w} + \boldsymbol{\mu}$. Therefore, the Lagrange dual function is given by:

$$\begin{aligned} g(\lambda) &= \inf_{\mathbf{z}} \mathcal{L}(\mathbf{z}, \lambda) \\ &= \mathcal{L}(\mathbf{z}^*, \lambda) \\ &= -\frac{\lambda^2}{4}\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w} + \lambda \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} \end{aligned}$$

Consequently, the *dual* problem of eq.(36) is:

$$\max_{\lambda \geq 0} g(\lambda)$$

Again, by optimality arguments (i.e. $\partial g/\partial\lambda = 0$), we have that g is maximized at $\lambda^* = 2\frac{\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}}{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}$. Note that $\lambda^* \geq 0$ since $\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} > 0$. Finally:

$$\begin{aligned} d^2 &= \max_{\lambda \geq 0} g(\lambda) \\ &= g(\lambda^*) \\ &= \frac{(\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu})^2}{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}} \\ &\equiv \mathcal{F}(\mathbf{w}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \end{aligned}$$

B Moment Generating Function of the Square of a Sub-Gaussian Variable

Let s be a sub-Gaussian variable with parameter σ_s and mean $\mu_s = \mathbb{E}[s]$. By sub-Gaussianity, we know that the moment generating function is bounded as follows:

$$(\forall t \in \mathbb{R}) \ \mathbb{E}[e^{t(s-\mu_s)}] \leq e^{\frac{1}{2}t^2\sigma_s^2}$$

Our goal is to find a similar bound for the moment generating function of the sub-exponential variable $v = s^2$. Let $\Gamma(r)$ be the Gamma function, the moments of the sub-Gaussian variable s are bounded as follows:

$$(\forall r \geq 0) \ \mathbb{E}[|s|^r] \leq r 2^{r/2} \sigma_s^r \Gamma(r/2)$$

Let $\mu_v = \mathbb{E}[v]$. By power series expansion and since $\Gamma(r) = (r-1)!$ for an integer r, we have:

$$\begin{aligned} \mathbb{E}[e^{t(v-\mu_v)}] &= 1 + t\mathbb{E}[v-\mu_v] + \sum_{r=2}^{\infty} \frac{t^r \mathbb{E}[(v-\mu_v)^r]}{r!} \\ &\leq 1 + \sum_{r=2}^{\infty} \frac{t^r \mathbb{E}[|s|^{2r}]}{r!} \\ &\leq 1 + \sum_{r=2}^{\infty} \frac{t^r 2r 2^r \sigma_s^{2r} \Gamma(r)}{r!} \\ &= 1 + \sum_{r=2}^{\infty} t^r 2^{r+1} \sigma_s^{2r} \\ &= 1 + \frac{8t^2\sigma_s^4}{1 - 2t\sigma_s^2} \end{aligned}$$

By making $|t| \leq 1/(4\sigma_s^2)$, we have $1/(1-2t\sigma_s^2) \leq 2$. Finally, since $(\forall\alpha)\ 1 + \alpha \leq e^{\alpha}$, we have that for a sub-Gaussian variable s with parameter σ_s:

$$(\forall |t| \leq 1/(4\sigma_s^2)) \ \mathbb{E}[e^{t(s^2 - \mathbb{E}[s^2])}] \leq e^{16t^2\sigma_s^4} \tag{37}$$

Thus, we obtained a bound for the moment generating function of the sub-exponential variable s^2, that is similar to that of sub-Gaussian variables but holds only for a small range of t.